Research Article

Multiple Positive Solutions of Semilinear Elliptic Problems in Exterior Domains

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Assume that \( q \) is a positive continuous function in \( \mathbb{R}^N \) and satisfies the suitable conditions. We prove that the Dirichlet problem

\[-\Delta u + u = q(z)|u|^{p-2}u \text{ in } \Omega,\]

\( u \in H^1_0(\Omega), \)

where \( \Omega \) is an unbounded domain \( \mathbb{R}^N \). Let \( q \) be a positive continuous function in \( \mathbb{R}^N \) and satisfy

\[
\lim_{|z| \to \infty} q(z) = q_\infty > 0, \quad q(z) \neq q_\infty.
\]
Associated with (1.1) and (1.2), we define the functional $a$, $b$, $b^\infty$, $J$, and $J^\infty$, for $u \in H^1_0(\Omega)$

\[
a(u) = \int_\Omega \left( |\nabla u|^2 + u^2 \right) dz = \|u\|_{H^1_0}^2,
\]

\[
b(u) = \int_\Omega q(z)u^p dz,
\]

\[
b^\infty(u) = \int_\Omega q^\infty u^p dz,
\]

\[
J(u) = \frac{1}{2} a(u) - \frac{1}{p} b(u_+),
\]

\[
J^\infty(u) = \frac{1}{2} a(u) - \frac{1}{p} b^\infty(u_+),
\]

where $u_+ = \max\{|u,0| \geq 0$. By Rabinowitz [1, Proposition B.10], the functionals $a$, $b$, $b^\infty$, $J$, and $J^\infty$ are of $C^2$.

It is well known that (1.1) admits infinitely many solutions in a bounded domain. Because of the lack of compactness, it is difficult to deal with this problem in an unbounded domain. Lions [2, 3] proved that if $q(z) \geq q_\infty > 0$, then (1.1) has a positive ground state solution in $\mathbb{R}^N$. Bahri and Li [4] proved that there is at least one positive solution of (1.1) in $\mathbb{R}^N$ when $\lim_{|z| \to \infty} q(z) = q_\infty > 0$ and $q(z) \geq q_\infty - C \exp(-\delta|z|)$ for $\delta > 2$. Zhu [5] has studied the multiplicity of solutions of (1.1) in $\mathbb{R}^N$ as follows. Assume $N \geq 5$, $\lim_{|z| \to \infty} q(z) = q_\infty$, $q(z) \geq q_\infty > 0$, and there exist positive constants $C$, $\gamma$, $R_0$ such that $q(z) \geq q_\infty + C/|z|^\gamma$ for $|z| \geq R_0$, then (1.1) has at least two nontrivial solutions (one is positive and the other changes sign). Esteban [6, 7] and Cao [8] have studied the multiplicity of solutions of $-\Delta u + u = q(z)|u|^{p-2}u$ with Neumann condition in an exterior domain $\mathbb{R}^N \setminus \overline{D}$, where $D$ is a $C^{1,1}$ bounded domain in $\mathbb{R}^N$. Hirano [9] proved that if $\|q - q_\infty\|_\infty$ is sufficiently small and $q(z) \geq q_\infty(1 + C \exp(-\delta|z|))$ for $0 < \delta < 1$, then (1.1) admits at least three nontrivial solutions (one is positive and the other changes sign) in $\mathbb{R}^N$. Recently, under the same conditions, Lin [10] showed that (1.1) admits at least two positive solutions and one nodal solution in an exterior domain. Let $q(z) = a(z) + \mu b(z)$. Wu [11] showed that for sufficiently small $\mu$, if $a$ and $b$ satisfy some hypotheses, then (1.1) has at least three positive solutions in $\mathbb{R}^N$.

In this paper, we consider the multiplicity of positive solutions of (1.1) in an exterior domain. If $q$ satisfies the suitable conditions ($\|q - q_\infty\|_\infty$ is sufficiently small and $q(z) \geq q_\infty + C \exp(-\delta|z|)$ for $0 < \delta < 2$), then we can show that (1.1) admits at least three positive solutions in an exterior domain. First, in Section 3, we use the concentration-compactness argument of Lions [2, 3] to obtain the “ground-state solution” (see Theorem 3.7). In Section 4, we study the idea of category in Adachi-Tanaka [12] and Bahri-Li minimax method to get that there are at least three positive solutions of (1.1) in $\mathbb{R}^N \setminus \overline{D}$ (see Theorems 4.10 and 4.15).

2. Existence of (PS)—Sequences

Let $\Omega$ be an unbounded domain in $\mathbb{R}^N$. We define the Palais-Smale (denoted by (PS)) sequences, (PS)-values, and (PS)-conditions in $H^1_0(\Omega)$ for $J$ as follows.
Lemma 2.2. Let \( u \in H_0^1(\Omega) \) be a critical point of \( J \), then \( u \) is a nonnegative solution of (1.1). Moreover, if \( u \neq 0 \), then \( u \) is positive in \( \Omega \).

Proof. Suppose that \( u \in H_0^1(\Omega) \) satisfies \( \langle J'(u), \varphi \rangle = 0 \) for any \( \varphi \in H_0^1(\Omega) \), that is,

\[
\int_\Omega (\nabla u \nabla \varphi + u \varphi) = \int_\Omega q(z) u^{p-1} \varphi \quad \text{for any } \varphi \in H_0^1(\Omega). \tag{2.1}
\]

Thus, \( u \) is a weak solution of \(-\Delta u + u = q(z)u^{p-1} \) in \( \Omega \). Since \( q > 0 \) in \( \mathbb{R}^N \), by the maximum principle, \( u \) is nonnegative. If \( u \neq 0 \), we have that \( u \) is positive in \( \Omega \).

Define

\[
\alpha(\Omega) = \inf_{u \in M(\Omega)} J(u), \tag{2.2}
\]

where \( M(\Omega) = \{ u \in H_0^1(\Omega) \setminus \{0\} \mid a(u) = b(u) \} \) and

\[
\alpha^\infty(\Omega) = \inf_{u \in M^\infty(\Omega)} J^\infty(u), \tag{2.3}
\]

where \( M^\infty(\Omega) = \{ u \in H_0^1(\Omega) \setminus \{0\} \mid a(u) = b^\infty(u) \} \).

Lemma 2.3. Let \( \beta \in \mathbb{R} \) and let \( \{ u_n \} \) be a \((PS)_\beta\)-sequence in \( H_0^1(\Omega) \) for \( J \). Then,

(i) \( \{ u_n \} \) is a bounded sequence in \( H_0^1(\Omega) \),

(ii) \( a(u_n) = b(u_n) + o_n(1) = (2p/(p-2))\beta + o_n(1) \) as \( n \to \infty \) and \( \beta \geq 0 \).

By Chen et al. [13] and Chen and Wang [14], we have the following lemmas.

Lemma 2.4. (i) For each \( u \in H_0^1(\Omega) \setminus \{0\} \) with \( u_+ \neq 0 \), there exists the unique number \( s_u > 0 \) such that \( s_u u \in M(\Omega) \) and \( \sup_{s \geq 0} J(su) = J(s_u u) \).

(ii) Let \( \beta > 0 \) and \( \{ u_n \} \) a sequence in \( H_0^1(\Omega) \setminus \{0\} \) for \( J \) such that \( u_n \neq 0 \), \( J(u_n) = \beta + o_n(1) \) and \( a(u_n) = b(u_n) + o_n(1) \). Then, there is a sequence \( \{ s_n \} \) in \( \mathbb{R}^+ \) such that \( s_n = 1 + o_n(1) \), \( \{ s_n u_n \} \) in \( M(\Omega) \) and \( J(s_n u_n) = \beta + o_n(1) \) as \( n \to \infty \).

Lemma 2.5. There exists a positive constant \( c \) such that \( \|u\|_{H^1} \geq c > 0 \) for each \( u \in M(\Omega) \). Moreover, \( \alpha(\Omega) > 0 \).

Lemma 2.6. Let \( \Omega_1 \subsetneq \Omega_2 \). If \( J \) satisfies the \((PS)_{\alpha(\Omega_1)}\)-condition or \( \alpha(\Omega_1) \) is a critical value, then \( \alpha(\Omega_2) < \alpha(\Omega_1) \).

Proof. See Chen et al. [13] or Lin et al. [15].
Remark 2.7. The above definitions and lemmas hold not only for $f^\infty$ and $M^\infty(\Omega)$ but also for $\alpha^\infty(\Omega)$.

Lemma 2.8. Every minimizing sequence $\{u_n\}$ in $M^\infty(\Omega)$ of $\alpha^\infty(\Omega)$ is a $(PS)_{\alpha^\infty(\Omega)}$-sequence in $H_0^1(\Omega)$ for $J$. Moreover, $\alpha^\infty(\Omega)$ is a $(PS)$-value.

3. Existence of Ground State Solution

From now on, let $\Omega = \mathbb{R}^N \setminus \overline{D}$ be an exterior domain, where $D$ is a $C^{1,1}$ bounded domain in $\mathbb{R}^N$. By Lions [2, 3], Struwe [16], and Lien et al. [17], we have the following decomposition lemmas.

Lemma 3.1 (Palais-Smale Decomposition Lemma for $J$). Assume that $q$ is a positive continuous function in $\mathbb{R}^N$ and $\lim_{|z| \to \infty} q(z) = q_\infty > 0$. Let $\{u_n\}$ be a $(PS)_J$-sequence in $H_0^1(\Omega)$ for $J$. Then, there are a subsequence $\{u_n\}$, a nonnegative integer $l$, sequences $\{z_n^i\}_{i=1}^\infty$ in $\mathbb{R}^N$, functions $u$ in $H_0^1(\Omega)$, and $w^i \neq 0$ in $H^1(\mathbb{R}^N)$ for $1 \leq i \leq l$ such that

$$
\begin{align*}
&\left|z_n^i - z_n^j\right| \to \infty \quad \text{for } 1 \leq i, j \leq l, \; i \neq j, \\
&-\Delta u + u = q(z)|u|^{p-2}u \quad \text{in } \Omega, \\
&-\Delta w^i + w^i = q_\infty|w^i|^{p-2}w^i \quad \text{in } \mathbb{R}^N, \\
&u_n = u + \sum_{i=1}^l w^i \left( \cdot - z_n^i \right) + o_n(1) \quad \text{strongly in } H^1(\mathbb{R}^N), \\
&J(u_n) = J(u) + \sum_{i=1}^l J^\infty(w^i) + o_n(1).
\end{align*}
$$

Lemma 3.2 (Palais-Smale Decomposition Lemma for $J^\infty$). Let $\{u_n\}$ be a $(PS)_J$-sequence in $H_0^1(\Omega)$ for $J^\infty$. Then, there are a subsequence $\{u_n\}$, a nonnegative integer $l$, sequences $\{z_n^i\}_{i=1}^\infty$ in $\mathbb{R}^N$, functions $u$ in $H_0^1(\Omega)$, and $w^i \neq 0$ in $H^1(\mathbb{R}^N)$ for $1 \leq i \leq l$ such that

$$
\begin{align*}
&\left|z_n^i - z_n^j\right| \to \infty \quad \text{for } 1 \leq i, j \leq l, \; i \neq j, \\
&-\Delta u + u = q_\infty|u|^{p-2}u \quad \text{in } \Omega, \\
&-\Delta w^i + w^i = q_\infty|w^i|^{p-2}w^i \quad \text{in } \mathbb{R}^N, \\
&u_n = u + \sum_{i=1}^l w^i \left( \cdot - z_n^i \right) + o_n(1) \quad \text{strongly in } H^1(\mathbb{R}^N), \\
&J^\infty(u_n) = J^\infty(u) + \sum_{i=1}^l J^\infty(w^i) + o_n(1).
\end{align*}
$$
Lemma 3.3. (i) $\alpha^\infty(\Omega) = \alpha^\infty(\mathbb{R}^N)$ (denoted by $\alpha^\infty$).

(ii) Let $\{u_n\} \subset M(\Omega)$ be a (PS)$_\beta$-sequence in $H^1_0(\Omega)$ for $J$ with $0 < \beta < \alpha^\infty$.

Then, there exist a subsequence $\{u_{n_k}\}$ and a nonzero $u_0 \in H^1_0(\Omega)$ such that $u_n \rightharpoonup u_0$ strongly in $H^1_0(\Omega)$, that is, $J$ satisfies the (PS)$_\beta$-condition in $H^1_0(\Omega)$. Moreover, $u_0$ is a positive solution of (1.1) such that $J(u_0) = \beta$.

Proof. (i) Since $\Omega$ is an exterior domain, by Lien et al. [17], $\Omega$ is a ball-up domain (for any $r > 0$, there exists $z \in \Omega$ such that $B^N(z; r) \subset \Omega$) and $\alpha^\infty(\Omega) = \alpha^\infty(\mathbb{R}^N)$.

(ii) Since $\{u_n\} \subset M(\Omega)$ is a (PS)$_\beta$-sequence in $H^1_0(\Omega)$ for $J$ with $0 < \beta < \alpha^\infty$, by Lemma 2.3, $\{u_n\}$ is bounded. Thus, there exist a subsequence $\{u_{n_k}\}$ and $u_0 \in H^1_0(\Omega)$ such that $u_{n_k} \rightharpoonup u_0$ weakly in $H^1_0(\Omega)$. It is easy to check that $u_0$ is a solution of (1.1). Applying Palais-Smale Decomposition Lemma 3.1, we get

$$\alpha^\infty > \beta = J(u_n) \geq l\alpha^\infty.$$  \hspace{1cm} (3.3)

Then, $l = 0$ and $u_0 \neq 0$. Hence, $u_n \rightharpoonup u_0$ strongly in $H^1_0(\Omega)$ and $J(u_0) = \beta$. Moreover, by Lemma 2.2, $u_0$ is positive in $\Omega$. \hfill $\blacksquare$

It is well known that there is the unique (up to translation), positive, smooth, and radially symmetric solution $w$ of (1.2) in $\mathbb{R}^N$ such that $J^\infty(w) = \alpha^\infty$. (See Bahri and Lions [18], Gidas et al. [19, 20] and Kwong [21]). Recall the facts

(i) for any $\varepsilon > 0$, there exist constants $C_0, C_0' > 0$ such that for all $z \in \mathbb{R}^N$

$$w(z) \leq C_0 \exp(-|z|), \quad |\nabla w(z)| \leq C_0' \exp(-(1 - \varepsilon)|z|),$$  \hspace{1cm} (3.4)

(ii) for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$w(z) \geq C_\varepsilon \exp(-(1 + \varepsilon)|z|) \quad \forall z \in \mathbb{R}^N.$$  \hspace{1cm} (3.5)

Suppose $D \subset B^N(0; R) = \{z \in \mathbb{R}^N \mid |z| < R\}$ for some $R > 0$. Let $\psi_R : \mathbb{R}^N \rightarrow [0, 1]$ be a $C^\infty$-function on $\mathbb{R}^N$ such that $0 \leq \psi_R \leq 1$, $|\nabla \psi_R| \leq c$ and

$$\psi_R(z) = \begin{cases} 1 & \text{for } |z| \geq R + 1, \\ 0 & \text{for } |z| \leq R. \end{cases}$$  \hspace{1cm} (3.6)

We define

$$w_{\Xi}(z) = \psi_R(z)w(z - \Xi) \quad \forall \Xi \in \mathbb{R}^N.$$  \hspace{1cm} (3.7)

Clearly, $w_{\Xi}(z) \in H^1_0(\Omega)$.

We need the following lemmas to prove that $\sup_{t \geq 0} J(tw_{\Xi}) < \alpha^\infty$ for sufficiently large $|\Xi|$. 

Lemma 3.4. Let $E$ be a domain in $\mathbb{R}^N$. If $f : E \to \mathbb{R}$ satisfies

$$\int_E |f(z)e^{\sigma|z|}| dz < \infty \quad \text{for some } \sigma > 0,$$

then

$$\left(\int_E f(z)e^{-\sigma|z-\overline{z}|} dz\right)e^{\sigma|\overline{z}|} = \int_E f(z)e^{\sigma((z,\overline{z})/|\overline{z}|)} dz + o(1) \quad \text{as } |\overline{z}| \to \infty. \quad (3.9)$$

Proof. Since $\sigma|\overline{z}| \leq \sigma|z| + \sigma|z-\overline{z}|$, we have

$$\left|f(z)e^{-\sigma|z-\overline{z}|}e^{\sigma|\overline{z}|}\right| \leq \left|f(z)e^{\sigma|z|}\right|. \quad (3.10)$$

Since $-\sigma|z-\overline{z}| + \sigma|\overline{z}| = \sigma((z,\overline{z})/|\overline{z}|) + o(1)$ as $|\overline{z}| \to \infty$, then the lemma follows from the Lebesque-dominated convergence theorem. \qed

Next, assume that $q$ is a positive continuous function in $\mathbb{R}^N$ and satisfies (q1) and

$$q(z) \geq q_\infty + C \exp(-\delta|z|) \quad \text{for some } C > 0 \text{ and } 0 < \delta < 2. \quad (q2)$$

Then, we have the following lemmas.

Lemma 3.5. (i) There exists a number $t_0 > 0$ such that for $0 \leq t < t_0$ and each $w_\overline{z} \in H^1_0(\Omega)$, we have

$$J(tw_\overline{z}) < \alpha_\infty. \quad (3.11)$$

There exists a number $t_1 > 0$ such that for any $t > t_1$ and $|\overline{z}| \geq R + 2$, we have

$$J(tw_\overline{z}) < 0. \quad (3.12)$$

Proof. (i) Since $\alpha_\infty > 0 = J(0)$, $J$ is continuous in $H^1_0(\Omega)$ and $|w_\overline{z}|$ is bounded in $H^1_0(\Omega)$, then there exists $t_0 > 0$ such that for $0 \leq t < t_0$ and each $w_\overline{z} \in H^1_0(\Omega)$

$$J(tw_\overline{z}) < \alpha_\infty. \quad (3.13)$$
For $|z| \geq R + 2$: since $0 \leq \psi_R \leq 1$, $|\nabla \psi_R| \leq c$ and $q(z) \geq q_\infty$, we have that

$$J(tw) = \frac{t^2}{2} \int_\Omega \left[ |(\nabla \psi_R(z) w(z - \overline{z}))|^2 + (\psi_R(z) w(z - \overline{z}))^2 \right] dz$$

$$- \frac{t^2}{p} \int_\Omega q(z) (\psi_R(z) w(z - \overline{z}))^p dz$$

$$\leq \frac{t^2}{2} \int_{\mathbb{R}^N} \left[ |(\nabla \psi_R(z) w(z - \overline{z}) + \psi_R \nabla w(z - \overline{z})|^2 + \nabla w(z - \overline{z})^2 \right] dz$$

$$- \frac{t^2}{p} \int_{\mathbb{R}^N} q(z) (\nabla w(z - \overline{z}))^p dz$$

$$\leq \frac{t^2}{2} \int_{\mathbb{R}^N} \left[ |\nabla w(z) + |\nabla w(z)||^2 + w(z)^2 \right] dz - \frac{t^2}{p} \int_{B(0,1)} q(z) (\nabla w(z))^p dz. \quad (3.14)$$

Hence, there exists $t_1 > 0$ such that

$$J(tw) < 0 \quad \text{for any } t > t_1, \ |z| \geq R + 2. \quad (3.15)$$

**Lemma 3.6.** There exists a number $R_1 > R + 2 > 0$ such that for any $|z| \geq R_1$, we obtain

$$\sup_{t \geq 0} J(tw) < \alpha^\infty. \quad (3.16)$$

**Proof.** Applying the above lemma, we only need to show that there exists a number $R_1 > R + 2 > 0$ such that for any $|z| \geq R_1$,

$$\sup_{t \in \mathbb{R}} J(tw) < \alpha^\infty. \quad (3.17)$$

For $t_0 \leq t \leq t_1$, since

$$|\nabla (\psi_R \nabla w(z - \overline{z}))|^2 = |\nabla \psi_R|^2 w(z - \overline{z})^2 + q_R^2 |\nabla w(z - \overline{z})|^2 + 2\psi_R \nabla w(z - \overline{z}) \nabla \psi_R \nabla w(z - \overline{z}), \quad (3.18)$$
then we have

\[ f(tw) = \frac{t^2}{2} \int_{\mathbb{R}^N} \left\{ \left| \nabla (q_R(z)w(z-\bar{z})) \right|^2 + \left| (q_R(z)w(z-\bar{z})) \right|^2 \right\} dz \]

\[ - \frac{t^p}{p} \int_{\mathbb{R}^N} q(z) [q_R(z)w(z-\bar{z})]^p dz \quad (\because \text{the definition of } q_R) \]

\[ \leq \frac{t^2}{2} \int_{\mathbb{R}^N} \left[ \left| \nabla w(z-\bar{z}) \right|^2 + \omega(z-\bar{z})^2 \right] dz - \frac{t^p}{p} \int_{\mathbb{R}^N} q_\infty w(z-\bar{z})^p dz \]

\[ + \frac{t^2}{2} \int_{\mathbb{R}^N} \left[ \left| \nabla q_R \right|^2 \omega(z-\bar{z})^2 + 2q_R \omega(z-\bar{z}) \nabla q_R \nabla w(z-\bar{z}) \right] dz \]

\[ - \frac{t^p}{p} \int_{\mathbb{R}^N} \left[ q(z)q_R^p \omega(z-\bar{z})^p - q_\infty \omega(z-\bar{z})^p \right] dz \quad (\because (3.18) \text{ and } 0 \leq q_R \leq 1) \]

\[ \leq \alpha^\infty + \frac{t^2}{2} \int_{\mathbb{R}^N} \left[ \left| \nabla \omega(z-\bar{z}) \right|^2 + 2\omega(z-\bar{z}) \left| \nabla q_R \right| \left| \nabla w(z-\bar{z}) \right| \right] dz \]

\[ - \frac{t^p}{p} \int_{|z| \geq R+1} (q(z) - q_\infty) \omega(z-\bar{z})^p dz \]

\[ + \frac{t^p}{p} \int_{|z| \leq R+1} q_\infty \omega(z-\bar{z})^p dz \quad \left( \because \sup_{t \geq 0} f^\infty(tw) = \alpha^\infty \text{ and the definition of } q_R \right). \]

(3.19)

Since the support of \( \nabla q_R \) is bounded, then

\[ \int_{\text{supp}(\nabla q_R)} \left| \nabla q_R \right|^2 \omega(z-\bar{z})^2 dz \leq C_1 \exp(-2|\bar{z}|), \]

(3.20)

\[ \int_{\text{supp}(\nabla q_R)} \left| \omega(z-\bar{z}) \right| \left| \nabla q_R \right| \left| \nabla w(z-\bar{z}) \right| dz \leq C_2 \exp(-|2 - \epsilon| |\bar{z}|). \]

Similarly, we have

\[ \int_{|z| \leq R+1} q_\infty \omega(z-\bar{z})^p dz \leq C_3 \exp(-p|\bar{z}|). \]

(3.21)
Since \( q(z) \geq q_\infty + C \exp(-\delta|z|) \) for some \( 0 < \delta < 2 \), by Lemma 3.4, there exists \( R'_1 > R + 2 > 0 \) such that for any \( |\Xi| > R'_1 \)

\[
\int_{|\Xi| \leq R + 1} (q(z) - q_\infty) w(z - \Xi)^p dz \geq C'_p \exp(- \min\{\delta, p(1 + \varepsilon)\} |\Xi|)
\]

(3.22)

\[
\geq C'_p \exp(-\delta|\Xi|).
\]

Choosing \( 0 < \varepsilon < 2 - \delta \) and using (3.20)–(3.22), there exists \( R_1 > R'_1 \) such that for \( |\Xi| \geq R_1 \), we have

\[
\sup_{t \in \mathcal{S} \cap S_1} J(tw_\Xi) < a^\infty,
\]

(3.23)

that is, \( \sup_{t \geq 0} J(tw_\Xi) < a^\infty \).

Using the Ekeland variational principle (or see Stuart [22]), there is a \((\text{PS})_{a(\Omega)}\) sequence \( \{u_n\} \subset M(\Omega) \) for \( J \). Then, we apply Lemma 3.3(ii) to obtain the existence of positive ground state solution of (1.1) in \( \Omega \).

**Theorem 3.7.** Assume that \( q \) is a positive continuous function in \( \mathbb{R}^N \) and satisfies \((q1)\) and \((q2)\). Then, there exists at least one positive ground state solution \( u_0 \) of (1.1) in \( \Omega \).

**Proof.** Since \( w_\Xi \in H^1_0(\Omega) \), by Lemma 2.4(i), there exists \( s_\Xi > 0 \) such that \( s_\Xi w_\Xi \in M(\Omega) \). Thus, by Lemma 3.6, \( a(\Omega) \leq J(s_\Xi w_\Xi) \leq \sup_{t \geq 0} J(tw_\Xi) < a^\infty \) for \( |\Xi| \geq R_1 \). Using the Ekeland variational principle, there is a \((\text{PS})_{a(\Omega)}\)-sequence \( \{u_n\} \subset M(\Omega) \) for \( J \). Apply Lemma 3.3(ii), there exists at least one positive solution \( u_0 \) of (1.1) in \( \Omega \) such that \( J(u_0) = a(\Omega) \).

\[ \square \]

### 4. Existence of Multiple Solutions

In this section, we use two methods to obtain the existence of multiple positive solutions of (1.1) in an exterior domain. Part I: we study the idea of category to prove Theorem 4.10. Part II: we study the Bahri-Li minimax method to prove Theorem 4.15.

**Lemma 4.1.** Assume that \( q \) is a positive continuous function in \( \mathbb{R}^N \). If \( q \) satisfies \((q1)\), \((q2)\) and \((m/2)q_\infty \geq q(z) \) where \( m > 2 \), then there exists \( m_0 > 2 \) such that for \( m \leq m_0 \), we obtain that \( 2a(\Omega) > a^\infty \).

**Proof.** Since \( q(z) \geq q_\infty \), by Lions [2, 3], let \( w_0 \in H^1(\mathbb{R}^N) \) be a positive solution of \(-\Delta w_0 + w_0 = q(z)|w_0|^{p-2}w_0 \) in \( \mathbb{R}^N \) and \( J(w_0) = a(\mathbb{R}^N) \). By Lemma 2.4(i) and Remark 2.7, there exists \( s_0 > 0 \) such that \( s_0 w_0 \in M^\infty(\mathbb{R}^N) \) and \( J^\infty(s_0 w_0) \geq a^\infty \) and

\[
\int_{\mathbb{R}^N} \left[ |\nabla (s_0 w_0)|^2 + (s_0 w_0)^2 \right] dz = \int_{\mathbb{R}^N} q_\infty (s_0 w_0)^p dz \geq \frac{2p}{p - 2} a^\infty.
\]

(4.1)
Moreover, we have
\[
1 = \frac{\int_{\mathbb{R}^N} |\nabla w_0|^2 + w_0^2 \, dz}{\int_{\mathbb{R}^N} q(z) w_0^p \, dz} < \frac{\int_{\mathbb{R}^N} |\nabla w_0|^2 + w_0^2 \, dz}{\int_{\mathbb{R}^N} q_{20} w_0^p \, dz} = s_0^{p-2} < \frac{\int_{\mathbb{R}^N} (m/2) q_{20} w_0^p \, dz}{\int_{\mathbb{R}^N} q_{20} w_0^p \, dz} = \frac{m}{2}.
\] (4.2)

Hence, using the above inequalities, we get
\[
\alpha(\mathbb{R}^N) = J(w_0) = \sup_{s \geq 0} J(sw_0) > J(s_0 w_0)
= J^\infty(s_0 w_0) - \frac{1}{p} \int_{\mathbb{R}^N} (q(z) - q_{20})(s_0 w_0)^p \, dz
\geq \alpha^\infty - \frac{1}{p} \left( \frac{m}{2} - 1 \right) \int_{\mathbb{R}^N} q_{20}(s_0 w_0)^p \, dz
= \alpha^\infty - \frac{s_0^2}{p} \left( \frac{m}{2} - 1 \right) \int_{\mathbb{R}^N} (|\nabla w_0|^2 + w_0^2) \, dz
> \alpha^\infty - \frac{1}{p} \left( \frac{m}{2} - 1 \right) \left( \frac{m}{2} \right)^{2/(p-2)} \frac{2p}{p-2} \alpha(\mathbb{R}^N),
\] (4.3)

that is, \([1 + ((m-2)/(p-2))(m/2)^{2/(p-2)})\alpha(\mathbb{R}^N)] > \alpha^\infty. \) Choose some \(m_0 > 2\) such that for \(2 < m \leq m_0, \) then \(2\alpha(\mathbb{R}^N) > \alpha^\infty. \) By Lemma 2.6 and Theorem 3.7, \(2\alpha(\Omega) > 2\alpha(\mathbb{R}^N) > \alpha^\infty. \)

**Lemma 4.2.** There exists a number \(\delta_0 > 0\) such that if \(u \in M^\infty(\Omega)\) and \(J^\infty(u) \leq \alpha^\infty + \delta_0,\) then
\[
\int_{\mathbb{R}^N} \frac{z}{|z|} \left( |\nabla u|^2 + u^2 \right) \, dz \neq 0.
\] (4.4)

**Proof.** On the contrary, there exists a sequence \(\{u_n\}\) in \(M^\infty(\Omega)\) such that \(J^\infty(u_n) = \alpha^\infty + o_n(1)\) as \(n \to \infty\) and
\[
\int_{\mathbb{R}^N} \frac{z}{|z|} \left( |\nabla u_n|^2 + u_n^2 \right) \, dz = 0 \quad \forall n.
\] (4.5)

By Lemma 2.8, \(\{u_n\}\) is a \((PS)_{\alpha^\infty}\)-sequence in \(H^1_0(\Omega)\) for \(J^\infty. \) Since \(\alpha^\infty(\Omega) = \alpha^\infty(\mathbb{R}^N), \) Lien et al. [17] proved that (1.2) does not have any ground state solution in an exterior domain, that is, \(\inf_{\nu \in \text{M}^\infty(\Omega)} J^\infty(\nu) = \alpha^\infty(\Omega)\) is not achieved. Applying the Palais-Smale Decomposition Lemma 3.2, we have that there exists a sequence \(\{z_n\}\) in \(\mathbb{R}^N\) such that \(|z_n| \to \infty\) as \(n \to \infty\) and
\[
u_n(z) = \frac{w(z - z_n) + o_n(1)}{ \text{strongly in } H^1(\mathbb{R}^N)}. \] (4.6)
where \( w \) is the positive solution of (1.2) in \( \mathbb{R}^N \). Suppose the subsequence \( z_n / |z_n| \to z_0 \) as \( n \to \infty \), where \( z_0 \) is a unit vector in \( \mathbb{R}^N \). Then, by the Lebesgue dominated convergence theorem, we have

\[
\bar{\Omega} = \int_{\mathbb{R}^N} \frac{z}{|z|} (|\nabla u_n|^2 + u_n^2) \, dz
\]
\[
= \int_{\mathbb{R}^N} \frac{z + z_n}{|z + z_n|} (|\nabla w|^2 + w^2) \, dz + o_n(1)
\]
\[
= \left( \frac{2p}{p-2} \right) a^\infty z_0 + o_n(1),
\]

which is a contradiction.

Using the results of Lemma 2.4(i), let \( K(u) = J(s_u u) = \sup_{s \geq 0} J(su) \) for each \( u \in H^1_0(\Omega) \setminus \{0\} \) with \( u_+ \neq 0 \). For \( c \in \mathbb{R} \), we denote

\[
[K \leq c] = \{ u \in \Sigma \mid K(u) \leq c \},
\]

where \( \Sigma = \{ u \in H^1_0(\Omega) \mid u_+ \neq 0 \text{ and } ||u||_{H^1} = 1 \} \). Then, we have the following lemma.

**Lemma 4.3.** (i) \( K \in C^1(\Sigma, \mathbb{R}) \) and

\[
\langle K'(u), \varphi \rangle = s_u \langle J'(s_u u), \varphi \rangle
\]

for all \( \varphi \in T_u \Sigma = \{ \varphi \in H^1_0(\Omega) \mid \langle \varphi, u \rangle = 0 \} \).

(ii) \( u \in \Sigma \) is a critical point of \( K(u) \) if and only if \( s_u u \in H^1_0(\Omega) \) is a critical point of \( J \).

**Proof.** (i) For \( u \in \Sigma \), it is easy to check that

\[
\frac{d}{ds} J(su)|_{s=s_u} = 0,
\]
\[
\frac{d^2}{ds^2} J(su)|_{s=s_u} = a(u) - (p-1)s_u^{p-2}b(u) = (2-p)a(u) < 0.
\]

Then, using the implicit function theorem to obtain that \( s_u \in C^1(\Sigma, (0, \infty)) \). Therefore, \( K(u) = J(s_u u) \in C^1(\Sigma, \mathbb{R}) \). Since \( s_u u \in M(\Omega) \), we can get \( \langle J'(s_u u), u \rangle = 0 \). Thus,

\[
\langle K'(u), \varphi \rangle = \langle J'(s_u u), s_u \varphi \rangle + \langle J'(s_u u), \langle \varphi, u \rangle u \rangle
\]
\[
= s_u \langle J'(s_u u), \varphi \rangle \quad \forall \varphi \in T_u \Sigma.
\]

(ii) By (i), \( K'(u) = 0 \) if and only if \( \langle J'(s_u u), \varphi \rangle = 0 \) for all \( \varphi \in T_u \Sigma \). Since \( H^1_0(\Omega) \) is a Hilbert space and \( \langle J'(s_u u), u \rangle = 0 \), so it is equivalent to \( J'(s_u u) = 0 \) in \( H^{-1}(\Omega) \). \( \square \)
Lemma 4.4. Assume that $q$ is a positive continuous function in $\mathbb{R}^N$ and satisfies $(q1)$ and for $m > 2$ and $0 < \delta < 2$

$$\frac{m}{2} q_\infty \geq q(z) \geq q_\infty + C \exp(-\delta |z|) \quad \text{where} \quad 0 < C \leq \frac{m-2}{2} q_\infty. \quad (4.12)$$

We have that there exists a number $m_0 \geq m_1 > 2$ ($m_0$ is defined in Lemma 4.1) such that if $m \leq m_1$, then

$$\int_{\mathbb{R}^N} \frac{z}{|z|} (|\nabla u|^2 + u^2) dz \neq 0 \quad \text{for any} \quad u \in [K < \alpha^\infty]. \quad (4.13)$$

Proof. By the assumptions of $q$, Lemmas 2.4(i) and 3.6, the set $[K < \alpha^\infty]$ is nonempty. For any $u \in [K < \alpha^\infty]$, $u \in \Sigma$, $s_\alpha u \in M(\Omega)$ and $J(s_\alpha u) < \alpha^\infty$, we get $J(s_\alpha u) \geq \alpha(\Omega)$ and

$$\frac{2p}{p-2} \alpha(\Omega) \leq s_\alpha^p \int_{\Omega} q(z) u_\alpha^p dz < \frac{2p}{p-2} \alpha^\infty. \quad (4.14)$$

Since $2\alpha(\Omega) > \alpha^\infty$ (by Lemma 4.1), then we have

$$\frac{p}{p-2} \alpha^\infty < \frac{2p}{p-2} \alpha(\Omega) \leq s_\alpha^p \|q\|_\infty \int_{\Omega} u_\alpha^p dz \leq \left( \frac{2p}{p-2} \alpha^\infty \right)^{p/2} \|q\|_\infty \int_{\Omega} u_\alpha^p dz. \quad (4.15)$$

By Lemma 4.2 (i) and Remark 2.7, there exists $t_\infty > 0$ such that $t_\infty u \in M^\infty(\Omega)$, then by (4.15), we have

$$t_\infty^2 = t_\infty^p \int_{\Omega} q_\infty u_\infty^p dz > t_\infty^p q_\infty \left( \frac{p-2}{2p\alpha^\infty} \right)^{(p-2)/2} \frac{1}{mq_\infty}, \quad (4.16)$$

that is,

$$m^{1/(p-2)} \sqrt{\frac{2p\alpha^\infty}{p-2}} > t_\infty. \quad (4.17)$$

Since $u \in [K < \alpha^\infty]$ and by the definitions of $J$ and $J_\infty$,

$$\alpha^\infty > J(s_\alpha u) = \sup_{s \geq 0} J(su) \geq J(t_\infty u)$$

$$= \frac{1}{2} a(t_\infty u) - \frac{1}{p} \int_{\Omega} q(z)t_\infty^p u_\infty^p dz \quad (4.18)$$

$$= J_\infty(t_\infty u) - \frac{1}{p} \int_{\Omega} (q(z) - q_\infty) t_\infty^p u_\infty^p dz.$$
From (4.17) and (4.18), we have

\[
J^\infty(t_\infty u) < a^\infty + \frac{1}{p} \int_{\Omega} (q(z) - q_\infty) t_\infty u^p_t \, dz
\]

\[
\leq a^\infty + \frac{1}{pq_\infty} \left( \frac{m-2}{2} \right) q_\infty t_\infty^2
\]

\[
< a^\infty + \frac{m-2}{p-2} m^{2/(p-2)} a^\infty.
\]  \hspace{1cm} (4.19)

Hence, there exists \( m_0 \geq m_1 > 2 \) such that if \( 2 < m < m_1 \), then

\[
J^\infty(t_\infty u) \leq a^\infty + \delta_0, \quad \text{where } t_\infty u \in M^\infty(\Omega). \hspace{1cm} (4.20)
\]

By Lemma 4.2, we obtain

\[
\int_{R^N} \frac{z}{|z|} \left[ |\nabla (t_\infty u)|^2 + (t_\infty u)^2 \right] \, dz \neq 0,
\]  \hspace{1cm} (4.21)

or

\[
\int_{R^N} \frac{z}{|z|} (|\nabla u|^2 + u^2) \, dz \neq 0.
\]  \hspace{1cm} (4.22)

We try to show that for a sufficiently small \( \sigma > 0 \)

\[
\text{cat}(\{K \leq a^\infty - \sigma\}) \geq 2. \hspace{1cm} (4.23)
\]

To prove (4.23), we need some preliminaries. Recall the definition of Lusternik-Schnirelman category.

**Definition 4.5.** (i) For a topological space \( X \), we say a nonempty, closed subset \( A \subset X \) is contractible to a point in \( X \) if and only if there exists a continuous mapping

\[
\eta : [0,1] \times A \rightarrow X
\]  \hspace{1cm} (4.24)

such that for some \( x_0 \in X \) and

\[
\begin{align*}
\eta(0,x) &= x, \quad \forall x \in A, \\
\eta(1,x) &= x_0, \quad \forall x \in A.
\end{align*}
\]  \hspace{1cm} (4.25)

Lemma 4.6. Suppose that $X$ is a Hilbert manifold and $\Psi \in C^1(X, \mathbb{R})$. Assume that there are $c_0 \in \mathbb{R}$ and $k \in \mathbb{N}$.

(i) $\Psi(x)$ satisfies the (PS)$_c$-condition for $c \leq c_0$.
(ii) $\text{cat}([x \in X | \Psi(x) \leq c_0]) \geq k$.

Then, $\Psi(x)$ has at least $k$ critical points in $\{x \in X; \Psi(x) \leq c_0\}$.

Proof. See Ambrosetti [23, Theorem 2.3].

Lemma 4.7. Let $N \geq 1$, $S^{N-1} = \{z \in \mathbb{R}^N \ | \ |z| = 1\}$, and let $X$ be a topological space. Suppose that there are two continuous maps

$$F : S^{N-1} \to X, \quad G : X \to S^{N-1}$$

(4.27)

such that $G \circ F$ is homotopic to the identity map of $S^{N-1}$, that is, there exists a continuous map $\zeta : [0, 1] \times S^{N-1} \to S^{N-1}$ such that

$$\zeta(0, z) = (G \circ F)(z) \quad \text{for each } z \in S^{N-1},$$

$$\zeta(1, z) = z \quad \text{for each } z \in S^{N-1}.$$

(4.28)

Then,

$$\text{cat}(X) \geq 2.$$  

(4.29)

Proof. See Adachi and Tanaka [12, Lemma 2.5].

From the result of Lemma 4.4, for $2 < m \leq m_1$, let $q$ satisfy the condition

$$\frac{m}{2} q(x) \geq q(z) \geq q(x) + C \exp(-\delta|z|) \quad \text{where } 0 < C \leq \frac{m-2}{2} q(x) \text{ and } 0 < \delta < 2. \quad (q'_2)$$

In this section, assume that $q$ is a positive continuous function in $\mathbb{R}^N$ and satisfies $(q1)$, and $(q'_2)$. Let $\tilde{z} \in S^{N-1}$ and $w_n(z) = qR(z)w(z - n\tilde{z}) \in H_0^1(\Omega)$ for each $n \in \mathbb{N}$. By Lemma 2.4(i),
there exist unique numbers \((n, \tilde{z}) > 0\) such that \(s(n, \tilde{z})w_n \in M(\Omega)\). We define a map \(F_n : S^{N-1} \to H^1_0(\Omega)\) by

\[
F_n(\tilde{z})(z) = \frac{s(n, \tilde{z})w_n(z)}{\|s(n, \tilde{z})w_n(z)\|_{H^1}} \quad \text{for} \quad \tilde{z} \in S^{N-1}.
\] (4.30)

Then, we have the following lemma.

**Lemma 4.8.** There are \(n_0 \in \mathbb{N}\) and a sequence \(\{\sigma_n\}\) in \(\mathbb{R}^+\) such that

\[
F_n\left(S^{N-1}\right) \subseteq [K \leq \alpha^\infty - \sigma_n] \quad \text{for each} \quad n \geq n_0.
\] (4.31)

**Proof.** Since there exists a unique number \(s(n, \tilde{z}) > 0\) such that \(s(n, \tilde{z})w_n \in M(\Omega)\), and by the definition of \(K\), then we obtain that there exists \(t_n > 0\) such that

\[
K\left(\frac{s(n, \tilde{z})w_n(z)}{\|s(n, \tilde{z})w_n(z)\|_{H^1}}\right) = J\left(t_n \frac{s(n, \tilde{z})w_n(z)}{\|s(n, \tilde{z})w_n(z)\|_{H^1}}\right),
\] (4.32)

where \(t_n = \|s(n, \tilde{z})w_n(z)\|_{H^1}\). By Lemma 3.6, there is \(n_0 \in \mathbb{N}\) such that \(J(s(n, \tilde{z})w_n) \leq \sup_{t \geq 0} J(tw_n) < \alpha^\infty\) for each \(n \geq n_0\). Thus, the conclusion holds. \(\Box\)

Applying Lemma 4.4, we obtain

\[
\int_{\mathbb{R}^N} \frac{z}{|z|}\left(\nabla u^2 + u^2\right)dz \neq 0 \quad \text{for any} \quad u \in [K \in \alpha^\infty].
\] (4.33)

Now, we define

\[
G : [K < \alpha^\infty] \to S^{N-1}
\] (4.34)

by

\[
G(u) = \frac{\int_{\mathbb{R}^N}(z/|z|)\left(\nabla u^2 + u^2\right)dz}{\left|\int_{\mathbb{R}^N}(z/|z|)\left(\nabla u^2 + u^2\right)dz\right|}.
\] (4.35)

**Lemma 4.9.** For each \(n \geq n_0\), the map

\[
G \circ F_n : S^{N-1} \to S^{N-1}
\] (4.36)

is homotopic to the identity.

**Proof.** Define

\[
\zeta_n(\theta, \tilde{z}) : [0, 1] \times S^{N-1} \to S^{N-1}
\] (4.37)
by

\[ \xi_n(\theta, \bar{z}) = \begin{cases} 
G \left( \frac{(1 - 2\theta) s(n, \bar{z}) \psi_R w(z - n\bar{z}) + 2\theta \psi_R w(z - n\bar{z})}{\| (1 - 2\theta) s(n, \bar{z}) \psi_R w(z - n\bar{z}) + 2\theta \psi_R w(z - n\bar{z}) \|_{H^1}} \right) & \text{for } \theta \in \left[ 0, \frac{1}{2} \right), \\
G \left( \frac{\psi_R w(z - (n/2(1 - \theta))\bar{z})}{\| \psi_R w(z - (n/2(1 - \theta))\bar{z}) \|_{H^1}} \right) & \text{for } \theta \in \left[ \frac{1}{2}, 1 \right), \\
\bar{z} & \text{for } \theta = 1. 
\end{cases} \]

(4.38)

We need to show that \( \lim_{\theta \to 1^-} \xi_n(\theta, \bar{z}) = \bar{z} \) and

\[ \lim_{\theta \to 1/2} \xi_n(\theta, \bar{z}) = G \left( \frac{\psi_R w(z - n\bar{z})}{\| \psi_R w(z - n\bar{z}) \|_{H^1}} \right). \]  

(4.39)

(a) \( \lim_{\theta \to 1^-} \xi_n(\theta, \bar{z}) = \bar{z} \) : for \( 1/2 < \theta < 1 \), since

\[
\int_{\mathbb{R}^N} \frac{z}{|z|} \left( \left| \nabla \left[ \psi_R w \left( z - \frac{n}{2(1 - \theta)} \bar{z} \right) \right] \right|^2 + \psi_R^2 w \left( z - \frac{n}{2(1 - \theta)} \bar{z} \right)^2 \right) dz \\
= \int_{\mathbb{R}^N} \frac{z + (n/2(1 - \theta))\bar{z}}{|z + (n/2(1 - \theta))\bar{z}|} \left( |\nabla w(z)|^2 + w(z)^2 \right) dz + o(1) \\
= \left( \frac{2p}{p - 2} \right) \alpha^\theta \bar{z} + o(1) \quad \text{as } \theta \to 1^-,
\]

and \( \| \psi_R w(z - (n/2(1 - \theta))\bar{z}) \|_{H^1}^2 = (2p/(p - 2))\alpha^\theta + o(1) \) as \( \theta \to 1^- \), then \( \lim_{\theta \to 1^-} \xi_n(\theta, \bar{z}) = \bar{z} \).

(b) By the continuity of \( G \), it is easy to check that

\[ \lim_{\theta \to 1/2} \xi_n(\theta, \bar{z}) = G \left( \frac{\psi_R w(z - n\bar{z})}{\| \psi_R w(z - n\bar{z}) \|_{H^1}} \right). \]  

(4.41)

Thus, \( \xi_n(\theta, \bar{z}) \in C([0, 1] \times S^{N-1}, S^{N-1}) \) and

\[ \xi_n(0, \bar{z}) = G(F_n(\bar{z})) \quad \forall \bar{z} \in S^{N-1}, \]

(4.42)

\[ \xi_n(1, \bar{z}) = \bar{z} \quad \forall \bar{z} \in S^{N-1}, \]

provided \( n \geq n_0 \). This completes the proof.\( \Box \)
Theorem 4.10. Assume that \( q \) is a positive continuous function in \( \mathbb{R}^N \) and satisfies (q1) and (q2). Then, \( J(u) \) has at least two critical points in

\[ [K < \alpha^\infty], \]  

and there exists at least two positive solutions of (1.1) in \( \Omega \).

Proof. Applying Lemmas 4.7 and 4.9, we have for \( n \geq n_0 \)

\[ \text{cat}([K \leq \alpha^\infty - \sigma_n]) \geq 2. \]  

Next, we need to show that \( K \) satisfies the (PS)\( \beta \)-condition for \( 0 < \beta \leq \alpha^\infty - \sigma_n \). Let \( \{u_n\} \subset \Sigma \) satisfy \( K(u_n) = \beta + o_n(1) \) and

\[ \|K'(u_n)\|_{J^{-1}(\beta)} = \sup \{ \langle K'(u_n), \varphi \rangle | \varphi \in T_{u_n} \Sigma \text{ and } \|\varphi\|_{H^1} = 1 \} = o_n(1) \text{ as } n \to \infty. \]  

Since \( K(u_n) = J(s_n u_n) = \beta + o_n(1) \) as \( n \to \infty \) and \( s_n u_n \in M(\Omega) \), then

\[ s_n^2 = \frac{2p}{p - 2} \beta + o_n(1). \]  

Using (4.9) and \( \langle J'(s_n u_n), u_n \rangle = 0 \) to obtain that

\[ \|J'(s_n u_n)\|_{H^{-1}} = o_n(1) \text{ as } n \to \infty. \]  

Hence, \( \{s_n u_n\} \subset M(\Omega) \) is a (PS)\( \beta \)-sequence for \( J \). By Lemma 3.3(ii), \( K \) satisfies the (PS)\( \beta \)-condition for \( 0 < \beta \leq \alpha^\infty - \sigma_n \). Now, we apply Lemma 4.6 to get that \( K \) has at least two critical points in \( [K < \alpha^\infty] \). Moreover, by Lemmas 4.3(ii) and 2.2, there are at least two positive solutions of (1.1) in \( \Omega \).

Recall that there exist a unique \( s_u > 0 \) and a unique \( s_u^\infty > 0 \) such that \( s_u u \in M(\Omega) \) and \( s_u^\infty u \in M^\infty(\Omega) \). Then, we have the following results.

Lemma 4.11. For each \( u \in \Sigma \), we have that

\[ \left( \frac{p - m}{p - 2} \right) J^\infty(s_u^\infty u) \leq J(s_u u) \leq J^\infty(s_u^\infty u), \text{ where } m > 2. \]  


Lemma 4.13. Let $\alpha \geq q(\infty) \geq q$, where $m > 2$, we obtain that for each $u \in \Sigma$ and

\[ J(s_u u) \leq J^\infty(s_u u) \leq \sup_{s \geq 0} J^\infty(s_u u) = J^\infty(s_u^\infty u), \]

\[ J(s_u u) = \sup_{s \geq 0} J(su) \geq J(s_u^\infty u) = \frac{1}{2} \|s_u^\infty u\|^{2}_{H^1} - \frac{1}{p} \int_{\Omega} q(z)(s_u^\infty u_s)^p dz \]

\[ \geq \frac{1}{2} \int_{\Omega} q_{\infty}(s_u^\infty u_s)^p dz - \frac{1}{p} \int_{\Omega} \frac{m}{2} q_{\infty}(s_u^\infty u_s)^p dz \]

\[ = \left( \frac{1}{2} - \frac{m}{2p} \right) \int_{\Omega} q_{\infty}(s_u^\infty u_s)^p dz = \left( \frac{p - m}{p - 2} \right) J^\infty(s_u^\infty u). \]

\[ \square \]

Let

\[ K(u) = \max_{s \geq 0} J(su) = J(s_u u) > 0, \]

\[ K^\infty(u) = \max_{s \geq 0} J^\infty(s_u u) = J(s_u^\infty u) > 0, \]

where $s_u u \in M(\Omega)$ and $s_u^\infty u \in M^\infty(\Omega)$. Bahri-Li’s minimax argument [4] also works for $K$.

Let

\[ \Gamma = \left\{ g \in C\left( B_r(0), \Sigma \right) \left| g|_{\partial B_r(0)} = \frac{q_R(z)w(z - y)}{\|q_R(z)w(z - y)\|_{H^1}} \right. \right\} \text{ for large } r = |y|. \]

Then, we define

\[ \gamma(\Omega) = \inf_{g \in \Gamma} \sup_{y \in B_r(0)} K(g(y)), \]

\[ \gamma^\infty(\Omega) = \inf_{g \in \Gamma} \sup_{y \in B_r(0)} K^\infty(g(y)). \]

Lemma 4.12. $\alpha^\infty < \gamma^\infty(\Omega) < 2\alpha^\infty$.

Proof. Bahri and Li [4] proved that (1.2) admits at least one positive solution $u$ in $\Omega$ and $J^\infty(u) = \gamma^\infty(\Omega) < 2\alpha^\infty$. Lien et al. [17] proved that (1.2) does not have any positive ground state solution in $\Omega$ and $\alpha^\infty(\Omega) = a^\infty(\mathbb{R}^N) = \alpha^\infty$. Hence, $\alpha^\infty < \gamma^\infty(\Omega) < 2\alpha^\infty$. \( \square \)

The following minimax lemma is given in Shi [24] to unify the mountain pass lemma of Ambrosetti and Rabinowitz [25] and the saddle point theorem of Rabinowitz [26].

Lemma 4.13. Let $V$ be a compact metric space, $V_0 \subset V$ a closed set, $X$ a Banach space, $\chi \in C(V_0, X)$ and let us define the complete metric space $M$ by

\[ M = \left\{ g \in C(V, X) \mid g(s) = \chi(s) \text{ if } s \in V_0 \right\} \]

(4.53)
Boundary Value Problems

with the usual distance $d$. Let $\varphi \in C^1(X, \mathbb{R})$ and let us define

$$c = \inf_{g \in M} \max_{s \in V} \varphi(g(s)), \quad c_1 = \max_{\lambda \in \mathbb{R}} \varphi.$$  \hfill (4.54)

If $c > c_1$, then for each $\varepsilon > 0$ and each $g \in M$ such that

$$\max_{s \in V} \varphi(g(s)) \leq c + \varepsilon,$$

there exists $v \in X$ such that

$$c - \varepsilon \leq \varphi(v) \leq \max_{s \in V} \varphi(g(s)),$$

$$\text{dist}(v, g(V)) \leq \varepsilon^{1/2},$$

$$\|\varphi'(v)\| \leq \varepsilon^{1/2}.$$ \hfill (4.56)

Lemma 4.14. Assume that $q$ is a positive continuous function in $\mathbb{R}^N$. If $q$ satisfies (q1) and (q2). Let $\{u_n\} \subset M(\Omega)$ be a $(PS)_\beta$-sequence in $H_0^1(\Omega)$ for $J$ with $\alpha^\infty < \beta < \alpha^{\infty} + \alpha(\Omega)$. Then, there exist a subsequence $\{u_n\}$ and a nonzero $u_0 \in H_0^1(\Omega)$ such that $u_n \to u_0$ strongly in $H_0^1(\Omega)$, that is, $J$ satisfies the $(PS)_\beta$-condition in $H_0^1(\Omega)$. Moreover, $u_0$ is a positive solution of (1.1) such that $J(u_0) = \beta$.

Proof. The proof is similar to Lemma 3.3(ii). Applying Palais-Smale Decomposition Lemma 3.1, we get

$$\alpha^\infty + \alpha(\Omega) > \beta = J(u_n) \geq l\alpha^\infty + \alpha(\Omega) \quad \text{(or } \geq l\alpha^{\infty}).$$ \hfill (4.57)

Since $w$ is the unique (up to translation), positive solution of (1.2) in $\mathbb{R}^N$ and $J^\infty(w) = \alpha^\infty > \alpha(\Omega)$, then $l = 0$ and $u_0 \neq 0$. Hence, $u_n \to u_0$ strongly in $H_0^1(\Omega)$ and $J(u_0) = \beta$. Moreover, by Lemma 2.2, $u_0$ is positive in $\Omega$. \hfill \Box

Theorem 4.15. Assume that $q$ is a positive continuous function in $\mathbb{R}^N$. If $q$ satisfies (q1) and there exists a number $m' > 2$ such that for any $2 < m \leq m'$,

$$\frac{m}{2} q_{\infty} \geq q(z) \geq q_{\infty} + C \exp(-\delta|z|), \quad \text{where } 0 < C \leq \frac{m-2}{2} q_{\infty} \text{ and } 0 < \delta < 2,$$ \hfill (q_{2}')

then (1.1) admits at least three positive solutions in $\Omega$.

Proof. Applying Lemma 4.11(iii) to obtain

$$\left(\frac{p-m}{p-2}\right)^{\alpha^\infty} \leq \alpha(\Omega) \leq \alpha^{\infty},$$

$$\left(\frac{p-m}{p-2}\right)^{\gamma^\infty(\Omega)} \leq \gamma(\Omega) \leq \gamma^{\infty}(\Omega).$$ \hfill (4.58)
Since \( \alpha^\infty < \gamma^\infty(\Omega) < 2\alpha^\infty \), given \( 0 < \varepsilon < (2\alpha^\infty - \gamma^\infty(\Omega)) / 2 \), there is a number \( \min\{m_1, p\} \geq m_2 > 2 \) such that for any \( 2 < m \leq m_2 \), we have

\[
\gamma^\infty(\Omega) < \alpha^\infty + \alpha(\Omega) \leq 2\alpha^\infty. \tag{4.59}
\]

Choosing some \( \min\{m_2, p\} \geq m' > 2 \) such that for any \( 2 < m \leq m' \), we get

\[
\alpha^\infty < \gamma(\Omega) \leq \gamma^\infty(\Omega) < \alpha^\infty + \alpha(\Omega) \leq 2\alpha^\infty. \tag{4.60}
\]

By Lemma 3.6, for any \( t \geq 0 \), we have

\[
J(t\psi_R(z)(z - y)) \leq \alpha^\infty + o(1) \quad \text{as} \quad |y| \to \infty. \tag{4.61}
\]

Then,

\[
K\left(\frac{\psi_R(z)(z - y)}{\|\psi_R(z)(z - y)\|_{H^1}}\right) = f\left(\frac{t\psi_R(z)(z - y)}{\|\psi_R(z)(z - y)\|_{H^1}}\right)
\leq \alpha^\infty + o(1) \quad \text{as} \quad |y| \to \infty,
\]

that is, \( \gamma(\Omega) > K(\psi_R(z)(z - y)/\|\psi_R(z)(z - y)\|_{H^1}) \) for large \( r = |y| \). Applying Lemma 4.3 and the minimax Lemma 4.13 to obtain that \( \gamma(\Omega) \) is a (PS)-value in \( H^1_0(\Omega) \) for \( J \). Hence, by Lemmas 2.2 and 4.14, we have that there exists a positive solution \( u \) of (1.1) in \( \Omega \) such that \( J(u) = \gamma(\Omega) \). From the result of Theorem 4.10, (1.1) admits at least three positive solutions in \( \Omega \).

\[\square\]

References


