Fixed Points of Log-Linear Discrete Dynamics

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In this paper we study the fixed points of the Log-linear discrete dynamics. We show that almost all Log-linear dynamics have at most two fixed points which is a generalization of Sonis’s result.

Keywords: Log-linear discrete dynamics, Fixed points

1 INTRODUCTION

The log-linear discrete dynamics

\[ f_i(x_1, \ldots, x_n) = \frac{c_i x_1^{a_{i1}} \cdots x_n^{a_{in}}}{\sum_{j=1}^{n} c_j x_1^{a_{j1}} \cdots x_n^{a_{jn}}}, \quad i = 1, \ldots, n, \]

have been studied originally as a socio-spatial dynamic model by Dendrinos and Sonis [1]. Many interesting phenomena, for example strange attractors, pitch folk like bifurcations and invariant circles [1–5] have been found to be contained in them.

The log-linear dynamics maps depict a family of dynamics defined systematically by matrix \( A = (a_{ij}) \) and vector \( \vec{c} = (c_1, \ldots, c_n)^T \); like other such families of dynamics (for instance the Lotka–Volterra dynamics) they are a definitive object of mathematical studies. Therefore a thorough analysis of the log-linear dynamics is necessary because of the importance not only from an applicational view point but also from a pure mathematical view point.

In this paper we investigate the fixed points of the dynamics as our first step of a more extended mathematical study of the log-linear discrete dynamics. We define a real valued function on \( R \), which plays a key role in counting the number of the fixed points found in the map, and we prove that almost all dynamics have at most two fixed points. This result is a generalization of Sonis’s result [4].

2 DEFINITIONS AND NOTATIONS

We begin with some notations and definitions.

For an \( n \)-dimensional vector \( \vec{x} = (x_1, \ldots, x_n)^T \), let \( (\vec{x})_i \) be the \( i \)-th component of \( \vec{x} \), i.e., \( (\vec{x})_i = x_i \).
Let
\[ E = \text{diag}(1, \ldots, 1), \]
the \( n \) dimensional unit matrix,
\[ \vec{u} = (1, \ldots, 1)^T \in \mathbb{R}^n, \]
\[ R^{n+} = \{ \vec{x} \in \mathbb{R}^n \mid x_i > 0 \text{ for } i = 1, \ldots, n \}, \]
\[ \Delta^{n-1} = \{ \vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{u} = 1, x_i > 0 \]
for \( i = 1, \ldots, n \).

For an \( n \times n \) matrix \( A = (a_{ij}) \), \( \vec{a}_i \) denotes the \( i \)th column vector of \( A \), i.e.,
\[ \vec{a}_i = (a_{i1}, \ldots, a_{in})^T, \quad A = (\vec{a}_1, \ldots, \vec{a}_n). \]

Given an \( n \times n \) matrix \( A = (a_{ij}) \) and \( n \) positive real numbers \( c_1, \ldots, c_n \), we define a vector \( \vec{\gamma} \) and functions \( g_i, \vec{g}, f_i \) and \( \vec{f} \) defined on \( R^{n+} \) as follows:
\[ \vec{\gamma} = (\log c_1, \ldots, \log c_n)^T, \]
\[ g_i(\vec{x}) = c_i x_1^{a_{i1}} \cdots x_n^{a_{in}}, \quad i = 1, \ldots, n, \]
\[ \vec{g}(\vec{x}) = (g_1(\vec{x}), \ldots, g_n(\vec{x}))^T, \]
\[ g(\vec{x}) = \vec{g} \cdot \vec{u} = \sum_{i=1}^n g_i(\vec{x}), \]
\[ f_i(\vec{x}) = \frac{g_i(\vec{x})}{g(\vec{x})}, \]
\[ \vec{f}(\vec{x}) = (f_1(\vec{x}), \ldots, f_n(\vec{x}))^T. \]

Since \( \vec{f}(\vec{x}) \cdot \vec{u} = 1 \), the map \( \vec{f} \) gives dynamics on the \((n-1)\)-simplex \( \Delta^{n-1} \).

We call this dynamics the log-linear discrete dynamics.

For a vector \( \vec{d} = (d_1, \ldots, d_n) \in \mathbb{R}^n \), let
\[ A[\vec{d}] = (\vec{a}_1 + d_1 \vec{u}, \ldots, \vec{a}_n + d_n \vec{u}). \]

If we modify a matrix \( A \) to a matrix \( A[\vec{d}] \), then the function \( g_i(\vec{x}) \) becomes
\[ c_i x_1^{a_{i1}+d_i} \cdots x_n^{a_{in}+d_n} = g_i(\vec{x}) x_1^{d_i} \cdots x_n^{d_n} \]
and the function \( g(\vec{x}) \) becomes
\[ g(\vec{x}) x_1^{d_1} \cdots x_n^{d_n}. \]

This implies that the dynamics \( \vec{f} \) do not change under the modification \( A \) to \( A[\vec{d}] \).

Therefore as the canonical form of a matrix \( A \), we can consider, for example \([1]\),
\[ \begin{bmatrix} 0, \ldots, 0 \\ * \mid * \end{bmatrix}, \]
\[ \begin{bmatrix} * \mid * \end{bmatrix}. \]

However we will not restrict a matrix \( A \) in the canonical form, to keep a free hand for perturbations in the set of \( n \times n \) matrices \( M(n) \).

Let \( V = \{ A \in M(n) \mid \det(A-E) = 0 \} \) and \( \tilde{M}(n) = M(n) - V = \{ A \in M(n) \mid \det(A-E) \neq 0 \} \). Then since \( \det(A-E) \) is a polynomial function of \( a_{ij} \)’s, \( V \) is a \((n^2-1)\)-dimensional surface in \( n^2 \)-dimensional space \( M(n) \). Hence \( V \) is a thin set in \( M(n) \) and almost all matrices belong to \( \tilde{M}(n) \). Moreover even if \( A \) is in \( V \), one can modify \( A \) to \( A[\vec{d}] \) in \( \tilde{M}(n) \) except for the few and rare cases discussed later.

Suppose that \( A \in \tilde{M}(n) \). We define functions of a positive variable \( t \) as follows:
\[ \varphi_i(t) = \frac{t^{(B \vec{a})_i}}{e^{(B \vec{\gamma})}}, \quad i = 1, \ldots, n, \]
\[ \vec{\varphi}(t) = (\varphi_1(t), \ldots, \varphi_n(t)), \]
and
\[ \Phi(t) = \vec{\varphi}(t) \cdot \vec{u}, \]
where \( B = (A-E)^{-1} \). Note that \( \Phi(t) \) is not a constant function and that \( \vec{\varphi}(t) \in \Delta^{n-1} \) if and only if \( \Phi(t) = 1 \).

3 FIXED POINTS

Suppose that \( \vec{x} \) is a fixed point of \( \vec{f} \), that is \( \vec{f}(\vec{x}) = \vec{x} \). We can find this fixed point of \( \vec{f} \) by
solving the nonlinear equation system
\[ g_1(\vec{x}) = x_1 g(\vec{x}), \]
\[ \vdots \]
\[ g_n(\vec{x}) = x_n g(\vec{x}), \]
\[ \vec{x} \cdot \vec{u} = 1. \]

However we note that it is difficult to solve this nonlinear equation system even numerically.

The following theorem shows that we can find all fixed points of \( f \) by solving a single nonlinear equation,
\[ \Phi(t) = 1, \quad t > 0 \quad (*) \]
whose numerical solutions can be easily obtained.

**Theorem 1** Let \( A \in \tilde{M}(n) \). Suppose that the equation \( (*) \) has \( m \) distinct solutions \( t_1, \ldots, t_m \). Then \( f \) has just \( m \) fixed points \( \varphi(t_1), \ldots, \varphi(t_m) \).

**Proof** Suppose that \( f(\vec{x}) = \vec{x} \). Then \( g_i(\vec{x}) = x_i t_i, \quad i = 1, \ldots, n \), where \( t = g(\vec{x}) \) i.e.,
\[ c_1 x_1^{a_1} \cdots x_n^{a_n} = x_i t_i, \quad i = 1, \ldots, n. \]
Taking logarithms on both sides, we have
\[ \gamma_i + \sum_j a_j \log x_j = \log x_i + \log t, \quad i = 1, \ldots, n, \]
\[ (A - E)(\log x_1, \ldots, \log x_n)^T = -\vec{\gamma} + (\log t) \vec{u}. \]
Since \( A - E \) has the inverse matrix \( B \), one has
\[ (\log x_1, \ldots, \log x_n)^T = -B \vec{\gamma} + \log t \cdot B \vec{u}, \]
\[ \log x_i = -(B \vec{\gamma})_i + \log t \cdot (B \vec{u})_i, \quad i = 1, \ldots, n. \]
Therefore one obtains
\[ x_i = \frac{e^{(B \vec{u})_i}}{e^{(B \vec{u})_1} + \cdots + e^{(B \vec{u})_n}}, \quad i = 1, \ldots, n. \]
Since \( \vec{x} \cdot \vec{u} = 1, \) \( t \) is a solution of \( \Phi(t) = 1 \).

Conversely we show that if \( t \) is a solution of the equation \( (*) \), then \( \Phi(t) = 1 \).

First we notice that \( AB = B + E \) since \( E = (A - E)B = AB - B \). Then
\[ g_i(\varphi(t)) = c_i(\varphi(t))^{a_i} \cdots (\varphi_n(t))^{a_n} \]
\[ = c_i \left( \frac{t (B \vec{u})_i}{e^{(B \vec{u})_1} + \cdots + e^{(B \vec{u})_n}} \right)^{a_i} \cdots \left( \frac{t (B \vec{u})_n}{e^{(B \vec{u})_1} + \cdots + e^{(B \vec{u})_n}} \right)^{a_n} \]
\[ = \frac{t^{a_0} (B \vec{u})_1 + \cdots + a_n (B \vec{u})_n}{e^{a_0 (B \vec{u})_1} + \cdots + e^{a_n (B \vec{u})_n}} \]
\[ = \frac{t^{a_0} (B \vec{u})_1 + \cdots + a_n (B \vec{u})_n}{e^{a_0 (B \vec{u})_1} + \cdots + e^{a_n (B \vec{u})_n}} = c_i \left( \frac{t (B \vec{u})_i}{e^{(B \vec{u})_1} + \cdots + e^{(B \vec{u})_n}} \right)^{a_i} \]
\[ = t \varphi_i(t), \quad i = 1, \ldots, n, \]
and
\[ g(\varphi(t)) = \vec{g}(\varphi(t)) \cdot \vec{u} \]
\[ = (t \varphi_i(t)) \cdot \vec{u} = t \Phi(t). \]

Hence
\[ f_i(\varphi(t)) = \frac{g_i(\varphi(t))}{g(\varphi(t))} = \frac{\varphi_i(t)}{\Phi(t)}, \quad i = 1, \ldots, n. \]
Therefore if \( i \) is a solution of the equation, then
\[ f_i(\varphi(i)) = \frac{\varphi_i(i)}{\Phi(i)} = \varphi_i(i), \quad i = 1, \ldots, n, \]
so that \( \varphi(i) \) is a fixed point of \( f \).

Finally if \( i \) and \( \tilde{i} \) are distinct solutions of the equation, then \( \varphi_1(i) \neq \varphi_1(\tilde{i}) \) since \( \varphi_1(t) \) is a monotone function. Hence \( \varphi(t_1), \ldots, \varphi(t_m) \) are distinct.

In Section 5 we give Example 5 in which the coefficients \( c_1, c_2, c_3 \) are all equal to 1. Then the equation has no solution. In general:

**Proposition 1** Suppose that \( A \in \tilde{M}(n) \) and \( c_1 = \cdots = c_n = 1 \). Then the equation \( (*) \) has:
1. one solution if \( (B \vec{u})_1 > 0, \ldots, (B \vec{u})_n > 0 \),
2. one solution if \( (B \vec{u})_1 < 0, \ldots, (B \vec{u})_n < 0 \),
3. no solution if \( (B \vec{u})_i \geq 0, (B \vec{u})_j \leq 0 \) for some \( 1 \leq i, j \leq n \).
**Proof** In case (1) (resp. (2)), \( \Phi(t) \) is an increasing (resp. decreasing) function and

\[
\lim_{t \to +0} \Phi(t) = 0 \quad (\text{resp. } \infty), \quad \lim_{t \to -\infty} \Phi(t) = \infty \quad (\text{resp. } 0).
\]

Therefore the equation has unique solution. In case (3)

\[
\Phi(t) > \varphi(t) = \frac{t^{(b)}}{e^{(b)}} = t^{(b)} \geq 1 \quad \text{for any } t \geq 1
\]

and

\[
\Phi(t) > \varphi(t) = \frac{t^{(b)}}{e^{(b)}} = t^{(b)} \geq 1 \quad \text{for any } t < 1
\]

since \( \varphi = 0 \). Therefore the equation has no solution.

**4 THE NUMBER OF FIXED POINTS**

In this section we prove that almost all log-linear dynamics have at most two fixed points.

We first prove:

**Lemma 1** Suppose that

\[
h(t) = a_1 t^{\alpha_1} + a_2 t^{\alpha_2} + \cdots + a_n t^{\alpha_n},
\]

where \( \alpha_1 > \alpha_2 > \cdots > \alpha_n \) and \( \alpha_n = 0 \).

1. If \( a_1, \ldots, a_n > 0 \), then \( h(t) > 0 \) for all \( t > 0 \).
2. If \( a_1, \ldots, a_k > 0 \), \( a_{k+1}, \ldots, a_n < 0 \) for some \( k (1 \leq k < n) \), then

\[
\begin{align*}
h(t) &< 0, \quad 0 < t < t_0, \\
h(t) &> 0, \quad t = t_0, \\
h(t) &> 0, \quad t_0 < t
\end{align*}
\]

for some \( t_0 > 0 \).

**Proof** Note that

\[
\lim_{t \to -\infty} h(t) = \infty.
\]

Lemma 1 is true when \( n = 2 \). We may therefore proceed by induction, assuming Lemma 1 true for \( n \). Let

\[
h(t) = a_1 t^{\alpha_1} + a_2 t^{\alpha_2} + \cdots + a_n t^{\alpha_n} + a_{n+1} t^{\alpha_{n+1}}
\]

(\( \alpha_1 > \alpha_2 > \cdots > \alpha_n > \alpha_{n+1} = 0 \)).

Then

\[
h'(t) = a_1 \alpha_1 t^{\alpha_1 - 1} + a_2 \alpha_2 t^{\alpha_2 - 1} + \cdots + a_n \alpha_n t^{\alpha_n - 1}
\]

since \( \alpha_{n+1} = 0 \). We write \( h'(t) \) in the form

\[
h'(t) = t^{\alpha_{n+1}} k(t),
\]

where

\[
k(t) = b_1 t^{\beta_1} + \cdots + b_n t^{\beta_n},
\]

\[
b_1 = a_1 \alpha_1, \ldots, b_n = a_n \alpha_n,
\]

\[
\beta_1 = \alpha_1 - \alpha_n, \ldots, \beta_n = \alpha_n - \alpha_n = 0.
\]

Note that \( \beta_1 > \beta_2 > \cdots > \beta_n = 0 \). If \( a_1, \ldots, a_{n+1} > 0 \), then \( b_1, \ldots, b_n > 0 \), so \( k(t) > 0 \), \( t > 0 \) by the assumption. Since \( h(0) = a_{n+1} > 0 \) and \( h'(t) > 0 \) for all \( t > 0 \), \( h(t) > 0 \) for all \( t > 0 \), so that (1) holds.

If \( a_1, \ldots, a_n > 0 \) and \( a_{n+1} < 0 \), then \( b_1, \ldots, b_n > 0 \), so \( h(0) = a_{n+1} < 0 \) and \( h'(t) > 0 \) for all \( t > 0 \). Since

\[
\lim_{t \to -\infty} h(t) = \infty,
\]

there exists \( t_0 > 0 \) such that:

\[
\begin{align*}
h(t) &< 0, \quad 0 < t < t_0, \\
h(t) &= 0, \quad t = t_0, \\
h(t) &> 0, \quad t_0 < t
\end{align*}
\]

If \( a_1, \ldots, a_k > 0 \) and \( a_{k+1}, \ldots, a_{n+1} < 0 \) for some \( k (1 \leq k < n) \), then \( b_1, \ldots, b_k > 0 \) and \( b_{k+1}, \ldots, b_n < 0 \). Hence there exists \( t_0 > 0 \) such that:

\[
\begin{align*}
h'(t) &< 0, \quad 0 < t < t_0, \\
h'(t) &= 0, \quad t = t_0, \\
h'(t) &> 0, \quad t_0 < t
\end{align*}
\]
Moreover since \( h(0) = a_{n+1} < 0, \ h(t) < 0 \) for \( 0 < t \leq t_0 \).

Since \( h'(t) > 0 \) for all \( t > t_0 \) and \( \lim_{t \to \infty} h(t) = \infty \), there exists \( t'_0 > t_0 > 0 \) such that:

\[
\begin{align*}
  h(t) &< 0, \quad 0 < t < t'_0, \\
  h(t_0) &< 0, \quad t = t'_0, \\
  h(t) &> 0, \quad t'_0 < t.
\end{align*}
\]

Therefore (2) holds.

**Theorem 2** Almost all log-linear dynamics have at most two fixed points.

**Proof** It suffices to show that the equation (\( \ast \))

\[
\Phi(t) = 1, \quad t > 0
\]

has at most two solutions.

Without the loss of generality, we may write

\[
\Phi(t) = a_1 t^{\alpha_1} + \cdots + a_l t^{\alpha_l} + \text{const.}
\]

where \( a_1, \ldots, a_l > 0 \) and \( \alpha_1 > \alpha_2 > \cdots > \alpha_l \).

If \( \alpha_1, \ldots, \alpha_l > 0 \) (resp. \( \leq 0 \)), then by the same arguments as the proof of Proposition 1, equation (\( \ast \)) has a unique solution.

Suppose the \( \alpha_1, \ldots, \alpha_k > 0 \) and \( \alpha_{k+1}, \ldots, \alpha_l < 0 \) for some \( k \ (1 \leq k < l) \). Then

\[
\Phi'(t) = a_1 \alpha_1 t^{\alpha_1 - 1} + \cdots + a_k \alpha_k t^{\alpha_k - 1}
\]

\[
= \alpha_1 a_1 t^{\alpha_1 - 1} (b_1 + \beta_1 t^\beta_1 + \cdots + b_l t^\beta_l),
\]

where

\[
\beta_1 = \alpha_1 - \alpha_l, \quad \beta_2 = \ldots, \beta_l = \alpha_l - \alpha_l,
\]

\[
b_1 = 1, \quad b_2 = \frac{a_2 a_1}{\alpha_2 a_1}, \ldots, b_l = \frac{a_l a_1}{\alpha_l a_1}.
\]

Note that \( \beta_1 > \cdots > \beta_l = 0, b_1, \ldots, b_k > 0 \) and \( b_{k+1}, \ldots, b_l < 0 \). By Lemma 1, there exist \( t_0 > 0 \) such that:

\[
\begin{align*}
  \Phi'(t) &< 0, \quad 0 < t < t_0, \\
  \Phi'(t_0) &= 0, \quad t = t_0, \\
  \Phi'(t) &> 0, \quad t_0 < t.
\end{align*}
\]

Therefore \( \Phi(t) \) is monotonically decreasing for \( t < t_0 \) and \( \Phi(t) \) is monotonically increasing for \( t > t_0 \).

Since

\[
\lim_{t \to t_0^-} \Phi(t) = \infty, \quad \lim_{t \to \infty} \Phi(t) = \infty,
\]

the number of solutions is 0, 1 or 2 depending on the value of \( \Phi(t_0) \). Hence the number of the fixed points is at most two.

**Remark** We suppose in Theorems 1 and 2 that \( A - E \) is invertible. As the coefficients of \( A \) are taken randomly, the probability that \( A - E \) is noninvertible is zero. However, when the coefficients are restricted to integers, or when one changes an entry of \( A \) continuously, one often has to consider a matrix \( A \) with \( \det(A - E) = 0 \). So we will study the case \( A - E \) when it is noninvertible.

Suppose that \( \det(A - E) = 0 \). In this case one may try to modify \( A \) to \( A[\tilde{d}] \) so that \( \det(A[\tilde{d}]) - E \neq 0 \).

Let \( C = (c_{ij}) = A - E \). Since

\[
\det(A[\tilde{d}] - E) = \det(C[\tilde{d}]) = \det(C)
\]

\[
+ d_1 \det(\bar{u}, \bar{e}_2, \ldots, \bar{e}_n)
\]

\[
+ \ldots + d_n \det(\bar{e}_1, \ldots, \bar{e}_{n-1}, \bar{u})
\]

one can choose \( \tilde{d} \) so that \( \det(A[\tilde{d}] - E) \neq 0 \) except for the case where

\[
\det(\bar{u}, \bar{e}_2, \ldots, \bar{e}_n) = \cdots = \det(\bar{e}_1, \ldots, \bar{e}_{n-1}, \bar{u}) = 0.
\]

**Example** Let

\[
A = \begin{pmatrix}
0 & 0 & 0 \\
0 & 2 & 1 \\
1 & 2 & 3
\end{pmatrix}.
\]

Then

\[
\det(A - E) = \det\begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 1 \\
1 & 2 & 2
\end{pmatrix} = 0.
and
\[
\det \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix} = \det \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \det \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix} = 0.
\]

So one cannot modify \( A \) to \( A[\bar{d}] \) with \( \det(A[\bar{d}]) = -E \neq 0 \). For this example, one can get fixed points by simple calculations.

Suppose that \( c_1 = 1 \). Then \( \bar{x} = (x_1, x_2, x_3)^T \) is a fixed point if and only if
\[ x_1 + x_2 + x_3 = 1, \quad x_1, x_2, x_3 > 0, \]
\[ c_2x_1x_2x_3 = c_3(x_1x_2x_3)^2 = 1. \]

This system of equations has no solution except for the case where
\[ c_3 = c_2^2, \quad c_2 > 27, \]
in which case the fixed points make a closed curve in the 2-simplex.

5 EXAMPLE

In this section we give some numerical examples illustrating the forms of the function \( \Phi(t) \).

Example 1:
\[ c_1 = 1, \quad c_2 = 1, \quad c_3 = 1, \quad A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 0 \\ -3 & -1 & 3 \end{pmatrix}. \]

Then
\[ \Phi(t) = t + t^2 + t^3 \]
is monotonically increasing and the equation has one solution.

Example 2:
\[ c_1 = 1, \quad c_2 = 1, \quad c_3 = 1, \quad A = \begin{pmatrix} -1 & -1 & 1 \\ -1 & 1 & 0 \\ 3 & 1 & -1 \end{pmatrix}. \]

Then
\[ \Phi(t) = t + t^2 + t^3 \]
is monotonically decreasing and the equation has one solution.

Example 3:
\[ c_1 = 1, \quad c_2 = 7, \quad c_3 = 50, \quad A = \begin{pmatrix} 0 & 1 & -2 \\ 3 & 0 & 2.5 \\ 2 & 0.5 & 0 \end{pmatrix}. \]

Then
\[ \Phi(t) = 0.209128t^{0.2} + 0.123576t^{0.488889} + \frac{0.768706}{t^{0.355556}} \]
has one minimum \((< 1)\) and the equation has two solutions.

Example 4:
\[ c_1 = 1, \quad c_2 = 7, \quad c_3 = 50, \quad A = \begin{pmatrix} 0 & 1 & -2 \\ 3 & 0 & -2.57419151135 \\ 2 & 0.5 & 0 \end{pmatrix}. \]

Then
\[ \Phi(t) = 0.209128t^{0.2} + 0.124635t^{0.500423} + \frac{0.771993}{t^{0.349789}} \]
has one minimum (1) and the equation has one solution.

Example 5:

\[ c_1 = 1, \quad c_2 = 1, \quad c_3 = 1, \]

\[ A = \begin{pmatrix} 0 & 1 & -2 \\ 3 & 0 & -2.5 \\ 2 & 0.5 & 0 \end{pmatrix}. \]

Then

\[ \Phi(t) = t^{0.2} + t^{0.488889} + \frac{1}{t^{0.355556}} \]

has one minimum (1) and the equation has no solution.

References


