Research Article

On the Higher-Order $q$-Euler Numbers and Polynomials with Weight $\alpha$

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Received 9 May 2011; Accepted 19 June 2011

Academic Editor: Cengiz Çinar

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The main purpose of this paper is to present a systemic study of some families of higher-order $q$-Euler numbers and polynomials with weight $\alpha$. In particular, by using the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$, we give a new concept of $q$-Euler numbers and polynomials with weight $\alpha$.

1. Introduction

Let $p$ be a fixed odd prime. Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C},$ and $\mathbb{C}_p$, will, respectively, denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$ (see [1–14]). When one speaks of $q$-extension, $q$ can be regarded as an indeterminate, complex number $q \in \mathbb{C}$, or $p$-adic number $q \in \mathbb{C}_p$; it is always clear from context. If $q \in \mathbb{C}$, we assume $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|1 - q|_p < 1$ (see [1–14]).

In this paper, we use the notation of $q$-number as follows:

$$[x]_q = \frac{1 - q^x}{1 - q} \quad (1.1)$$

(see [1–14]). Note that $\lim_{q \to 1} [x]_q = x$ for any $x$ with $|x|_p \leq 1$ in the $p$-adic case.
Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) \, d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) (-q)^x,$$

(1.2)

$$= \lim_{N \to \infty} \frac{[2]_q}{2} \sum_{x=0}^{N-1} f(x) (-q)^x$$

(see [4–7]).

From (1.2), we note that

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0),$$

(1.3)

where $f_1(x) = f(x + 1)$.

It is well known that the ordinary Euler polynomials are defined by

$$\frac{2}{e^t + 1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!},$$

(1.4)

with the usual convention of replacing $E^n(x)$ by $E_n(x)$.

In the special case, $x = 0$ and $E_n(0) = E_n$ are called the $n$th Euler numbers (see [1–14]).

By (1.5), we get the following recurrence relation as follows:

$$E_0 = 1, \quad (E + 1)^n + E = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases}$$

(1.5)

Recently, $(h, q)$-Euler numbers are defined by

$$E^{(h)}_{0,q} = \frac{2}{1 + q^{h'}}, \quad q^h \left(qE^{(h)}_q + 1\right)^n + E^{(h)}_n = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases}$$

(1.6)

with the usual convention about replacing $(E^{(h)}_q)^n$ by $E^{(h)}_n$ (see [1–12]).

Note that $\lim_{q \to 0} E^{(h)}_{n,q} = E_n$.

For $\alpha \in \mathbb{N}$, the weight $q$-Euler numbers are also defined by

$$\tilde{E}^{(\alpha)}_{0,q} = 1, \quad q^h \left(q^{\alpha} \tilde{E}^{(\alpha)}_q + 1\right)^n + \tilde{E}^{(\alpha)}_n = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases}$$

(1.7)

with the usual convention about replacing $(\tilde{E}^{(\alpha)}_q)^n$ by $\tilde{E}^{(\alpha)}_n$ (see [4]).
The purpose of this paper is to present a systemic study of some families of higher-order $q$-Euler numbers and polynomials with weight $\alpha$. In particular, by using the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$, we give a new concept of $q$-Euler numbers and polynomials with weight $\alpha$.

2. Higher-Order $q$-Euler Numbers and Polynomials with Weight $\alpha$

For $h \in \mathbb{Z}$, $\alpha, k \in \mathbb{N}$, and $n \in \mathbb{Z}_+$, let us consider the expansion of higher-order $q$-Euler polynomials with weight $\alpha$ as follows:

$$
\tilde{E}^{(a)}_{n,q}(h, k | x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} x_1 + \cdots + x_k + x \frac{n!}{q^{x_1(h-1)} \cdots x_k(h-k)} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k).
$$

From (1.2) and (2.1), we note that:

$$
\tilde{E}^{(a)}_{n,q}(h, k | x) = \frac{[2]^k_q}{(1 - q^a)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{q^{alx}}{(1 + q^{al+h}) \cdots (1 + q^{al+h-k+1})}.
$$

In the special case, $x = 0$, $\tilde{E}^{(a)}_{n,q}(h, k | 0) = \tilde{E}^{(a)}_{n,q}(h, k)$ are called the higher-order $q$-Euler numbers with weight $\alpha$.

By (2.1), we get

$$
\tilde{E}^{(a)}_{n,q}(h, k) = (q^a - 1) \tilde{E}^{(a)}_{n+1,q}(h - \alpha, k) + \tilde{E}^{(a)}_{n,q}(h - \alpha, k).
$$

From (2.1) and (2.2), we have

$$
\tilde{E}^{(a)}_{0,q}(ma, k + 1) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{j=1}^{k+1} (ma-j) x_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_{k+1}) = \sum_{l=0}^{m} \binom{m}{l} (q^a - 1)^l \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} x_1 + \cdots + x_{k+1} \frac{q^{\sum_{j=1}^{k+1} j x_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_{k+1})} = \sum_{l=0}^{m} \binom{m}{l} (q^a - 1)^l \tilde{E}^{(a)}_{l,q}(0, k + 1) = \frac{[2]^k_{q^m}}{(1 + q^{am})(1 + q^{am-1}) \cdots (1 + q^{am-k})}.
$$
From (2.1), we can derive the following equation:

\[
\sum_{j=0}^{i} \binom{i}{j} (q^a - 1)^j \int \cdots \int [x_1 + \cdots + x_k]^{n-j} q^{(h-a-1)x_1 + \cdots + (h-a-k)x_k} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)
\]

\[
= \int \cdots \int [x_1 + \cdots + x_k]^{n-j} q^{(h-1)x_1 + \cdots + (h-k)x_k} q^{(a_1 + \cdots + x_k) (i-1)} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)
\]

(2.5)

\[
= \sum_{j=0}^{i-1} (q^a - 1)^j \binom{i-1}{j} \tilde{E}_{n-i+j,q}^{(a)}(h,k).
\]

By (2.1), (2.2), (2.3), and (2.4), we see that

\[
\sum_{j=0}^{i} (q^a - 1)^j \binom{i}{j} \tilde{E}_{n-i+j,q}^{(a)}(h-\alpha,k) = \sum_{j=0}^{i-1} (q^a - 1)^j \binom{i-1}{j} \tilde{E}_{n-i+j,q}^{(a)}(h,k).
\]

(2.6)

Therefore, we obtain the following theorem.

**Theorem 2.1.** For \(a, k \in \mathbb{N}\) and \(n, i \in \mathbb{Z}_+\), one has

\[
\sum_{j=0}^{i} \binom{i}{j} (q^a - 1)^j \tilde{E}_{n-i+j,q}^{(a)}(h-\alpha,k) = \sum_{j=0}^{i-1} (q^a - 1)^j \binom{i-1}{j} \tilde{E}_{n-i+j,q}^{(a)}(h,k).
\]

(2.7)

By simple calculation, we easily see that

\[
\sum_{j=0}^{m} \binom{m}{j} (q^a - 1)^j \tilde{E}_{j,q}^{(a)}(0,k) = \frac{[2]_q^k}{(1 + q^{am}) (1 + q^{am-1}) \cdots (1 + q^{am-k+1})}.
\]

(2.8)

### 3. Polynomials \(\tilde{E}_{n,q}^{(a)}(0, k \mid x)\)

We now consider the polynomials \(\tilde{E}_{n,q}^{(a)}(0, k \mid x)\) (in \(q^a\)) by

\[
\tilde{E}_{n,q}^{(a)}(0, k \mid x) = \int \cdots \int [x + x_1 + \cdots + x_k]^{n-j} q^{\sum_{j=1}^{k} j x_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k).
\]

(3.1)

By (3.1), we get

\[
(q^a - 1)^n \tilde{E}_{n,q}^{(a)}(0, k \mid x) = [2]_q^k \sum_{l=0}^{n} \binom{n}{l} q^{alx} (-1)^{n-l} \frac{1}{(1 + q^{al}) \cdots (1 + q^{al-k+1})}.
\]

(3.2)
From (3.1) and (3.2), we can derive the following equation:

\[
\int_{\mathbb{Z}_q} \cdots \int_{\mathbb{Z}_q} q^{\sum_{j=1}^{k} (an-j) x_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) = \sum_{j=0}^{n} \binom{n}{j} \alpha_j^1 \frac{1}{q-1} \tilde{E}_{j,q}^{(a)}(0,k | x),
\]

\[
\int_{\mathbb{Z}_q} \cdots \int_{\mathbb{Z}_q} q^{\sum_{j=1}^{k} (an-j) x_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) = \frac{[2]^k q^{an}}{(1 + q^an) \cdots (1 + q^an-k+1)}.
\]  

(3.3)

Therefore, by (3.2) and (3.3), we obtain the following theorem.

**Theorem 3.1.** For \( \alpha \in \mathbb{N} \) and \( n, k \in \mathbb{Z}_q \), one has

\[
\tilde{E}_{n,q}^{(a)}(0,k | x) = \frac{[2]^k q^{an}}{[\alpha]^n_q (1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{alx} \frac{1}{(-q^{an-k+1} : q)_k},
\]

(4.3)

\[
\sum_{l=0}^{n} \binom{n}{l} [\alpha]^l_q (q-1)^l \tilde{E}_{l,q}^{(a)}(0,k | x) = \frac{q^{an} [2]^k q^{nk}}{(-q^{an-k+1} : q)_k},
\]

where \( (a : q)_0 = 1 \) and \( (a : q)_k = (1-a)(1-aq) \cdots (1-aq^{k-1}) \).

Let \( d \in \mathbb{N} \) with \( d \equiv 1 \mod(2) \). Then we have

\[
\int_{\mathbb{Z}_q} \cdots \int_{\mathbb{Z}_q} \left[ x + \sum_{j=1}^{k} x_j \right] q^d \sum_{j=1}^{k} \alpha_j \frac{1}{q^{d \sum_{j=1}^{k} \alpha_j}}
\]

\[
= \frac{[d]^n_q}{[d]^k_{-q}} \sum_{a_1, \ldots, a_k=0}^{d-1} q^{-\sum_{i=1}^{k} (1-a_i) (-1)^{\sum_{j=1}^{k} a_j}}
\]

\[
\times \int_{\mathbb{Z}_q} \cdots \int_{\mathbb{Z}_q} \left[ x + \sum_{j=1}^{k} \frac{a_j}{d} + \sum_{j=1}^{k} x_j \right] \frac{1}{q^{d \sum_{j=1}^{k} \alpha_j}}
\]

\[
= \frac{[d]^n_q}{[d]^k_{-q}} \sum_{a_1, \ldots, a_k=0}^{d-1} q^{-\sum_{i=1}^{k} (1-a_i) (-1)^{\sum_{j=1}^{k} a_j}} \tilde{E}_{n,q}^{(a)}(0,k | x).
\]

Thus, by (3.5), we obtain the following theorem.

**Theorem 3.2.** For \( d \in \mathbb{N} \) with \( d \equiv 1 \mod(2) \), one has

\[
\tilde{E}_{n,q}^{(a)}(0,k | x) = \frac{[d]^n_q}{[d]^k_{-q}} \sum_{a_1, \ldots, a_k=0}^{d-1} q^{-\sum_{i=1}^{k} (1-a_i) (-1)^{\sum_{j=1}^{k} a_j}} \tilde{E}_{n,q}^{(a)}(0,k | \frac{x + a_1 + \cdots + a_k}{d}).
\]

(3.6)
Moreover,

$$\tilde{E}^{(a)}_{n,q}(0, k \mid dx) = \frac{[d]_q^n}{[d]_{-q}^k} \sum_{a_1, \ldots, a_k=0}^{d-1} q^{-\sum_{j=1}^{k} (j-1)a_j} (-1)^{\sum_{j=1}^{k} a_j} \tilde{E}^{(a)}_{n,q}(0, k \mid x + \frac{a_1 + \cdots + a_k}{d}) .$$

(3.7)

By (3.1), we get

$$\tilde{E}^{(a)}_{n,q}(0, k \mid x) = \sum_{l=0}^{n} \binom{n}{l} [x]_{q^l}^{n-l} q^{alx} \tilde{E}^{(a)}_{l,q}(0, k) ,$$

(3.8)

where \( \tilde{E}^{(a)}_{n,q}(0, k \mid 0) = \tilde{E}^{(a)}_{n,q}(0, k) .

Thus, we note that

$$\tilde{E}^{(a)}_{n,q}(0, k \mid x + y) = \sum_{l=0}^{n} \binom{n}{l} [y]_{q^l}^{n-l} q^{aly} \tilde{E}^{(a)}_{l,q}(0, k \mid x) .$$

(3.9)

4. Polynomials \( \tilde{E}^{(a)}_{n,q}(h, 1 \mid x) \)

Let us define polynomials \( \tilde{E}^{(a)}_{n,q}(h, 1 \mid x) \) as follows:

$$\tilde{E}^{(a)}_{n,q}(h, 1 \mid x) = \int_{\mathbb{Z}_p} [x + x_1]_{q^h}^{n} q^{x_1(h-1)} d\mu_{-q}(x_1) .$$

(4.1)

From (4.1), we have

$$\tilde{E}^{(a)}_{n,q}(h, 1 \mid x) = \frac{[2]_q}{(1 - q^h)} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{alx} \frac{1}{1 + q^{alx+h}} .$$

(4.2)

By the calculation of the fermionic p-adic q-integral on \( \mathbb{Z}_p \), we see that

$$q^{ax} \int_{\mathbb{Z}_p} [x + x_1]_{q^h}^{n} q^{x_1(h-1)} d\mu_{-q}(x_1)$$

$$= (q^a - 1) \int_{\mathbb{Z}_p} [x + x_1]_{q^h}^{n+1} q^{x_1(h-a-1)} d\mu_{-q}(x_1) + \int_{\mathbb{Z}_p} [x + x_1]_{q^h}^{n} q^{x_1(h-a-1)} d\mu_{-q}(x_1) .$$

(4.3)

Thus, by (4.3), we obtain the following theorem.

Theorem 4.1. For \( \alpha \in \mathbb{N} \) and \( h \in \mathbb{Z} \), one has

$$q^{ax} \tilde{E}^{(a)}_{n,q}(h, 1 \mid x) = (q^a - 1) \tilde{E}^{(a)}_{n+1,q}(h - \alpha - 1, 1 \mid x) + \tilde{E}^{(a)}_{n,q}(h - \alpha - 1, 1 \mid x) .$$

(4.4)
It is easy to show that

\[
\tilde{E}_{n,q}^{(a)}(h, 1 | x) = \int_{\mathbb{Z}_q} [x + x_1]_q^n q^{x_1(h-1)} d\mu_{-q}(x_1)
\]

\[
= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} \int_{\mathbb{Z}_q} [x_1]_q^n q^{x_1(h-1)} d\mu_{-q}(x_1)
\]

\[
= \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^{lx} \tilde{E}_{n,q}^{(a)}(h, 1)
\]

\[
= \left( q^{ax} \tilde{E}_{\bar{q}}^{(a)}(h, 1) + [x]_q^n \right), \text{ for } n \geq 1,
\]

with the usual convention about replacing \((\tilde{E}_{\bar{q}}^{(a)}(h, 1))^n \) by \(\tilde{E}_{n,q}^{(a)}(h, 1)\).

From \(qI_q(f_1) + I_{-q}(f) = [2]_{q^f}(0)\), we have

\[
q^h \int_{\mathbb{Z}_p} [x + x_1 + 1]_q^n q^{x_1(h-1)} d\mu_{-q}(x_1) + \int_{\mathbb{Z}_q} [x + x_1]_q^n q^{x_1(h-1)} d\mu_{-q}(x_1) = [2]_q[x]_q^n.
\]

(4.6)

By (4.3) and (4.6), we get

\[
q^h \tilde{E}_{n,q}^{(a)}(h, 1 | x + 1) + \tilde{E}_{n,q}^{(a)}(h, 1 | x) = [2]_q[x]_q^n.
\]

(4.7)

For \(x = 0\) in (4.7), we have

\[
q^h \tilde{E}_{n,q}^{(a)}(h, 1 | 1) + \tilde{E}_{n,q}^{(a)}(h, 1) = \begin{cases} [2]_q^n, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases}
\]

(4.8)

Therefore, by (4.8), we obtain the following theorem.

**Theorem 4.2.** For \(h \in \mathbb{Z}\) and \(n \in \mathbb{Z}_+\), one has

\[
q^h \left( q^{a} \tilde{E}_{\bar{q}}^{(a)}(h, 1) + 1 \right)^n + \tilde{E}_{n,q}^{(a)}(h, 1) = \begin{cases} [2]_q^n, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases}
\]

(4.9)

with the usual convention about replacing \((\tilde{E}_{\bar{q}}^{(a)}(h, 1))^n \) by \(\tilde{E}_{n,q}^{(a)}(h, 1)\).

From the fermionic \(p\)-adic \(q\)-integral on \(\mathbb{Z}_p\), we easily get

\[
\tilde{E}_{0,q}^{(a)}(h, 1) = \int_{\mathbb{Z}_p} q^{x_1(h-1)} d\mu_{-q}(x_1) = [2]_q [2]_{q^f}.
\]

(4.10)
By (4.1), we see that

\[
\tilde{E}_{n,q}^{(a)}(h, 1 | 1 - x) = \int_{\mathbb{Z}_p} [1 - x + x_1]^n q^{-x_1(h-1)} d\mu_{-q^{-1}}(x_1)
\]

\[
= (-1)^n q^{an+1-h} \frac{[2]_q}{(1 - q^n)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{sl} \frac{1}{1 + q^{al+h}}
\]

\[
= (-1)^n q^{an+1-h} \tilde{E}_{n,q}^{(a)}(h, 1 | x).
\] (4.11)

Therefore, by (4.11), we obtain the following theorem.

**Theorem 4.3.** For \( \alpha \in \mathbb{N}, h \in \mathbb{Z}, \) and \( n \in \mathbb{Z}_+, \) one has

\[
\tilde{E}_{n,q}^{(a)}(h, 1 | 1 - x) = (-1)^n q^{an+1-h} \tilde{E}_{n,q}^{(a)}(h, 1 | x).
\] (4.12)

In particular, for \( x = 1, \) one gets

\[
\tilde{E}_{n,q}^{(a)}(h, 1) = (-1)^n q^{an+1-h} \tilde{E}_{n,q}^{(a)}(h, 1 | 1)
\]

\[
= (-1)^n q^{an+1-h} \tilde{E}_{n,q}^{(a)}(h, 1) \quad \text{if} \quad n \geq 1.
\] (4.13)

Let \( d \in \mathbb{N} \) with \( d \equiv 1 \pmod{2}. \) Then one has

\[
\int_{\mathbb{Z}_p} q^{x_1(h-1)} [x + x_1]^n d\mu_{-q}(x_1)
\]

\[
= \frac{[d]_q^n}{[d]_{-q}} \sum_{a=0}^{d-1} q^{ha} (-1)^a \int_{\mathbb{Z}_p} \frac{x + a}{d} d\mu_{-q^a}(x).
\] (4.14)

Therefore, by (4.14), we obtain the following theorem.

**Theorem 4.4** (Multiplication formula). For \( d \in \mathbb{N} \) with \( d \equiv 1 \pmod{2}, \) we have

\[
\tilde{E}_{n,q}^{(a)}(h, 1 | x) = \frac{[d]_q^n}{[d]_{-q}} \sum_{a=0}^{d-1} q^{ha} (-1)^a \tilde{E}_{n,q}^{(a)} \left( h, 1 | \frac{x + a}{d} \right).
\] (4.15)
5. Polynomials \( \overline{E}_{n,q}^{(a)}(h, k \mid x) \) and \( k = h \)

In (2.1), we know that

\[
\overline{E}_{n,q}^{(a)}(h, k \mid x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_q^n q^{(h-1)x_1 + \cdots + (h-k)x_k} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k).
\] (5.1)

Thus, we get

\[
(q^a - 1)^n \overline{E}_{n,q}^{(a)}(h, k \mid x) = [2]^q \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} \frac{q^{al}x}{(1 + q^{al+h}) \cdots (1 + q^{al+h-k+1})}.
\]

\[

q^h \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_q^n q^{(h-1)x_1 + \cdots + (h-k)x_k} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)

= - \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_q^n q^{(h-1)x_1 + \cdots + (h-k)x_k} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)

+ [2]^q \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_2 + \cdots + x_k]_q^n q^{(h-2)x_2 + \cdots + (h-k)x_k} d\mu_{-q}(x_2) \cdots d\mu_{-q}(x_k).
\] (5.2)

Therefore, by (2.1) and (5.2), we obtain the following theorem.

**Theorem 5.1.** For \( h \in \mathbb{Z}, \ \alpha \in \mathbb{N}, \) and \( n \in \mathbb{Z}_+, \) one has

\[
q^h \overline{E}_{n,q}^{(a)}(h, k \mid x + 1) + \overline{E}_{n,q}^{(a)}(h, k \mid x) = [2]^q \overline{E}_{n,q}^{(a)}(h - 1, k - 1 \mid x).
\] (5.3)

Note that

\[
q^{ax} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_q^n q^{hx_1 + \cdots + (h+1-k)x_k} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)

= (q^a - 1) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_q^n q^{(h-a)x_1 + \cdots + (h+1-a-k)x_k} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)

+ \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_q^n q^{(h-a)x_1 + \cdots + (h+1-a-k)x_k} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)

= (q^a - 1) \overline{E}_{n+1,q}^{(a)}(h + 1 - \alpha, k \mid x) + \overline{E}_{n,q}^{(a)}(h + 1 - \alpha, k \mid x).
\] (5.4)

Therefore, by (5.4), we obtain the following theorem.
Theorem 5.2. For \( n \in \mathbb{Z}_+ \), one has

\[
q^{ax} \widetilde{E}^{(a)}_{n,q}(h + 1, k \mid x) = (q^a - 1) \widetilde{E}^{(a)}_{n+1,q}(h + 1 - a, k \mid x) + \widetilde{E}^{(a)}_{n,q}(h + 1 - a, k \mid x). \tag{5.5}
\]

Let \( d \in \mathbb{N} \) with \( d \equiv 1 \pmod{2} \). Then we get

\[
\sum_{\mathbb{Z}_p} \cdots \sum_{\mathbb{Z}_p} \left[ x + \sum_{j=1}^{k} x_j \right]^n q^{\sum_{i=1}^{k} (h-j)x_i} d\mu_{-q^a}(x_1) \cdots d\mu_{-q^a}(x_k)
\]

\[
= \frac{[d]_{q^a}^n}{[d]_{-q}^k} \sum_{d_1, \ldots, d_k=0}^{d-1} q^{h \sum_{i=1}^{k} a_j - \sum_{j=2}^{k} (j-1)a_j - \sum_{j=1}^{k} a_j} (-1)^{\sum_{j=1}^{k} a_j} \sum_{\mathbb{Z}_p} \cdots \sum_{\mathbb{Z}_p} \left[ x + \sum_{j=1}^{k} x_j \right]^n q^{d \sum_{i=1}^{k} (h-j)x_i} d\mu_{-q^a}(x_1) \cdots d\mu_{-q^a}(x_k). \tag{5.6}
\]

Therefore, by (5.6), we obtain the following theorem.

Theorem 5.3. For \( d \in \mathbb{N} \) with \( d \equiv 1 \pmod{2} \), one has

\[
\widetilde{E}^{(a)}_{n,q}(h, k \mid dx) = \frac{[d]_{q^a}^n}{[d]_{-q}^k} \sum_{d_1, \ldots, d_k=0}^{d-1} q^{h \sum_{i=1}^{k} a_j - \sum_{j=2}^{k} (j-1)a_j - \sum_{j=1}^{k} a_j} \sum_{\mathbb{Z}_p} \cdots \sum_{\mathbb{Z}_p} \left[ x + \sum_{j=1}^{k} x_j \right]^n q^{d \sum_{i=1}^{k} (h-j)x_i} d\mu_{-q^a}(h, k \mid x + \frac{a_1 + \cdots + a_k}{d}). \tag{5.7}
\]

Let \( \widetilde{E}^{(a)}_{n,q}(k, k \mid x) = \widetilde{E}^{(a)}_{n,q}(k \mid x) \). Then we get

\[
(q^a - 1)^n \widetilde{E}^{(a)}_{n,q}(k \mid x),
\]

\[
= \sum_{l=0}^{n} \binom{n}{l} (-1)^{n-l} q^{al \cdot dx} \frac{[2]_{q^a}^k}{(1 + q^{al+k}) \cdots (1 + q^{al+1})}
\]

\[
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[ k - x + x_1 + \cdots + x_k \right]^n q^{-(k-1)x_1 - \cdots - (k-k)x_k} d\mu_{-q^a}(x_1) \cdots d\mu_{-q^a}(x_k)
\]

\[
= q^{a\left(\frac{k+1}{2}\right)-k} \frac{[2]_{q^a}^k}{(1 - q^{-a})^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^{l} q^{al \cdot dx} \frac{1}{(1 + q^{al+1}) \cdots (1 + q^{al+k})}
\]

\[
= (-1)^{n} q^{na} q^{\left(\frac{k+1}{2}\right)-k} \frac{[2]_{q^a}^k}{(1 - q^{-a})^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^{l} q^{al \cdot dx} \frac{1}{(1 + q^{al+1}) \cdots (1 + q^{al+k})}
\]

\[
= (-1)^{n} q^{a\left(n+\left(\frac{k+1}{2}\right)\right)-k} \widetilde{E}^{(a)}_{n,q}(k \mid x).
\]
Therefore, by (5.8), we obtain the following theorem.

**Theorem 5.4.** For \( n \in \mathbb{Z}_+ \), one has

\[
\tilde{E}_{n,q}^{(a)}(k \mid k - x) = (-1)^n q^{a(n + \binom{n+1}{2}) - k} \tilde{E}_{n,q}^{(a)}(k \mid x). 
\] (5.9)

Let \( x = k \) in Theorem 5.4. Then we see that

\[
\tilde{E}_{n,q}^{(a)}(k \mid 0) = (-1)^n q^{a\binom{n+1}{2} - k} \tilde{E}_{n,q}^{(a)}(k \mid k). 
\] (5.10)

From (4.6) and Theorem 5.1, we note that

\[
q^k \tilde{E}_{n,q}^{(a)}(k \mid x + 1) + \tilde{E}_{n,q}^{(a)}(k \mid x) = [2]_q \tilde{E}_{n,q}^{(a)}(k - 1 \mid x). 
\] (5.11)

It is easy to show that

\[
(q^a - 1)^n \tilde{E}_{n,q}^{(a)}(k \mid 0) = \sum_{l=0}^{n} \binom{n}{l} (-1)^{l+n} \frac{[2]_q^k}{(1 + q^{a+l}) \cdots (1 + q^{a+l+k})}. 
\] (5.12)

By simple calculation, we get

\[
\sum_{l=0}^{n} \binom{n}{l} (q^a - 1)^l \int_{\mathbb{Z}_q} \cdots \int_{\mathbb{Z}_q} [x_1 + \cdots + x_k]^l q^{\sum_{i=1}^l (k - l) x_i} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) 
\]

\[
= [2]_q^k 
\]

\[
(1 + q^{a+n+k}) (1 + q^{a+n+k-1}) \cdots (1 + q^{a+n+1}). 
\] (5.13)

From (5.13), we note that

\[
\sum_{l=0}^{n} \binom{n}{l} (q^a - 1)^l \tilde{E}_{l,q}^{(a)}(k \mid 0) = \frac{[2]_q^k}{(1 + q^{a+n+k}) (1 + q^{a+n+k-1}) \cdots (1 + q^{a+n+1})}, 
\]

\[
\tilde{E}_{n,q}^{(a)}(k \mid x) = \int_{\mathbb{Z}_q} \cdots \int_{\mathbb{Z}_q} [x + x_1 + \cdots + x_k]^n q^{\sum_{i=1}^n (k - l) x_i - \sum_{i=1}^n (k - l) x_i} d\mu_{-q}(x_1) 
\]

\[
\cdots d\mu_{-q}(x_k) 
\]

\[
= \sum_{l=0}^{n} \binom{n}{l} q^{a l} \tilde{E}_{l,q}^{(a)}(k \mid 0) [x]^{n-l} 
\]

\[
= \left( q^{a} \tilde{E}_{q}^{(a)}(k \mid 0) + [x]_q^n \right) \quad \text{for} \ n \in \mathbb{Z}_+, 
\]

with the usual convention about replacing \((\tilde{E}_{q}^{(a)}(k \mid 0))^{n}\) by \(\tilde{E}_{n,q}^{(a)}(k \mid 0)\).
Put $x = 0$ in (5.11); we get
\[
q^k \tilde{E}_{n,q}^{(a)} (k \mid 1) + \tilde{E}_{n,q}^{(a)} (k \mid 0) = [2]_q \tilde{E}_{n,q}^{(a)} (k - 1 \mid 0).
\]
(5.15)

Thus, we have
\[
q^k \left( q^n \tilde{E}_{q}^{(a)} (k \mid 0) + 1 \right)^n + \tilde{E}_{n,q}^{(a)} (k \mid 0) = [2]_q \tilde{E}_{n,q}^{(a)} (k - 1 \mid 0),
\]
with the usual convention about replacing $(\tilde{E}_{q}^{(a)} (k \mid 0))^n$ by $\tilde{E}_{n,q}^{(a)} (k \mid 0)$.

**Acknowledgment**

This work was supported by the Dong-A University research fund.

**References**


