Research Article

A Recurrence Formula for $D$ Numbers $D_{2n}^{(2n-1)}$

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we establish a recurrence formula for $D$ numbers $D_{2n}^{(2n-1)}$. A generating function for $D$ numbers $D_{2n}^{(2n)}$ is also presented.

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1. Introduction and Results

The Bernoulli polynomials $B_n^{(k)}(x)$ of order $k$, for any integer $k$, may be defined by (see [1–4])

$$
\left(\frac{t}{e^t - 1}\right)^k e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}, \quad |t| < 2\pi.
$$

(1.1)

The numbers $B_n^{(k)} = B_n^{(k)}(0)$ are the Bernoulli numbers of order $k$, $B_n^{(1)} = B_n$ are the ordinary Bernoulli numbers (see [2, 5]). By (1.1), we can get (see [4, page 145])

$$
\frac{d}{dx} B_n^{(k)}(x) = n B_n^{(k-1)}(x),
$$

(1.2)

$$
B_n^{(k+1)}(x) = \frac{k - n}{k} B_n^{(k)}(x) + (x - k) \frac{n}{k} B_n^{(k)}(x),
$$

(1.3)

$$
B_n^{(k+1)}(x + 1) = \frac{nx}{k} B_{n-1}^{(k)}(x) - \frac{n - k}{k} B_n^{(k)}(x),
$$

(1.4)

where $n \in \mathbb{N}$, with $\mathbb{N}$ being the set of positive integers.
The numbers \(B_n^{(n)}\) are called the Nörlund numbers (see [2, 4, 6]). A generating function for the Nörlund numbers \(B_n^{(n)}\) is (see [4, page 150])

\[
\frac{t}{(1 + t) \log(1 + t)} = \sum_{n=0}^{\infty} B_n^{(n)} \frac{t^n}{n!}.
\]  

(1.5)

The \(D\) numbers \(D_n^{(k)}\) may be defined by (see [4, 7, 8])

\[
(t \csc t)^k = \sum_{n=0}^{\infty} (-1)^n D_n^{(k)} \frac{t^{2n}}{(2n)!}, \quad |t| < \pi.
\]  

(1.6)

By (1.1), (1.6), and note that \(\csc t = 2i/(e^{it} - e^{-it})\) (where \(i^2 = -1\)), we can get

\[
D_n^{(k)} = 4^n B_n^{(k)} \left(\frac{k}{2}\right).
\]  

(1.7)

Taking \(k = 1, 2\) in (1.7), and note that \(B_n^{(1)}(1/2) = (2^{1-2n} - 1)\) \(B_{2n}, B_n^{(2)}(1) = (1 - 2n)\) \(B_{2n}\) (see [4, page 22, page 145]), we have

\[
D_n^{(1)} = (2 - 2^n) B_{2n}, \quad D_n^{(2)} = 4^n (1 - 2n) B_{2n}.
\]  

(1.8)

The \(D\) numbers \(D_n^{(k)}\) satisfy the recurrence relation (see [7])

\[
D_n^{(k)} = \frac{(2n - k + 2)(2n - k + 1)}{(k - 2)(k - 1)} D_n^{(k-2)} - \frac{2n(2n - 1)(k - 2)}{k - 1} D_{2n-2}^{(k-2)}.
\]  

(1.9)

By (1.9), we may immediately deduce the following (see [4, page 147]):

\[
D_n^{(2n+1)} = \frac{(-1)^n (2n)!}{4^n} \left(\begin{array}{c} 2n \\ n \end{array}\right), \quad D_n^{(2n+2)} = \frac{(-1)^n 4^n}{2n + 1} (n!)^2,
\]  

(1.10)

\[
D_n^{(2n+3)} = \frac{(-1)^n (2n)!}{2 \cdot 4^n} \left(\begin{array}{c} 2n + 2 \\ n + 1 \end{array}\right) \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n + 1)^2}\right)\]

(1.11)

The numbers \(D_n^{(2n)}\) are called the \(D\)-Nörlund numbers that satisfy the recurrence relation (see [7])

\[
\sum_{j=0}^{n} \frac{(-1)^j}{4^j (2j + 1)} \left(\begin{array}{c} 2j \\ j \end{array}\right) \frac{D_{2n-2j}^{(2n-2j)}}{(2n - 2j)!} = \frac{(-1)^n}{4^n} \left(\begin{array}{c} 2n \\ n \end{array}\right),
\]  

(1.12)

so we find \(D_0^{(0)} = 1, D_2^{(2)} = -2/3, D_4^{(4)} = 88/15, D_6^{(6)} = -3056/21, D_8^{(8)} = 319616/45, D_{10}^{(10)} = -18940160/33, \ldots\)
A generating function for the $D$-Nörlund numbers $D_{2n}^{(2n)}$ is (see [7])

$$\frac{t}{\sqrt{1 + t^2} \log(t + \sqrt{1 + t^2})} = \sum_{n=0}^{\infty} D_{2n}^{(2n)} \frac{t^{2n}}{(2n)!}, \quad |t| < 1. \quad (1.13)$$

These numbers $D_{2n}^{(2n)}$ and $D_{2n}^{(2n-1)}$ have many important applications. For example (see [4, page 246])

$$\int_0^{\pi/2} \frac{\sin t}{t} \, dt = \sum_{n=0}^{\infty} \frac{(-1)^n D_{2n}^{(2n)}}{(2n+1)!}, \quad \int_0^{\pi/2} \frac{\sin t}{t} \, dt = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} D_{2n}^{(2n-1)}}{2^{2n}(2n-1)(n)!^2}, \quad (1.14)$$

$$\frac{2}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} D_{2n}^{(2n-1)}}{(2n-1)(2n)!}.$$

The main purpose of this paper is to prove a recurrence formula for $D$ numbers $D_{2n}^{(2n-1)}$ and to obtain a generating function for $D$ numbers $D_{2n}^{(2n-1)}$. That is, we will prove the following main conclusion.

**Theorem 1.1.** Let $n \in \mathbb{N}$. Then

$$\sum_{j=1}^{n} \binom{2n}{2j} (-1)^{j-1} 4^{-j-1} ((j-1)!)^2 D_{2n-2j}^{(2n-2j)} = \frac{(-1)^{n-1} 2(2n)!}{4^n} \binom{2n-2}{n-1}, \quad (1.15)$$

so one finds $D_2^{(1)} = -1/3$, $D_4^{(3)} = 17/5$, $D_6^{(5)} = -1835/21$, $D_8^{(7)} = 195013/45$, $D_{10}^{(9)} = -3887409/11, \ldots$.

**Theorem 1.2.** Let $t$ be a complex number with $|t| < 1$. Then

$$\sum_{n=0}^{\infty} D_{2n}^{(2n-1)} \frac{t^{2n}}{(2n)!} = \frac{1}{\sqrt{1 + t^2}} \left( \frac{t}{\log(t + \sqrt{1 + t^2})} \right)^2. \quad (1.16)$$

**2. Proof of the Theorems**

**Proof of Theorem 1.1.** Note the identity (see [4, page 203])

$$B_{2n}^{(k)} \left( x + \frac{k}{2} \right) = \sum_{j=0}^{n} \binom{2n}{2j} D_{2n-2j}^{(k-2j)} \left( x^2 \right) \left( x^2 - 2^2 \right) \cdots \left( x^2 - (j-1)^2 \right), \quad (2.1)$$

we have

$$\frac{B_{2n}^{(k)}(x + k/2) - B_{2n}^{(k)}(k/2)}{x^2} = \sum_{j=1}^{n} \binom{2n}{2j} D_{2n-2j}^{(k-2j)} \left( x^2 - 1^2 \right) \left( x^2 - 2^2 \right) \cdots \left( x^2 - (j-1)^2 \right). \quad (2.2)$$
Therefore,

$$\lim_{x \to 0} \frac{B_{2n}^{(k)}(x + k/2) - B_{2n}^{(k)}(k/2)}{x^2} = \sum_{j=1}^{n} \binom{2n}{2j} \frac{D_{2n-2j}^{(k-2j)}}{2^{2n-2j}} (-1)^{j-1} ((j-1)!)^2.$$

(2.3)

By (2.3) and (1.2), we have

$$\lim_{x \to 0} \frac{2n(2n-1)B_{2n-2}^{(k)}(x + k/2)}{2} = \sum_{j=1}^{n} \binom{2n}{2j} \frac{D_{2n-2j}^{(k-2j)}}{2^{2n-2j}} (-1)^{j-1} ((j-1)!)^2.$$

(2.4)

That is,

$$n(2n-1)B_{2n-2}^{(k)} \left( \frac{k}{2} \right) = \sum_{j=1}^{n} \binom{2n}{2j} \frac{D_{2n-2j}^{(k-2j)}}{2^{2n-2j}} (-1)^{j-1} ((j-1)!)^2.$$

(2.5)

By (2.5) and (1.7), we have

$$D_{2n-2}^{(k)} = \frac{1}{n(2n-1)} \sum_{j=1}^{n} \binom{2n}{2j} (-1)^{j-1} 4^{j-1} ((j-1)!)^2 D_{2n-2j}^{(k-2j)}.$$

(2.6)

Setting $k = 2n-1$ in (2.6), and note (1.10), we immediately obtain Theorem 1.1. This completes the proof of Theorem 1.1.

Remark 2.1. Setting $k = 2n$ in (2.6), and note (1.10), we may immediately deduce the following recurrence formula for $D$-Nörlund numbers $D_{2n}^{(2n)}$:

$$\sum_{j=1}^{n} \binom{2n}{2j} (-1)^j 4^j ((j-1)!)^2 D_{2n-2j}^{(2n-2j)} = (-1)^n n4^n ((n-1)!)^2 \quad (n \in \mathbb{N}).$$

(2.7)

Proof of Theorem 1.2. Note the identity (see [9])

$$\sum_{n=0}^{\infty} (-1)^n n^4 (n!)^2 \frac{t^{2n}}{(2n)!} = \frac{1}{1 + t^2} \left( 1 - \frac{t}{\sqrt{1 + t^2}} \log \left( \frac{t + \sqrt{1 + t^2}}{1 - \sqrt{1 + t^2}} \right) \right),$$

(2.8)

where $|t| < 1$. We have

$$\sum_{n=0}^{\infty} (-1)^n n^4 (n!)^2 \frac{t^{2n+2}}{(2n+2)!} = \frac{1}{2} \left( \log \left( \frac{t + \sqrt{1 + t^2}}{1 - \sqrt{1 + t^2}} \right) \right)^2,$$

(2.9)
That is,
\[
\sum_{n=1}^{\infty} (-1)^{n-1} 4^{n-1} ((n-1)!)^2 \frac{t^{2n}}{(2n)!} = \frac{1}{2} \left( \log \left( t + \sqrt{1+t^2} \right) \right)^2. 
\] (2.10)

On the other hand,
\[
\sum_{n=1}^{\infty} (-1)^{n-1} 2(2n)! \frac{2^n}{4^n} \frac{t^{2n}}{(2n)!} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{2n}{4^n} \frac{t^{2n}}{(2n)!} = \frac{t^2}{2 \sqrt{1+t^2}}.
\] (2.11)

Thus, by (2.10), (2.11), and Theorem 1.1, we have
\[
\sum_{n=1}^{\infty} (-1)^{n-1} 4^{n-1} ((n-1)!)^2 \frac{t^{2n}}{(2n)!} \sum_{n=1}^{\infty} D_{2n}^{(2n-1)} \frac{t^{2n}}{(2n)!} = \sum_{n=1}^{\infty} (-1)^{n-1} 2(2n)! \frac{2n}{4^n} \frac{t^{2n}}{(2n)!}.
\] (2.12)

That is,
\[
\frac{1}{2} \left( \log \left( t + \sqrt{1+t^2} \right) \right)^2 \sum_{n=1}^{\infty} D_{2n}^{(2n-1)} \frac{t^{2n}}{(2n)!} = \frac{t^2}{2 \sqrt{1+t^2}}.
\] (2.13)

By (2.13), and note that
\[
\lim_{t \to 0} \frac{t}{\log \left( t + \sqrt{1+t^2} \right)} = 1, \quad D_{0}^{(-1)} = 1,
\] (2.14)

we immediately obtain Theorem 1.2. This completes the proof of Theorem 1.2.

\[\square\]

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References


