EXPLORING THE $q$-RIEMANN ZETA FUNCTION AND $q$-BERNOULLI POLYNOMIALS

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We study that the $q$-Bernoulli polynomials, which were constructed by Kim, are analytic continued to $\beta_s(z)$. A new formula for the $q$-Riemann zeta function $\zeta_q(s)$ due to Kim in terms of nested series of $\zeta_q(n)$ is derived. The new concept of dynamics of the zeros of analytic continued polynomials is introduced, and an interesting phenomenon of “scattering” of the zeros of $\beta_s(z)$ is observed. Following the idea of $q$-zeta function due to Kim, we are going to use “Mathematica” to explore a formula for $\zeta_q(n)$.

1. Introduction

Throughout this paper, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ will denote the ring of integers, the field of real numbers, and the complex numbers, respectively. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number, or a $p$-adic number. In the complex number field, we will assume that $|q| < 1$ or $|q| > 1$. The $q$-symbol $[x]_q$ denotes $[x]_q = (1 - q^x)/(1 - q)$.

In this paper, we study that the $q$-Bernoulli polynomials due to Kim (see [2, 8]) are analytic continued to $\beta_s(z)$. By those results, we give a new formula for the $q$-Riemann zeta function due to Kim (cf. [4, 6, 8]) and investigate the new concept of dynamics of the zeros of analytic continued polynomials. Also, we observe an interesting phenomenon of “scattering” of the zeros of $\beta_s(z)$. Finally, we are going to use a software package called “Mathematica” to explore dynamics of the zeros from analytic continuation for $q$-zeta function due to Kim.

2. Generating $q$-Bernoulli polynomials and numbers

For $h \in \mathbb{Z}$, the $q$-Bernoulli polynomials due to Kim were defined as

$$\sum_{n=0}^{\infty} \frac{\beta_n(x, h | q)}{n!} t^n = -t \sum_{l=0}^{\infty} q^{l(h+1)+x} e^{[l+x]_q t} + (1 - q) h \sum_{l=0}^{\infty} q^{bh} e^{[l+x]_q t},$$

for $x, q \in \mathbb{C}$ (cf. [6, 8]).
In the special case $x = 0$, $\beta_n(0, h \mid q) = \beta_n(h \mid q)$ are called $q$-Bernoulli numbers (cf. [1, 5, 7, 8]).

By (2.1), we easily see that

$$\beta_n(x, h \mid q) = \frac{1}{(1 - q)^n} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \frac{j + h}{[j + h]_q} q^{jx}, \quad (\text{cf. [2, 6]}),$$

(2.2)

where $\binom{n}{j}$ is a binomial coefficient.

In (2.1), it is easy to see that

$$q^h(q\beta(h \mid q) + 1)^n - \beta_n(h \mid q) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases}$$

(2.3)

with the usual convention of replacing $\beta^n(h \mid q)$ by $\beta_n(h \mid q)$.

By differentiating both sides with respect to $t$ in (2.1), we have

$$\beta_m(h \mid q) = -m \sum_{n=0}^{\infty} q^{hn}[n]_q^{m-1} - (q - 1)(m + h) \sum_{n=0}^{\infty} q^{hn}[n]_q^m.$$  

(2.4)

Expanding (2.1) as a series and matching the coefficients on both sides give

$$\begin{align*}
\beta_0(2 \mid q) &= \frac{2}{[2]_q}, \\
\beta_1(2 \mid q) &= \frac{2q + 1}{[2][3]_q}, \\
\beta_2(2 \mid q) &= \frac{2q^2}{[3][4]_q}, \\
\beta_3(2 \mid q) &= -\frac{q^2(q - 1)(2[3]_q + q)}{[3][4][5]_q}, \\
&\quad \ldots, \\
\beta_0(h \mid q) &= \frac{h}{[h]_q}, \\
\beta_1(h \mid q) &= -\frac{(1 + q + \cdots + q^{h-1}) + q(1 + q + \cdots + q^{h-2}) + \cdots + q^{h-1}}{[h][h+1]_q}, \\
&\quad \cdots.
\end{align*}$$

(2.5)

By (2.1), the $q$-Bernoulli polynomials can be written as

$$\beta_m(x, h \mid q) = \sum_{j=0}^{m} \binom{m}{j} [x]_q^{m-j} q^{jx} \beta_j(h \mid q).$$

(2.6)
In the case $h = 0$, $\beta_m(x, 0 \mid q)$ will be symbolically written as $\beta_{m,q}(x)$. Let $G_q(x, t)$ be the generating function of $q$-Bernoulli polynomials as follows:

$$G_q(x, t) = \sum_{n=0}^{\infty} \beta_{n,q}(x) \frac{t^n}{n!}. \quad (2.7)$$

Then we easily see that

$$G_q(x, t) = \frac{q - 1}{\log q} e^{t/(1-q)} - t \sum_{n=0}^{\infty} q^{n+x} e^{[n+x]_q} t^n, \quad |t| < 1, \text{ (cf. [2, 3, 4, 6]).} \quad (2.8)$$

For $x = 0$, $\beta_{n,q} = \beta_{n,q}(0)$ will be called $q$-Bernoulli numbers.

By (2.8), we easily see that

$$\beta_{m,q}(n) - \beta_{m,q} = m \sum_{l=0}^{n-1} q^{[l]} q^{m-1} \beta_{n,q}. \quad (2.9)$$

Thus, we have

$$\sum_{l=0}^{n-1} q^{[l]} q^{m-1} = \frac{1}{m} \sum_{l=0}^{m-1} \binom{m}{l} q^{nl} \beta_{l,q}[n] q^{m-l} + \frac{1}{m} (1 - q^{mn}) \beta_{m,q}. \quad (2.10)$$

3. Beautiful shape of $q$-Bernoulli polynomials

In this section, we display the shapes of the $q$-Bernoulli polynomials $\beta_m(x, 1|1/2)$. For $m = 1, 2, \ldots, 10$, we can draw a plot of $\beta_m(x, 1|1/2)$, respectively. This shows the ten plots combined into one. For $m = 1, \ldots, 10, q$, Figure 3.1 displays the shapes of the $q$-Bernoulli
polynomials $\beta_m(x,1\mid 1/2)$. We plot the zeros of $\beta_m(x,1\mid 1/2)$, $m = 40, m = 60$, and $x \in \mathbb{C}$ (Figure 3.2). We plot the zeros of $\beta_m(x,1\mid -1/2)$, $m = 40, m = 60$, and $x \in \mathbb{C}$ (Figure 3.3). We plot the zeros of $\beta_m(x,1\mid 11/10)$, $m = 40, m = 60$, and $x \in \mathbb{C}$ (Figure 3.4). We plot the zeros of $\beta_m(x,1\mid -11/10)$, $m = 40, m = 60$, and $x \in \mathbb{C}$ (Figure 3.5). Stacks of zeros of $\beta_n(x,1\mid 1/2)$, $1 \leq n \leq 60$, from a 3D structure are presented in Figure 3.6. The curve $\beta(s)$ runs through the points $\beta_{-n}(n\mid 1/2)$ (Figure 3.7). We draw the curve of $\beta_{-n}(n\mid q)$ and $\lim_{n \to \infty} n\zeta_q(n + 1)$, $q = 3/10$, $5/10$, $7/10$, $9/10$, $99/100$, $999/1000$ (Figures 3.8, 3.9, and 3.10).
4. $q$-Riemann zeta function

We display the plot of $\beta_q(s)$, $0.1 \leq s \leq 0.9$, $1.1 \leq q \leq 2$ (Figure 4.1). We display the plot of $\beta_q(s)$, $1.03 \leq s \leq 2$, $0.1 \leq q \leq 2$ (Figure 4.2). We draw the curve of $\zeta_q(n)$, $q = 7/10$, $9/10$ (Figure 4.3). We draw the curve of $\beta_q(s, w)$, $2 \leq s \leq 3$, $-0.5 \leq w \leq 0.5$, $q = 11/10$ (Figure 4.4).

The $q$-Riemann zeta function due to Kim was defined as

$$\zeta_q^{(h)}(s) = \frac{1-s+h}{1-s}(q-1)\sum_{n=1}^{\infty} \frac{q^{nh}}{[n]_q^{s-1}} + \sum_{n=1}^{\infty} \frac{q^{nh}}{[n]_q^s}, \quad \text{for } s, h \in \mathbb{C}, \quad \text{(cf. [6, 8])}. \quad (4.1)$$
For $k \in \mathbb{N}$, $h \in \mathbb{Z}$, it was known that

\[
\zeta_q^{(h)}(1-k) = -\frac{\beta_k(h \mid q)}{k}, \quad \text{(cf. [6, 8])}.
\]

In the special case $h = s - 1$, $\zeta_q^{(s-1)}(s)$ will be written as $\zeta_q(s)$. For $s \in \mathbb{C}$, we note that

\[
\zeta_q(s) = \sum_{n=1}^{\infty} \frac{q^{n(s-1)}}{[n]_q}, \quad \text{(cf. [6, 8])}.
\]
By (4.1), (4.2), and (4.3), we easily see that

$$
\zeta_q(1-k) = -\frac{\beta_k(-k|q)}{k}, \quad \text{for } k \in \mathbb{N}, \text{ (cf. [3, 4, 6]).}
$$

(4.4)

From the above analytic continuation of $q$-Bernoulli numbers, we consider

$$
\beta_n = \beta_n(-n|q) \longrightarrow \beta(s),
$$

$$
\zeta_q(-n) = -\frac{\beta_{n+1}(-n+1|q)}{n+1} \longrightarrow \zeta_q(-s) = -\frac{\beta(s+1)}{s+1} \implies \zeta_q(1-s) = -\frac{\zeta(s)}{s}.
$$

(4.5)
From relation (4.5), we can define the other analytic continued half of \( q \)-Bernoulli numbers,

\[
\beta(s) = -s\zeta_q(1 - s), \quad \beta(-s) = s\zeta_q(1 + s)
\]

\[
\Rightarrow \beta_{-n} = \beta_{-n}(n \mid q) = \beta(-n) = n\zeta_q(n + 1), \quad n \in \mathbb{N}.
\] (4.6)

The curve \( \beta(s) \) runs through the points \( \beta_{-n} \) and \( \lim_{n \to -\infty} \beta_{-n} = n\zeta_q(n + 1) = 0 \).

However, the curve \( \beta_{-n}(n \mid q) \) grows \( \sim n \) asymptotically as \( q \to 1 \), \( (-n) \to -\infty \).

\[
\zeta_q(m) = \sum_{n=1}^{\infty} \frac{q^n(m-1)}{[n]_q^n} \Rightarrow \lim_{m \to -\infty} \zeta_q(m) = 0.
\] (4.7)
Figure 4.2. The plot of $\beta_q(s)$, $1.03 \leq s \leq 2, 0.1 \leq q \leq 2$.

Figure 4.3. The curve of $\zeta_q(n)$, $q = 7/10, 9/10$.

Figure 4.4. The curve of $\beta(s, w)$, $2 \leq s \leq 3, -0.5 \leq w \leq 0.5, q = 11/10$. 
5. Analytic continuation of \( q \)-Bernoulli polynomials

For consistency with the redefinition of \( \beta_n = \beta(n) \) in (4.5) and (4.6),
\[
\beta_n(x) = \beta_n(x, -n \mid q) = \sum_{k=0}^{n} \binom{n}{k} \beta_k q^{kx} [x]_q^{n-k}.
\] (5.1)

The analytic continuation can be then obtained as
\[
\begin{align*}
n &\mapsto s \in \mathbb{R}, \quad x \mapsto w \in \mathbb{C}, \\
\beta_k &\mapsto \beta(k + s - [s] \mid q) = -(k + (s - [s])) \zeta_q (1 - (k + (s - [s]))) \\
\binom{n}{k} &\mapsto \frac{\Gamma(1+s)}{\Gamma(1+k + (s - [s])) \Gamma(1+[s] - k)}
\end{align*}
\]
\[
\Rightarrow \beta_n(s) \longrightarrow \beta(s, w \mid q) = \sum_{k=1}^{[s]} \frac{\Gamma(1+s) \beta(k + s - [s]) q^{(k+s-[s])w} [w]_q^{[s]-k}}{\Gamma(1+k + (s - [s])) \Gamma(1+[s] - k)}
\]
\[
= \sum_{k=0}^{[s]+1} \frac{\Gamma(1+s) \beta((k-1) + s - [s]) q^{((k-1+s-[s])w} [w]_q^{[s]+1-k}}{\Gamma((k + (s - [s])) \Gamma(2+[s] - k)}
\] (5.2)

where \([s]\) gives the integer part of \( s \), and so \( s - [s] \) gives the fractional part.

Deformation of the curve \( \beta(2, w) \) into the curve \( \beta(3, w) \) via the real analytic continuation \( \beta(s, w), 2 \leq s \leq 3, -0.5 \leq w \leq 0.5 \).

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