A Note on the Modified $q$-Bernstein Polynomials

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We propose the modified $q$-Bernstein polynomials of degree $n$ which are different $q$-Bernstein polynomials of Phillips (1997). From these modified $q$-Bernstein polynomials of degree $n$, we derive some recurrence formulae for the modified $q$-Bernstein polynomials.

1. Introduction

Let $C[0,1]$ denote the set of continuous function on $[0,1]$. For $f \in C[0,1]$, Bernstein introduced the following well-known linear positive operators in [1]:

\[ B_n(f : x) := \sum_{k=0}^{n} f \left( \frac{k}{n} \right) \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) B_{k,n}(x), \]  

(1.1)

where $\binom{n}{k} = \frac{n(n-1) \cdots (n-k+1)}{k!}$. Here $B_n(f : x)$ is called the *Bernstein operator of order $n$ for $f$. For $k, n \in \mathbb{Z}_+$, the *Bernstein polynomial of degree $n$* is defined by

\[ B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \]  

(1.2)
where \( x \in [0, 1] \). For example,

\[
B_{0,1}(x) = 1 - x, \quad B_{1,1}(x) = x, \\
B_{0,2}(x) = (1 - x)^2, B_{1,2}(x) = 2x(1 - x), B_{2,2}(x) = x^2, \ldots
\]

Also, \( B_{k,n}(x) = 0 \), for \( k > n \), because \( \binom{n}{k} = 0 \).

Some people have studied the Bernstein polynomials in the area of approximation theory (see [2] through [3]). Note that for \( k \in \mathbb{Z}_+ \) and \( x \in [0, 1] \),

\[
\frac{t^k e^{(1-x)t} x^k}{k!} = \frac{x^k}{k!} \left( \sum_{n=0}^{\infty} \frac{(1-x)^n t^n}{n!} \right) \\
= \frac{x^k}{k!} \sum_{n=0}^{\infty} \frac{(1-x)^n (n+1) \cdots (n+k)}{(n+k)!} t^{n+k} \\
= \sum_{n=k}^{\infty} \left( \binom{n}{k} x^k (1-x)^{n-k} \right) \frac{t^n}{n!} \\
= \sum_{n=k}^{\infty} B_{k,n}(x) \frac{t^n}{n!}
\]

Because \( B_{k,0}(x) = B_{k,1}(x) = \cdots = B_{k,k-1}(x) = 0 \), we obtain the generating function for \( B_{k,n}(x) \) as follows:

\[
F^{(k)}(t, x) := \frac{t^k e^{(1-x)t} x^k}{k!} = \sum_{n=0}^{\infty} B_{k,n}(x) \frac{t^n}{n!}
\]

(see [4, 5]), where \( k \in \mathbb{Z}_+ \) and \( x \in [0, 1] \). Notice that

\[
B_{k,n}(x) = \begin{cases} 
\binom{n}{k} x^k (1-x)^{n-k} & \text{if } n \geq k, \\
0 & \text{if } n < k,
\end{cases}
\]

for \( n, k \in \mathbb{Z}_+ \) (see [2]).

Let \( 0 < q < 1 \). Define the \( q \)-number of \( x \) by

\[
[x]_q := \frac{1 - q^x}{1 - q}.
\]

See [2] through [3] for details and related facts. Note that \( \lim_{q \to 1} [x]_q = x \). In [6], Phillips proposed a generalization of the classical Bernstein polynomials based on \( q \)-integers. In the last decade some new generalizations of well-known positive linear operators, based on \( q \)-integers were introduced and studied by several authors (see [1–13]). Recently, Simsek
and Acikgoz have also studied the $q$-extension of Bernstein-type polynomials [5]. Their $q$-Bernstein-type polynomials are given by

$$Y_n(k; x : q) = \binom{n}{k} \frac{(-1)^k k!}{(1-q)^{n-k}} \sum_{m,l=0}^{\infty} \sum_{j=0}^{n-k} \binom{k+l-1}{l} \binom{n-k}{k} \times \left(\frac{(-1)^j q^{i+j(1-x)} S(m,k)(x \ln q)^m}{m!}\right), \quad (1.8)$$

where $S(m,k)$ are the second-kind stirling number. In [5], we can find some interesting formulae related to $q$-extension of Bernstein polynomials which are different $q$-Bernstein polynomials of Phillips. In the conference of Jangjeon Mathematical Society which was held in IRAN (on Feb.2010), Acikgoz and Arci has introduced several-type Bernstein polynomials (see [2]). The Acikgoz paper [2] announced in the conference is actually what motivated us to write this paper. In this paper, we considered the $q$-extension of Bernstein polynomials which were introduced by Acikgoz at the conference of Jangjeon Mathematical Society on Feb. 2010. First, we consider the $q$-extension of the generating function of Bernstein polynomials in (1.5). Indeed, this generating function is also treated by Simsek and Acikgoz in a previous paper (see [5]). From this $q$-extension of the generating function for the Bernstein polynomials, we propose the modified $q$-Bernstein polynomials of degree $n$ which are different $q$-Bernstein polynomials of Phillips. By using the properties of the modified $q$-Bernstein polynomials, we obtain some recurrence formulae for the modified $q$-Bernstein polynomials of degree $n$.

**2. The Modified $q$-Bernstein Polynomials**

For $0 < q < 1$, consider the $q$-extension of (1.5) as follows:

$$F_q^{(k)}(t, x) := \frac{t^k e^{[1-x] t} [x]^k_q}{k!}$$

$$= \frac{[x]^k_q}{k!} \sum_{n=0}^{\infty} \frac{[1-x]^n_q}{n!} t^{n+k}$$

$$= \sum_{n=k}^{\infty} \binom{n}{k} [x]^k_q [1-x]^n_q \frac{t^n}{n!}, \quad (2.1)$$

where $k, n \in \mathbb{Z}_+$ and $x \in [0, 1]$. Note that $\lim_{q \to 1} F_q^{(k)}(t, x) = F^{(k)}(t, x)$. We define the modified $q$-Bernstein polynomials as follows:

$$F_q^{(k)}(t, x) = \frac{t^k e^{[1-x] t} [x]^k_q}{k!} = \sum_{n=0}^{\infty} B_{k,n}(x, q) \frac{t^n}{n!}, \quad (2.2)$$

where $k, n \in \mathbb{Z}_+$ and $x \in [0, 1]$. 
Remark. This generating function is also introduced by Simsek and Acikgoz in a previous paper (see [5]).

By comparing the coefficients of (2.1) and (2.2), we obtain the following theorem.

**Theorem 2.1.** For $k, n \in \mathbb{Z}_+$ and $x \in [0, 1]$,

$$B_{k,n}(x, q) = \begin{cases} \binom{n}{k} [x]_q^k [1-x]_q^{n-k}, & \text{if } n \geq k \\ 0, & \text{if } n < k. \end{cases} \quad (2.3)$$

For $0 \leq k \leq n$, we have

$$[1-x]_q B_{k,n-1}(x, q) + [x]_q B_{k-1,n-1}(x, q)$$

$$= [1-x]_q \left( \binom{n-1}{k} [x]_q^k [1-x]_q^{n-1-k} + [x]_q \binom{n-1}{k-1} [x]_q^{k-1} [1-x]_q^{n-k} \right)$$

$$= \binom{n-1}{k} [x]_q^k [1-x]_q^{n-k} + \binom{n-1}{k-1} [x]_q^{k-1} [1-x]_q^{n-k}$$

$$= \binom{n}{k} [x]_q^k [1-x]_q^{n-k}, \quad (2.4)$$

and the derivatives of the modified $q$-Bernstein polynomials of degree $n$ are also polynomials of degree $n-1$, that is,

$$\frac{d}{dx} B_{k,n}(x, q) = \left( \binom{n}{k} k [x]_q^{k-1} [1-x]_q^{n-k} \frac{\ln q}{q-1} q^x + \binom{n}{k} [x]_q^{k} (n-k) [1-x]_q^{n-k-1} \left( -\frac{\ln q}{q-1} \right) q^{1-x} \right)$$

$$= \frac{\ln q}{q-1} \left\{ \binom{n}{k} k [x]_q^{k-1} [1-x]_q^{n-k} q^x - \binom{n}{k} [x]_q^{k} (n-k) [1-x]_q^{n-k-1} q^{1-x} \right\}$$

$$= n \left( q^x B_{k-1,n-1}(x, q) - q^{1-x} B_{k,n-1}(x, q) \right) \frac{\ln q}{q-1}. \quad (2.5)$$

Therefore, we obtain the following recurrence formulae.

**Theorem 2.2** (recurrence formulae for $B_{k,n}(x, q)$). For $k, n \in \mathbb{Z}_+$ and for $x \in [0, 1]$,

$$[1-x]_q B_{k,n-1}(x, q) + [x]_q B_{k-1,n-1}(x, q) = B_{k,n}(x, q),$$

$$\frac{d}{dx} B_{k,n}(x, q) = n \left( q^x B_{k-1,n-1}(x, q) - q^{1-x} B_{k,n-1}(x, q) \right) \frac{\ln q}{q-1}. \quad (2.6)$$
Let \( f \) be a continuous function on \([0, 1]\). Then the \textit{modified \( q \)-Bernstein operator of order} \( n \) for \( f \) is defined by

\[
B_{n,q}(f : x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x, q),
\]

where \( 0 \leq x \leq 1 \), \( n \in \mathbb{Z}_+ \). We get from Theorem 2.1 and (2.7) that for \( f(x) = x \),

\[
B_{n,q}(f : x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} [x]_q^k [1 - x]_q^{n-k}
\]

\[
= [x]_q \left(1 - [1 - x]_q [x]_q (q - 1)\right)^{n-1}
\]

\[
= f\left([x]_q\right) \left(1 + (1 - q) [x]_q [1 - x]_q\right)^{n-1}.
\]

We also see from Theorem 2.1 that

\[
B_{n,q}(1 : x) = \sum_{k=0}^{n} B_{k,n}(x, q)
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} [x]_q^k [1 - x]_q^{n-k}
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} [x]_q^k \left(1 - q^{1-x} [x]_q\right)^{n-k}
\]

\[
= \left(1 + (1 - q) [x]_q [1 - x]_q\right)^n.
\]

The modified \( q \)-Bernstein polynomials are symmetric polynomials in the following sense:

\[
B_{n-k,n}(1-x, q) = \binom{n}{n-k} [1 - x]_q^{n-k} [x]_q^k = B_{k,n}(x, q).
\]

Therefore, we get the following theorem.

\textbf{Theorem 2.3.} For \( k, n \in \mathbb{Z}_+ \) and \( x \in [0,1] \),

\[
B_{n-k,n}(1-x, q) = B_{k,n}(x, q),
\]

\[
B_{n,q}(1 : x) = \left(1 + (1 - q) [x]_q [1 - x]_q\right)^n.
\]
For $\zeta \in \mathbb{C}$, $x \in [0, 1]$ and for $n \in \mathbb{Z}_+$, consider

$$\frac{n!}{2\pi i} \oint_C \frac{[x]_q \zeta^k}{k!} e^{([1-x]_q \zeta)} \frac{d\zeta}{\zeta^{n+1}},$$

(2.12)

where $C$ is a circle around the origin and integration is in the positive direction. We see from the definition of the modified $q$-Bernstein polynomials and the basic theory of complex analysis including Laurent series that

$$\oint_C \frac{[x]_q \zeta^k}{k!} e^{([1-x]_q \zeta)} \frac{d\zeta}{\zeta^{n+1}} = \sum_{m=0}^{\infty} \oint_C B_{k,m}(x,q) \frac{\zeta^m}{m!} \frac{d\zeta}{\zeta^{n+1}} = 2\pi i \left( \sum_{m=0}^{\infty} \oint_C B_{k,m}(x,q) \frac{\zeta^m}{m!} \frac{d\zeta}{\zeta^{n+1}} \right).$$

(2.13)

We get from (2.12) and (2.13) that

$$\frac{n!}{2\pi i} \oint_C \frac{[x]_q \zeta^k}{k!} e^{([1-x]_q \zeta)} \frac{d\zeta}{\zeta^{n+1}} = B_{k,n}(x,q),$$

(2.14)

and

$$\oint_C \frac{[x]_q \zeta^k}{k!} e^{([1-x]_q \zeta)} \frac{d\zeta}{\zeta^{n+1}} = \left[ \sum_{m=0}^{\infty} \frac{[1-x]_q^m}{m!} \oint_C \frac{\zeta^m}{m!} \frac{d\zeta}{\zeta^{n+1}} \right] = \left( \sum_{m=0}^{\infty} \frac{[1-x]_q^m}{m!} \right) \left( \sum_{m=0}^{\infty} \frac{\zeta^m}{m!} \frac{d\zeta}{\zeta^{n+1}} \right).$$

(2.15)

We also get from (2.12) and (2.15) that

$$\frac{n!}{2\pi i} \oint_C \frac{[x]_q \zeta^k}{k!} e^{([1-x]_q \zeta)} \frac{d\zeta}{\zeta^{n+1}} = \binom{n}{k} [x]_q^k [1-x]_q^{n-k}.$$ 

(2.16)

Therefore, we see from (2.14) and (2.16) that

$$B_{k,n}(x,q) = \binom{n}{k} [x]_q^k [1-x]_q^{n-k}.$$ 

(2.17)
Corollary 2.5. For

Thus, the following corollary holds.

Note that

\[
\frac{n-k}{n} B_{k,n}(x,q) + \frac{k+1}{n} B_{k+1,n}(x,q) \\
= \frac{(n-1)!}{k!(n-k-1)!} [x]_q^k [1-x]_q^{n-k} + \frac{(n-1)!}{k!(n-k-1)!} [x]_q^{k+1} [1-x]_q^{n-k-1} \\
= \left( 1 - x_1 + [x]_q \right) B_{k,n-1}(x,q) \\
= \left( 1 + [x]_q \left( 1 - q^{-1} \right) \right) B_{k,n-1}(x,q) \\
= \left( 1 + (1-q) [x]_q [1-x]_q \right) B_{k,n-1}(x,q).
\]  

Therefore, we can write the modified $q$-Bernstein polynomials as a linear combination of polynomials of higher order as follows.

Theorem 2.4. For $k, n \in \mathbb{Z}$, and $x \in [0, 1]$,

\[
\left( \frac{n+k-1}{n+1} \right) B_{k,n+1}(x,q) + \left( \frac{k+1}{n+1} \right) B_{k+1,n+1}(x,q) = \left( 1 + (1-q) [x]_q [1-x]_q \right) B_{k,n}(x,q). 
\]  

We easily see from (2.17) that for $n, k \in \mathbb{N}$,

\[
\left( \frac{n-k+1}{k} \right) \left( \frac{[x]_q}{[1-x]_q} \right) B_{k-1,n}(x,q) = \left( \frac{n-k+1}{k} \right) \left( \frac{[x]_q}{[1-x]_q} \right) \left( \frac{n}{k-1} \right) [x]_q [1-x]_q^{n-k+1} \\
= \frac{n!}{k!(n-k)!} [x]_q^k [1-x]_q^{n-k} \\
= B_{k,n}(x,q).
\]  

Thus, the following corollary holds.

Corollary 2.5. For $n, k \in \mathbb{N}$ and $x \in [0, 1]$,

\[
\left( \frac{n-k+1}{k} \right) \left( \frac{[x]_q}{[1-x]_q} \right) B_{k-1,n}(x,q) = B_{k,n}(x,q).
\]
Theorem 2.6. For $k, n \in \mathbb{Z}_+$,

\[
B_{k,n}(x, q) = \binom{n}{k} x^k [1 - x]^n_q
\]

\[
= \binom{n}{k} x^k [1 - q^{1-x} [x]_q]^{n-k}
\]

\[
= \binom{n}{k} x^k \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l q^{l(1-x)} [x]^l_q
\]

\[
= \sum_{l=0}^{n-k} \binom{k+l}{k} \binom{n}{k+l} (-1)^l q^{l(1-x)} [x]^{l+k}_q
\]

\[
= \sum_{j=k}^{n} \binom{n}{j} \binom{n}{j} (-1)^{j-k} q^{(j-k)(1-x)} [x]^{j}_q.
\] (2.22)

Therefore, we showed that the following theorem holds.

**Theorem 2.6.** For $k, n \in \mathbb{Z}_+$ and $x \in [0, 1]$,

\[
B_{k,n}(x, q) = \sum_{j=k}^{n} \binom{n}{j} \binom{n}{j} (-1)^{j-k} q^{(j-k)(1-x)} [x]^{j}_q.
\] (2.23)

It is possible to write $[x]_q^k$ as a linear combination of the modified $q$-Bernstein polynomials by using the degree evaluation formulae and mathematical induction. We easily see from the property of the modified $q$-Bernstein polynomials that

\[
\sum_{k=1}^{n} \binom{k}{n} B_{k,n}(x, q) = \sum_{k=1}^{n} \binom{n-1}{k-1} [x]^k_q [1 - x]^{n-k}_q
\]

\[
= \sum_{k=0}^{n-1} \binom{n-1}{k} [x]^{k+1}_q [1 - x]^{n-1-k}_q
\]

\[
= [x]_q ([x]_q + [1 - x]_q)^{n-1},
\] (2.24)

and that

\[
\sum_{k=2}^{n} \binom{k}{n} B_{k,n}(x, q) = \sum_{k=2}^{n} \binom{n-2}{k-2} [x]^k_q [1 - x]^{n-k}_q
\]

\[
= \sum_{k=0}^{n-2} \binom{n-2}{k} [x]^{k+2}_q [1 - x]^{n-2-k}_q
\]

\[
= [x]_q^2 ([x]_q + [1 - x]_q)^{n-2}.
\] (2.25)
Continuing this process, we obtain

\[
\sum_{k=j}^{n} \binom{k}{j} B_{k,n}(x,q) = [x]_q \left( [x]_q + [1 - x]_q \right)^{n-j},
\]  

(2.26)

for \( j \in \mathbb{N} \). Therefore, we obtain the following theorem.

**Theorem 2.7.** For \( n, j \in \mathbb{Z}_+ \) and \( x \in [0,1] \),

\[
\frac{1}{(1 - x)_q + [x]_q} \sum_{k=j}^{n} \binom{k}{j} B_{k,n}(x,q) = [x]_q^j.
\]  

(2.27)

For \( k \in \mathbb{N} \), the *Bernoulli polynomial of order* \( k \) is defined by

\[
\left( \frac{t}{e^t - 1} \right)^k e^{xt} = \left( \frac{t}{e^t - 1} \right) \times \cdots \times \left( \frac{t}{e^t - 1} \right) e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!},
\]  

(2.28)

and \( B_n^{(k)} = B_n^{(k)}(0) \) are called the \( n \text{th} \) *Bernoulli numbers of order* \( k \). It is well known that the *second kind stirling number* is defined by

\[
\frac{(e^t - 1)^k}{k!} = \sum_{n=0}^{\infty} S(n,k) \frac{t^n}{n!},
\]  

(2.29)

for \( k \in \mathbb{N} \). We note from (2.2) that

\[
\left( [x]_q t \right)^k e^{[1-x]_q t} = [x]_q^k \left( e^t - 1 \right)^k \left( \frac{t}{e^t - 1} \right)^k e^{[1-x]_q t}
\]

\[
= [x]_q^k \left( \sum_{m=0}^{\infty} S(m,k) \frac{t^m}{m!} \right) \left( \sum_{n=0}^{\infty} B_n^{(k)} \left( [1 - x]_q \right) \frac{t^n}{n!} \right)
\]

\[
= [x]_q^k \sum_{l=0}^{\infty} \sum_{n=0}^{l} \frac{B_n^{(k)} \left( [1 - x]_q \right) S(l - n, k)!}{n!(l - n)!} \frac{t^n}{n!}
\]  

(2.30)

We have from (2.2) and (2.30) that

\[
B_{k,l}(x,q) = [x]_q^k \sum_{n=0}^{l} \frac{B_n^{(k)} \left( [1 - x]_q \right) S(l - n, k)}{n!(l - n)!} \frac{t^n}{n!},
\]  

(2.31)

and \( B_{k,0}(x,q) = B_{k,1}(x,q) = \cdots = B_{k,k-1}(x,q) = 0 \).
Remark. The Equations (2.30) and (2.31) are already known by Simsek and Acikgoz in a previous paper [5, page 7].

Let $\Delta$ be the shift difference operator defined by $\Delta f(x) = f(x + 1) - f(x)$. We see from the iterative method that

$$\Delta^n f(0) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(k), \quad (2.32)$$

for $n \in \mathbb{N}$. We get from (2.29) and (2.32) that

$$\sum_{n=0}^{\infty} S(n, k) \frac{t^n}{n!} = \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} e^{lt} \left[ \sum_{n=0}^{\infty} \frac{\Delta^k q^n t^n}{k!} \frac{1}{n!} \right]$$

$$= \sum_{n=0}^{\infty} \frac{\Delta^k q^n t^n}{k!} \frac{1}{n!}.$$

By comparing the coefficients on both sides above, we have

$$S(n, k) = \frac{\Delta^k q^n}{k!}, \quad (2.34)$$

for $n, k \in \mathbb{Z}_+$. Thus, we get from (2.31) and (2.34) that

$$B_{k,l}(x, q) = [x]_q \sum_{n=0}^{l} \frac{(1/n) B_n^{(k)}[1 - x]_q}{k!} \frac{\Delta^k q^n}{k!}. \quad (2.35)$$

Let $(Eh)(x) = h(x + 1)$ be the shift operator. Then the $q$-difference operator is defined by

$$\Delta_q^n = \prod_{j=0}^{n-1} \left( E - q^j I \right), \quad (2.36)$$

where $I$ is an identity operator (see [7] through [11]). For $f \in C[0, 1]$ and $n \in \mathbb{N}$, we have

$$\Delta_q^n f(0) = \sum_{k=0}^{n} \binom{n}{k} q^{k(k+1)/2} f(n-k), \quad (2.37)$$

where $\binom{n}{k}_q$ is the Gaussian binomial coefficient defined by

$$\binom{x}{k}_q = \frac{[x]_q[x-1]_q \cdots [x-k+1]_q}{[k]_q!}. \quad (2.38)$$
Let $F_q(t)$ be the generating function of the $q$-extension of the second kind stirling number as follows:

$$
F_q(t) := \frac{q^{-\Delta_j}}{[k]_q!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} q^{\frac{k-j}{2}} e^{i(j)q} = \sum_{n=0}^{\infty} S(n, k : q) \frac{t^n}{n!},
$$

(2.39)

We have from (2.39) that

$$
S(n, k : q) = \frac{q^{-\Delta_j}}{[k]_q!} \sum_{j=0}^{k} (-1)^{j} q^{\frac{j}{2}} \binom{k}{j} [k - j]_q^n = \frac{q^{-\Delta_j}}{[k]_q!} \Delta_j^k 0^n,
$$

(2.40)

where $[k]_q! = [k]_q[k-1]_q \cdots [2]_q[1]_q$. It is not difficult to see that

$$
[x]_q^n = \sum_{k=0}^{n} q^{-\Delta_j} \binom{x}{k} [k]_q! S(n, k : q).
$$

(2.41)

See also [7] through [11] for details and related facts for above. Then, we get from (2.41) and Theorem 2.7 that

$$
\sum_{k=0}^{j} q^{-\Delta_j} \binom{x}{k} [k]_q! S(j, k : q) = \frac{1}{[1-x]_q + [x]_q} \sum_{k=j}^{n-j} q^{-\Delta_j} \binom{k}{j} B_{k,n}(x, q).
$$

(2.42)

Therefore, this completes the proof of the following theorem.

**Theorem 2.8.** For $n, j \in \mathbb{Z}_+$ and $x \in [0, 1]$,

$$
\frac{1}{[1-x]_q + [x]_q} \sum_{k=j}^{n-j} q^{-\Delta_j} \binom{k}{j} B_{k,n}(x, q) = \sum_{k=0}^{j} q^{-\Delta_j} \binom{x}{k} [k]_q! S(j, k : q).
$$

(2.43)

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