In this paper we are interested in the existence of a global attractor for the equation

$$u_t - \Delta_m u + \lambda |u|^{m-2} u + f(x,u) = g(x), \quad x \in \mathbb{R}^N, \quad t \in \mathbb{R}^+,$$

(1.1)

with initial data condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^N,$$

(1.2)

where the m-Laplace operator $\Delta_m u = \text{div}(|\nabla u|^{m-2} \nabla u)$, $2 \leq m < N$, $\lambda > 0$.

For the case $m = 2$, the existence of global $(L^2(\mathbb{R}^N), L^2(\mathbb{R}^N))$-attractor for (1.1)-(1.2) is proved by Wang in [1] under appropriate assumptions on $f$ and $g$. Recently, Khanmamedov [2] studied the existence of global $(L^2(\mathbb{R}^N), L^m(\mathbb{R}^N))$-attractor for (1.1)-(1.2) with $m^* = mN/(N-m)$. Yang et al. in [3] investigated the global $(L^2(\mathbb{R}^N), L^p(\mathbb{R}^N) \cap W^{1,m}(\mathbb{R}^N))$-attractor for (1.1)-(1.2) with $m^* = m^N/(N-m)$. In this paper, we prove the existence of a global $(L^p(\mathbb{R}^N), L^m(\mathbb{R}^N) \cap W^{1,m}(\mathbb{R}^N))$-attractor for any $p > m$. We prove the existence of the global $(L^p(\mathbb{R}^N), L^m(\mathbb{R}^N))$-attractor for any $p > m$.

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1. Introduction

In this paper we are interested in the existence of a global $(L^2(\mathbb{R}^N), L^p(\mathbb{R}^N))$-attractor for the $m$-Laplacian equation

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1. Introduction

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\( \mathcal{A}_p \) under the assumptions \( f(x,u)u \geq a_1|u|^p - a_2|u|^m - a_3(x) \) and \( f_u(x,u) \geq a_4(x) \) with the constants \( a_1, a_2 > 0 \) and the functions \( a_3, a_4 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \). We note that the global attractor \( \mathcal{A}_p \) in [3] is related to the \( p \)-order polynomial of \( u \) on \( f(x,u) \). In [4], we consider the existence of global \((L^2(\mathbb{R}^N), L^p(\mathbb{R}^N))\)-attractor for (1.1)-(1.2), which the term \( \lambda|u|^{m-2}u \) is replaced by \( \lambda u \). We derive \( L^\infty \) estimate of solutions by Moser’s technique as in [5–7], and due to this, we need not to make the assumption like \( f_u(x,u) \geq a_4(x) \) to show the uniqueness. For a typical example is \( f(x,u) = a(x)|u|^{a-2}u - h(x)|u|^{p-2}u \) with \( a(x) \geq h(x) \geq 0, \alpha > \beta \geq 2, \) \( h(x) \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \). In [4], we assume that \( f(x,u) \) satisfies

\[
0 \leq \int_0^u f(x,\eta)d\eta + L(x)|u| \leq k_2(f(x,u)u + L(x)|u|) \tag{1.3}
\]

with some \( k_2 > 0 \) and \( L(x) \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \).

Obviously, the nonlinear function \( f(x,u) = -h(x)|u|^{q-2}u \) with \( h(x) \geq 0, q \geq 1 \) does not satisfy the assumption (1.3).

In this paper, motivated by [2–4], we are interested in the global \((L^2(\mathbb{R}^N), L^p(\mathbb{R}^N))\)-attractor \( \mathcal{A}_p \) for the problem (1.1)-(1.2) with any \( p > m \), in which \( p \) is independent of the order of polynomial for \( u \) on \( f(x,u) \).

Our assumptions on \( f(x,u) \) is different from that in [2–4]. To obtain the continuity of solution of (1.1)-(1.2) in \( L^p(\mathbb{R}^N) \), \( p \geq 2 \), we derive \( L^\infty \) estimate of solutions by Moser’s technique as in [4, 6, 7]. We will prove that the existence of the global attractor \( \mathcal{A}_p \) in \( L^p(\mathbb{R}^N) \) under weaker conditions.

The paper is organized as follows. In Section 2, we derive some estimates and prove some lemmas for the solution of (1.1)-(1.2). By the a priori estimates in Section 2, the existence of global \((L^2(\mathbb{R}^N), L^p(\mathbb{R}^N))\)-attractor for (1.1)-(1.2) is established in Section 3.

## 2. Preliminaries

We denote by \( L^p \) and \( W^{1,m} \) the space \( L^p(\mathbb{R}^N) \) and \( W^{1,m}(\mathbb{R}^N) \), and the relevant norms by \( \| \cdot \|_p \) and \( \| \cdot \|_{1,m} \), respectively. It is well known that \( W^{1,m}(\mathbb{R}^N) = W^{1,m}_0(\mathbb{R}^N) \). In general, \( \| \cdot \|_E \) denotes the norm of the Banach space \( E \).

For the proof of our results, we will use the following lemmas.

**Lemma 2.1** ([8–10] (Gagliardo-Nirenberg)). Let \( \beta \geq 0, 1 \leq r \leq q \leq m(1 + \beta)N/(N - m) \) when \( N > m \) and \( 1 \leq r \leq q \leq \infty \) when \( N \leq m \). Suppose \( u \in L^r \) and \( |u|^p u \in W^{1,m} \). Then there exists \( C_0 \) such that

\[
\|u\|_q \leq C_0^{1/(\beta + 1)}\|u\|^{1-\theta}_r \left\| \nabla (|u|^p u) \right\|^{\theta/(\beta + 1)}_m \tag{2.1}
\]

with \( \theta = (1 + \beta)(r^{-1} - q^{-1})/(N - m - 1 + (1 + \beta)r^{-1}), \) where \( C_0 \) is a constant independent of \( q, r, \beta, \) and \( \theta \) if \( N \neq m \) and a constant depending on \( q/(1 + \beta) \) if \( N = m \).

**Lemma 2.2** ([7]). Let \( y(t) \) be a nonnegative differentiable function on \( (0,T] \) satisfying

\[
y'(t) + At^{\beta - 1}y^{1+\theta}(t) \leq Br^{-k}y(t) + Cr^{-\delta}, \quad 0 < t \leq T, \tag{2.2}
\]
with \( A, \theta > 0, \lambda \theta \geq 1, B, C \geq 0, k \leq 1, \) and \( 0 \leq \delta < 1. \) Then one has

\[
y(t) \leq A^{-1/\theta} \left( 2 \lambda + 2BT^{1-k} \right)^{1/\theta} + 2C \left( \lambda + BT^{1-k} \right)^{-1} t^{1-\delta}, \quad 0 < t \leq T. \tag{2.3}
\]

**Lemma 2.3** ([11]). Let \( y(t) \) be a nonnegative differential function on \((0, \infty)\) satisfying

\[
y'(t) + Ay^{1+\mu}(t) \leq B, \quad t > 0
\]

with \( A, \mu > 0, B \geq 0. \) Then one has

\[
y(t) \leq \left( BA^{-1} \right)^{1/(1+\mu)} + (A\mu t)^{-1/\mu}, \quad t > 0.
\]

First, the following assumptions are listed.

**\( \mathbf{A}_1 \)** Let \( f(x, u) \in C^1 (R^{N+1}) \), \( f(x, 0) = 0 \) and there exist the nontrivial nonnegative functions \( h(x) \in L^\infty \cap L^\infty \) and \( h_1(x) \in L^1 \), such that \( F(x, u) \leq k_1 f(x, u)u \) and

\[
-h(x)|u|^q \leq f(x, u)u \leq h(x)|u|^q + h_1(x),
\]

\[
(f(x, u) - f(x, v))(u - v) \geq -k_2 \left( 1 + |u|^{q-2} + |v|^{q-2} \right)|u - v|^2,
\]

where \( F(x, u) = \int_0^u f(x, s)ds, 2 \leq q < m, q_1 = m/(m-q) \) and some constants \( k_1, k_2 \geq 0. \)

**\( \mathbf{A}_2 \)** Let \( f(x, u) \in C^1 (R^{N+1}) \), \( f(x, 0) = 0 \) and there exists the nontrivial nonnegative function \( h_1(x) \in L^1 \), such that \( F(x, u) \leq k_1 f(x, u)u \) and

\[
a_1|u|^\alpha - a_2|u|^m \leq f(x, u)u \leq b_1|u|^\alpha + b_2|u|^m + h_1(x),
\]

\[
(f(x, u) - f(x, v))(u - v) \geq -k_4 \left( 1 + |u|^{\alpha-2} + |v|^{\alpha-2} \right)|u - v|^2,
\]

where \( a_2 < \lambda, m < \alpha < m + 2m/N, \) and \( a_1, b_1, b_2 > 0, k_1, k_2 \geq 0. \)

A typical example is \( f(x, u) = a(x)|u|^{\alpha-2}u - h(x)|u|^{\beta-2}u \) with \( a(x), h(x) \geq 0, \) and \( \alpha > \beta \geq m. \)

The assumption \( \mathbf{A}_2 \) is similar to \([3, (1.3) - (1.7)].\)

**Remark 2.4.** If \( f(x, u) = -h(x)|u|^{q-2}u, q > m, \) the problem \((1.1)-(1.2)\) has no nontrivial solution for some \( h(x) \geq 0, \) see [12].

We first establish the following theorem.

**Theorem 2.5.** Let \( g \in L^m \cap L^\infty \) and \( u_0 \in L^2. \) If \( \mathbf{A}_1 \) holds, then the problem \((1.1)-(1.2)\) admits a unique solution \( u(t) \) satisfying

\[
u(t) \in X = C \left( [0, \infty), L^2 \right) \cap L^m \left( [0, \infty), W^{1,m} \right) \cap L^\infty \left( [0, \infty), L^2 \right), \]

\[
u \in L^m \left( [0, \infty), W^{-1,m} \right).
\]
and the following estimates:

\[ \|u(t)\|_2^2 \leq C_0 \left( \|g\|^{m'}_m + \|h\|^q_{q_1} \right) t + \|u_0\|_2^2, \quad t \geq 0, \]  
(2.10)

\[ \|\nabla u(t)\|_m^m + \lambda \|u(t)\|_m^m \leq C_0 \left( \|g\|^{m'}_m + \|h\|^q_{q_1} + \|h_1\|_1 \right) + t^{-1}\|u_0\|_2^2, \quad t > 0, \]  
(2.11)

\[ \int_s^t \|u_t(\tau)\|_2^2 d\tau \leq C_0 \left( \|g\|^{m'}_m + \|h\|^q_{q_1} + \|h_1\|_1 \right) + s^{-1}\|u_0\|_2^2, \quad 0 < s \leq t, \]  
(2.12)

\[ \|u(t)\|_{\infty} \leq C_1 t^{-s_0}, \quad s_0 = N(2m + (m - 2)N)^{-1}, \quad 0 < t \leq T \]  
(2.13)

with \( m' = m/(m - 1) \). The constant \( C_0 \) depends only on \( m, q, \lambda \), and \( C_1 \) depends on \( h, g, u_0 \), and \( T \).

**Proof.** For any \( T > 0 \), the existence and uniqueness of solution \( u(t) \) for (1.1)-(1.2) in the class

\[ X_T \equiv C \left( [0, T], L^2 \right) \cap L^m \left( [0, T], W^{1, \infty} \right) \cap L^\infty \left( [0, T], L^2 \right) \]  
(2.14)

can be obtained by the standard Faedo-Galerkin method, see, for example, [10, Theorem 7.1, page 232], or by the pseudomonotone operator method in [2]. Further, we extend the solution \( u(t) \) for all \( t \geq 0 \) by continuity and bounded over \( L^2 \) such that \( u(t) \in X \).

In the following, we will derive the estimates (2.10)–(2.13). The solution is in fact given as limits of smooth solutions of approximate equations (see [5, 6]), we may assume for our estimates that the solutions under consideration are appropriately smooth. We begin with the estimate of \( \|u(t)\|_2 \).

We multiply (1.1) by \( u \) and integrate by parts to get

\[ \frac{1}{2} \frac{d}{dt} \|u(t)\|_2^2 + \|\nabla u(t)\|_m^m + \lambda \|u(t)\|_m^m = \int_{\mathbb{R}^N} (g(x) - f(x, u)) u \, dx. \]  
(2.15)

Since

\[ \int_{\mathbb{R}^N} f(x, u(t)) u(t) \, dx \leq \int_{\mathbb{R}^N} h(x)|u(t)|^q \, dx \leq \lambda_0 \|u(t)\|_m^m + C_0 \|h\|_{q_1}^q, \]  
(2.16)

with \( \lambda_0 = \lambda/4 \). We have from (2.15) that

\[ \frac{1}{2} \frac{d}{dt} \|u(t)\|_2^2 + \|\nabla u(t)\|_m^m + 2\lambda_0 \|u(t)\|_m^m \leq C_0 \left( \|g\|^{m'}_m + \|h\|^q_{q_1} \right). \]  
(2.17)

Integrating (2.17) with respect to \( t \), we obtain

\[ \frac{1}{2} \|u(t)\|_2^2 + \int_0^t \left( \|\nabla u(\tau)\|_m^m + 2\lambda_0 \|u(\tau)\|_m^m \right) d\tau \leq C_0 \left( \|g\|^{m'}_m + \|h\|^q_{q_1} \right) t + \frac{1}{2} \|u_0\|_2^2. \]  
(2.18)
This implies (2.10) and the existence of $t^* \in (0, t)$ such that

$$\| \nabla u(t^*) \|_m^m + 2\lambda_0 \| u(t^*) \|_m^m \leq C_0 \left( \| g \|_{m'}^m + \| h \|_{q_1}^q \right) + t^{-1} \| u_0 \|_2^2, \quad t > 0. \quad (2.19)$$

On the other hand, multiplying (1.1) by $u_t$ and integrating on $(s, t) \times \mathbb{R}^N$, we get

$$\int_s^t \| u_t(\tau) \|_2^2 d\tau + \frac{1}{m} \| \nabla u(t) \|_m^m + \frac{1}{m} \| u(t) \|_m^m + \int_{\mathbb{R}^N} (F(x, u(t)) - g(x)u(t)) dx$$

$$= \frac{1}{m} \| \nabla u(s) \|_m^m + \frac{1}{m} \| u(s) \|_m^m + \int_{\mathbb{R}^N} (F(x, u(s)) - g(x)u(s)) dx. \quad (2.20)$$

By (2.6), we have $F(x, u) \geq -h(x)|u|^q$ and

$$-\int_{\mathbb{R}^N} F(x, u(t)) dx \leq \int_{\mathbb{R}^N} h(x)|u(t)|^q dx \leq \varepsilon \| u(t) \|_m^m + C_0 \| h \|_{q_1}^q \quad (2.21)$$

with $0 < \varepsilon \leq \lambda/2m$. Similarly, we have the following estimates by Young’s inequality:

$$\int_{\mathbb{R}^N} |g(x)u(t)| dx \leq \varepsilon \| u(t) \|_m^m + C_0 \| g \|_{m'}^m,$$

$$\int_{\mathbb{R}^N} |g(x)u(S)| dx \leq \| u(S) \|_m^m + \| g \|_{m'}^m,$$

$$\int_{\mathbb{R}^N} F(x, u(s)) dx \leq k_1 \int_{\mathbb{R}^N} (h(x)|u(s)|^q + h_1(x)) dx$$

$$\leq C_0 \left( \| u(s) \|_m^m + \| h \|_{q_1}^q + \| h_1 \|_1 \right). \quad (2.22)$$

Then, we have from (2.20) that

$$\int_s^t \| u_t(\tau) \|_2^2 d\tau + \frac{1}{m} \| \nabla u(t) \|_m^m + \frac{1}{2m} \| u(t) \|_m^m \leq C_0 \left( \| \nabla u(s) \|_m^m + \| u(s) \|_m^m + M_1 \right), \quad (2.23)$$

where

$$M_1 = \| g \|_{m'}^m + \| h \|_{q_1}^q + \| h_1 \|_1. \quad (2.24)$$

Further, we let $s = t^*$ in (2.23) and obtain from (2.19) that

$$\| \nabla u(t) \|_m^m + \lambda \| u(t) \|_m^m \leq C_0 \left( M_1 + t^{-1} \| u_0 \|_2^2 \right), \quad t > 0,$$

$$\int_s^t \| u_t(\tau) \|_2^2 d\tau \leq C_0 \left( M_1 + s^{-1} \| u_0 \|_2^2 \right), \quad 0 < s < t. \quad (2.25)$$
Thus, the solution \( u(t) \) satisfies (2.10)-(2.12). We now derive (2.13) by Moser’s technique as in [5, 6]. In the sequel, we will write \( u^p \) instead of \( |u|^{p-1}u \) when \( p \geq 1 \). Also, let \( C \) and \( C_j \) be the generic constants independent of \( p \) changeable from line to line.

Multiplying (1.1) by \( |u|^{p-2}u \), \( p \geq 2 \), we get

\[
\frac{1}{p} \frac{d}{dt} \|u(t)\|_p^p + C_1 p^{-1} \| \nabla u^{(p+m-2)/m} \|_m^m + \lambda \|u(t)\|^{p+m-2}_{p+m-2} \leq \int_{\mathbb{R}^N} (g(x) - f(x, u)) |u|^{p-2}u \, dx.
\]  

(2.26)

It follows from Young’s inequality that

\[
\int_{\mathbb{R}^N} |g(x)||u|^{p-1} \, dx \leq \lambda_0 \|u\|_{p+m-2}^{p+m-2} + \lambda_0^{(1-p)/(m-1)} \|g\|_{\alpha_p}^{\alpha_p} \\
- \int_{\mathbb{R}^N} f(x, u)|u|^{p-2}u \, dx \leq \lambda_0 \|u\|_{p+m-2}^{p+m-2} + \lambda_0^{(2-p-q)/(m-q)} \|h\|_{\beta_p}^{\beta_p}
\]

(2.27)

with \( \lambda_0 = \lambda/4, \alpha_p = (p + m - 2)/(m - 1), \beta_p = (p + m - 2)/(m - q) \). Then, (2.26) becomes

\[
\frac{1}{p} \frac{d}{dt} \|u(t)\|_p^p + C_1 p^{-1} \| \nabla u^{(p+m-2)/m} \|_m^m + 2\lambda_0 \|u(t)\|^{p+m-2}_{p+m-2} \\
\leq \lambda_0^{(1-p)/(m-1)} \|g\|_{\alpha_p}^{\alpha_p} + \lambda_0^{(2-p-q)/(m-q)} \|h\|_{\beta_p}^{\beta_p}.
\]  

(2.28)

Let \( R > m/2, p_1 = 2, p_n = Rp_{n-1} - (m - 2), n = 2, 3, \ldots \). Then, by Lemma 2.1, we see

\[
\| \nabla u^{(p_n+m-2)/m} \|_m^m \geq C_0^{-m/\theta_n} \|u\|_{p_n}^{(p_n+m-2)(1-\theta_n)} \|u\|_{p_n}^{(p_n+m-2)\theta_n},
\]

(2.29)

where

\[
\theta_n = \frac{p_n + m - 2}{m} \left( \frac{1}{p_{n-1}} - \frac{1}{p_n} \right) \left( \frac{1}{N} - \frac{1}{m} + \frac{p_n + m - 2}{mp_{n-1}} \right)^{-1} = \frac{NR(1 - p_{n-1} p_n^{-1})}{m + N(R - 1)}.
\]

Inserting (2.29) into (2.28) \( p = p_n \), we find

\[
\frac{d}{dt} \|u(t)\|_{p_n}^p + C_1 C_0^{-m/\theta_n} p_n^{-2-m} \|u\|_{p_n}^{p_n + r_n} \|u\|_{p_n^{-1}}^{m-2-r_n} \leq p_n A_n,
\]

(2.31)

where \( r_n = (p_n + m - 2)\theta_n^{-1} - p_n \) and

\[
A_n = \lambda_0^{(2-p_n-q)/(m-q)} \|h\|_{\beta_n}^{\beta_n} + \lambda_0^{(1-p_n)/(m-1)} \|g\|_{\alpha_n}^{\alpha_n}
\]

(2.32)

with \( \lambda_n = (p_n + m - 2)/(m - 1), \mu_n = (p_n + m - 2)/(m - q), n = 1, 2, \ldots \).
We claim that there exist the bounded sequences \{\xi_n\} and \{s_n\} such that
\[\|u(t)\|_{p_n} \leq \xi_n t^{-s_n}, \quad 0 < t \leq T.\] (2.33)

Indeed, by (2.10), this holds for \(n = 1\) if we take \(s_1 = 0\), \(\xi_1 = M_1 T^{1/2} + \|u_0\|_2\). If (2.33) is true for \(n - 1\), then we have from (2.31) that
\[y'(t) + At^{\alpha-1}y^{1+\theta}(t) \leq p_n A_{n}, \quad 0 < t \leq T,\] (2.34)
where \(y(t) = \|u(t)\|^{p_n}_{p_n}, \tau_n = s_n p_n\) and
\[\theta = r_n p_n^{-1}, \quad s_n = (1 + s_{n-1}(r_n - m + 2)) r_n^{-1}, \quad A = C_1 C_0^{-m/\theta_n} p_n^{-2} t^{-m-2-r_n}.\] (2.35)

Applying Lemma 2.2 to (2.34), we have (2.33) for \(n\) with
\[\xi_n = \xi_{n-1} \left( C_1 C_0^{-m/\theta_n} p_n^{-m-1} s_n^{-1} \right)^{1/r_n} + \left( 2 A_n s_n^{-1} \right)^{1/p_n} T^{1+s_n}\] (2.36)
for \(n = 2, 3, \ldots\).

It is not difficult to show that \(s_n \to s_0 = N(2m + (m-2)N)^{-1}\), as \(n \to \infty\) and \(\{\xi_n\}\) is bounded, see [6]. Then, (2.13) follows from (2.33) as \(n \to \infty\).

We now consider the uniqueness and continuity of the solution for (1.1)-(1.2) in \(L^2\). Let \(u_1, u_2\) be two solutions of (1.1)-(1.2), which satisfy (2.10)-(2.13). Denote \(u(t) = u_1(t) - u_2(t)\). Then \(u(t)\) solves
\[u_t - (\Delta_m u_1 - \Delta_m u_2) + \lambda \left( |u_1|^{m-2} u_1 - |u_2|^{m-2} u_2 \right) = f(x, u_2) - f(x, u_1).\] (2.37)

Multiplying (2.37) by \(u\), we get from (2.7) and (2.13) that
\[\frac{1}{2} \frac{d}{dt} \|u(t)\|^2_{L^2} + \gamma_0 \|\nabla u(t)\|^2_{L^m} + \gamma_1 \|u(t)\|^m_{L^m} \leq k_2 \int_{\mathbb{R}^N} \left( 1 + |u_1|^{q-2} + |u_2|^{q-2} \right) u^2 dx\]
\[\leq k_2 \int_{\mathbb{R}^N} \left( 1 + \|u_1(t)\|^{q-2}_{L^\infty} + \|u_2(t)\|^{q-2}_{L^\infty} \right) u^2 dx \leq C_0 \left( 1 + t^{-\delta_0(q-2)} \right) \|u(t)\|^2_{L^2}\] (2.38)

with some \(\gamma_0, \gamma_1 > 0\). Since \(s_0(q-2) < 1\) and \(u(0) = 0\), (2.38) implies that \(\|u(t)\|_{L^2} \equiv 0\) in \([0, T]\) and \(u_1(t) = u_2(t)\) in \([0, T]\).

Further, let \(t > s \geq 0\). Note that
\[\|u(t) - u(s)\|^2_{L^2} = \int_{\mathbb{R}^N} \left( \int_s^t u_1(\tau) d\tau \right)^2 dx \leq \int_s^t \|u_1(\tau)\|^2_{L^2}(t - s).\] (2.39)
This shows that \(\|u(t) - u(s)\|^2_{L^2} \to 0\) as \(t \to s\) and \(u(t) \in C([0,T], L^2)\). Then the proof of Theorem 2.5 is completed. \(\square\)
Remark 2.6. By (2.23), we know that if \( u_0 \in W^{1,m} \), then
\[
\int_0^t \| u(t) \|_2^2 \, dt + \frac{1}{m} \| \nabla u(t) \|_m^m + \frac{1}{2m} \| u(t) \|_m^m \leq C_0 \| u_0 \|_{1,m}^m + M_1, \quad t \geq 0,
\] (2.40)
where \( M_1 \) is given in (2.24). Hence, we have

**Theorem 2.7.** Assume \((A_1)\) and \( g \in L^m \cap L^\infty \). Suppose also \( u_0(x) \in W^{1,m} \). Then, the unique solution \( u(t) \) in Theorem 2.5 also satisfies
\[
u(t) \in Y \equiv L^\infty \left( [0, +\infty), W^{1,m} \right), \quad u_i \in L^2 \left( [0, +\infty), L^2 \right),
\] (2.41)
and the estimate (2.40).

Now consider the assumption \((A_2)\). Since \( m < \alpha < m + 2m/N \), one has \( s_0(\alpha - 2) = N(\alpha - 2)/(2m + (m - 2)N) < 1 \). By a similar argument in the proof of Theorem 2.5, one can establish the following theorem.

**Theorem 2.8.** Assume \((A_2)\) and \( g \in L^m \cap L^\infty, u_0 \in L^2 \). Then the problem (1.1)-(1.2) admits a unique solution \( u(t) \) which satisfies
\[
u(t) \in X \equiv C \left( [0, \infty), L^2 \right) \cap L^m \left( [0, \infty), W^{1,m} \right) \cap L^\infty \left( [0, \infty), L^2 \right),
\] (2.42)
and the following estimates:
\[
\| u(t) \|_2^2 \leq C_0 t \| g \|_m^m + \| u_0 \|_2^2, \quad t \geq 0,
\]
\[
\| \nabla u(t) \|_m^m + \lambda \| u(t) \|_m^m + \| u(t) \|_a^a \leq C_0 \left( \| g \|_m^m + \| h_1 \|_1 \right) + t^{-1} \| u_0 \|_2^2, \quad t > 0,
\] (2.43)
\[
\int_s^t \| u(t) \|_2^2 \, dt \leq C_0 \left( \| g \|_m^m + \| h_1 \|_1 \right) + s^{-1} \| u_0 \|_2^2, \quad 0 < s \leq t,
\]
\[
\| u(t) \|_\infty \leq C_1 t^{-s_0}, \quad s_0 = N(2m + (m - 2)N)^{-1}, \quad 0 < t \leq T.
\]

Further, if \( u_0 \in W^{1,m} \), the unique solution \( u(t)(\in Y) \) satisfies
\[
\int_0^t \| u(t) \|_2^2 \, dt + \| \nabla u(t) \|_m^m + \| u(t) \|_m^m + \| u(t) \|_a^a \leq C_0 \left( \| u_0 \|_{1,m}^m + \| h_1 \|_1 + \| g \|_m^m \right),
\] (2.44)
where \( C_0 \) depends only on \( m, N, \lambda, \alpha, \) and \( C_1 \) on the given data \( g, h_1, u_0, \) and \( T > 0 \).

So, by Theorems 2.5–2.8, one obtains that the solution operator \( S(t)u_0 = u(t), t \geq 0 \) of the problem (1.1)-(1.2) generates a semigroup on \( L^2 \) or on \( W^{1,m} \), which has the following properties:
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(1) \(S(t) : L^2 \rightarrow L^2\) for \(t \geq 0\), and \(S(0)u_0 = u_0\) for \(u_0 \in L^2\) or \(S(t) : W^{1,m} \rightarrow W^{1,m}\) for \(t \geq 0\), and \(S(0)u_0 = u_0\) for \(u_0 \in W^{1,m}\);

(2) \(S(t + s) = S(t)S(s)\) for \(t, s \geq 0\);

(3) \(S(t)\theta \rightarrow S(s)\theta\) in \(L^2\) as \(t \rightarrow s\) for every \(\theta \in L^2\).

From Theorems 2.5–2.8, one has the following lemma.

**Lemma 2.9.** Suppose (A1) (or (A2)) and \(g \in L^m \cap L^{r_\infty}\). Let \(\mathcal{B}_0\) be a bounded subset of \(L^2\). Then, there exists \(T_0 = T_0(\mathcal{B}_0)\) such that \(S(t)\mathcal{B}_0 \subset \mathcal{D}\) for every \(t \geq T_0\), where

\[
\mathcal{D} = \left\{ u \in W^{1,m} \mid ||\nabla u||^m_m + \lambda ||u||^m_m \leq M_1 \right\}
\]

with \(M_1 = ||h||^0_m + ||h_1||_1 + ||g||^m_m\) if (A1) holds, and \(M_1 = ||h_1||_1 + ||g||^m_m\) if (A2) holds.

Now it is a position of Theorem 2.5 to establish some continuity of \(S(t)\) with respect to the initial data \(u_0\), which will be needed in the proof for the existence of attractor.

**Lemma 2.10.** Assume that all the assumptions in Theorem 2.5 are satisfied. Let \(S(t)\phi_n\) and \(S(t)\phi\) be the solutions of problem (1.1)–(1.2) with the initial data \(\phi_n\) and \(\phi\), respectively. If \(\phi_n \rightarrow \phi\) in \(L^p(p \geq 2)\) as \(n \rightarrow \infty\), then \(S(t)\phi_n\) uniformly converges to \(S(t)\phi\) in \(L^p\) for any compact interval \([0,T]\) as \(n \rightarrow \infty\).

**Proof.** Let \(u_n(t) = S(t)\phi_n\), \(u(t) = S(t)\phi\), \(n = 1,2,\ldots\). Then, \(w_n(t) = u_n(t) - u(t)\) solves

\[
w_n - (\Delta_m u_n - \Delta_m u) + \lambda \left( |u_n|^{m-2} u_n - |u|^{m-2} u \right) = f(x,u) - f(x,u_n)
\]

and \(w_n(x,0) = \phi_n(x) - \phi(x)\).

Multiplying (2.46) by \(|w_n|^{p-2}w_n\), we get from [8, Chapter 1, Lemma 4.4] and (2.13) that

\[
\frac{1}{p} \frac{d}{dt} \|w_n(t)\|_p^p + \gamma_0 \int_{\mathbb{R}^N} |\nabla w_n|^{m} |w_n|^{p-2} dx + \lambda \|w_n(t)\|_{p+m}^{p+m-2} \leq k_2 \int_{\mathbb{R}^N} \left( 1 + |u|^{q-2}(t) + |u_n|^{q-2}(t) \right) |w_n(t)|_p^p dx \leq C_0 \left( 1 + \|u(t)\|_{\infty}^{q-2} + \|u_n(t)\|_{\infty}^{q-2} \right) \|w_n(t)\|_p^p \leq C_0 \left( 1 + t^{-s_0(q-2)} \right) \|w_n(t)\|_p^p, \quad 0 \leq t \leq T,
\]

for some \(\gamma_0 > 0\), depending on \(m, N\). This implies that

\[
\|w_n(t)\|_p \leq \|w_n(0)\|_p \exp \left( C_0 \left( T + (1-s_0(q-2))^{-1} T^{1-s_0(q-2)} \right) \right) = \|\phi_n - \phi\|_p \exp \left( C_0 \left( T + (1-s_0(q-2))^{-1} T^{1-s_0(q-2)} \right) \right), \quad 0 \leq t \leq T,
\]

(2.48)
with \( s_0(q - 2) = N(q - 2)((m - 2)N + 2m)^{-1} < 1 \). Letting \( n \to \infty \), we obtain the desired result.

**Lemma 2.11.** Suppose that all the assumptions in Theorem 2.5 are satisfied. Let \( u(t) \) be the solution of (1.1)-(1.2) with \( u_0 \in L^2 \), \( \| u_0 \|_2 \leq M_0 \). Then, \( \exists T_0 > 0 \), such that for any \( p > m \), one has

\[
\| u(t) \|_p \leq A_p + B_p(t - T_0)^{-1/p_{\alpha_0}}, \quad t > T_0,
\]

(2.49)

where \( \alpha_0 = (m - 2 + m^2/N)/(p - m) \) and \( A_p, B_p > 0 \), which depend only on \( p, N, m \) and the given data \( \| g \|_{a_p}, \| h \|_{p_0}, M_0 \) with \( \alpha_p = (p + m - 2)/(m - 1), \beta_p = (p + m - 2)/(m - q) \).

**Proof.** Multiplying (1.1) by \( |u|^{p-2}u \), we have

\[
\frac{1}{p} \frac{d}{dt} \| u(t) \|_p^p + \gamma_p \| \nabla (|u|^{(p-2)/m}u) \|_m^m + \lambda \| u \|_{p+m-2}^{p+m-2} \leq \int_{\mathbb{R}^N} (g(x) - f(x, u))u|u|^{p-2}dx
\]

(2.50)

with \( \gamma_p = mm(p-1)(m + p - 2)^{-m} \). Note that

\[
\int_{\mathbb{R}^N} g(x)|u|^{p-2}u \, dx \leq \varepsilon \| u \|_{p+m-2}^{p+m-2} + C_p \| g \|_{a_p},
\]

\[
-\int_{\mathbb{R}^N} f(x, u)|u|^{p-2} \, dx \leq \int_{\mathbb{R}^N} h(x)|u|^{p+q-2} \, dx \leq \varepsilon \| u \|_{p+m-2}^{p+m-2} + C_p \| h \|_{p_0}^\beta_p
\]

with \( 0 < \varepsilon < \lambda/4 \). Then (2.50) becomes

\[
\frac{1}{p} \frac{d}{dt} \| u(t) \|_p^p + \gamma_p \| \nabla (|u|^{(p-2)/m}u) \|_m^m + \lambda \| u \|_{p+m-2}^{p+m-2} \leq C_p \left( \| h \|_{p_0}^\beta_p + \| g \|_{a_p} \right).
\]

(2.52)

By Lemma 2.1, we get

\[
\| \nabla (|u(t)|^{\tau_1}u(t)) \|_m^m \geq C_0 \| u(t) \|_{p}^{m(1+\tau_1)/\theta_1} \| u(t) \|_m^\theta_1,
\]

(2.53)

with

\[
\tau = \frac{p - 2}{m}, \quad \theta_1 = (1 + \tau) \left( \frac{1}{m} - \frac{1}{p} \right) \left( \frac{1}{N} + \frac{\tau}{m} \right)^{-1}, \quad \tau_1 = m \left( 1 - \theta_1^{-1} \right) (1 + \tau) < 0.
\]

(2.54)

By Lemma 2.9, \( \exists T_0 > 0 \), such that \( t \geq T_0, \| u(t) \|_m \leq M_1 \). Therefore, we have from (2.52) and (2.53) that

\[
\frac{1}{p} \frac{d}{dt} \| u(t) \|_p^p + C_0 M_1^\theta \| u(t) \|_{p}^{p(1+\alpha_0)} \leq A \equiv C_p \left( \| h \|_{p_0}^\beta_p + \| g \|_{a_p} \right), \quad t > T_0
\]

(2.55)
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with

\[ p(1 + \alpha_0) = \frac{m(1 + \tau)}{\theta_1}, \quad \tau_1 = m - 2 - p\alpha_0 < 0, \quad \alpha_0 = \frac{m - 2 + m^2/N}{p - m} > 0. \]  

(2.56)

It follows from (2.55) and Lemma 2.3 that

\[ |u(t)|^p \leq \left( AM_1^{-n} C_0^{-1} \right)^{1/(1+\alpha_0)} + (C_0 M_1^n a_0(t - T_0))^{-1/\alpha_0}, \quad t > T_0. \]  

(2.57)

This gives (2.49) and completes the proof of Lemma 2.11.

By Lemma 2.11, we now establish

**Lemma 2.12.** Assume that all the assumptions in Theorem 2.5 are satisfied. Let \( B_0 \) be a bounded set in \( L^2 \) and \( u(t) \) be a solution of (1.1)-(1.2) with \( u_0 \in B_0 \). Then, for any \( \eta > 0 \) and \( p > m \), \( \exists r_0 = r_0(\eta, B_0) \), \( T_1 = T_1(\eta, B_0) \), such that \( r \geq r_0, t \geq T_1 \),

\[ \int_{B_r^c} |u(t)|^p dx \leq \eta, \quad \forall u_0 \in B_0, \]  

(2.58)

where \( B_r^c = \{ x \in R^N \mid |x| \geq r \} \).

**Proof.** We choose a suitable cut-off function for the proof. Let

\[ \phi_0(s) = \begin{cases} 0, & 0 \leq s \leq 1; \\ (n - k)^{-1} \left( n(s - 1)^k - k(s - 1)^n \right), & 1 < s < 2; \\ 1, & s \geq 2; \end{cases} \]  

(2.59)

in which \( n(k > m) \) will be determined later. It is easy to see that \( \phi_0(s) \in C^1[0, \infty) \), \( 0 \leq \phi_0(s) \leq 1 \), \( 0 \leq \phi_0'(s) \leq \beta_0 \phi_0^{1-1/k}(s) \) for \( s \geq 0 \), where \( \beta_0 = k(n/(n - k))^{1/k} \). For every \( r > 0 \), denote \( \phi = \phi(r, x) = \phi_0(|x|/r), \quad x \in R^N \). Then

\[ |\nabla_x \phi(r, x)| \leq \frac{\beta_1}{r} \phi^{1-1/k}(r, x), \quad x \in R^N, \]  

(2.60)

with \( \beta_1 = N\beta_0 \).

Multiplying (1.1) by \( |u|^{p-2} u\phi \), \( p > m \), we obtain

\[ \int_{R^N} \frac{1}{p} \frac{d}{dt} |u|^p \phi \, dx + \int_{R^N} F u \nabla (|u|^{p-2} u\phi) \, dx + \frac{\lambda}{2} \int_{R^N} |u|^{p+m-2} \phi \, dx \]

\[ \leq C_p \left( \|h\|_{\beta_0}^{\beta_0}(B_r^c) + \|g\|_{\alpha_0}^{\alpha_0}(B_r^c) \right), \]  

(2.61)
where and in the sequel, we let \( \|f\|_p^p(\Omega) = \int_\Omega |f(x)|^p \, dx \). Note that

\[
D_1 = \int_{R^N} |\nabla u|^{p-2} \nabla u \nabla \left( |u|^{p-2} u \phi \right) \, dx = (p - 1) \int_{R^N} |u|^{p-2} |\nabla u|^m \phi \, dx + D_2
\]

(2.62)

with

\[
D_2 = \int_{R^N} |\nabla u|^{m-2} \nabla u \nabla \phi |u|^{p-2} u \, dx
\]

\[
\leq \int_{R^N} |\nabla u|^{m-1} |\nabla \phi| |u|^{p-1} \, dx
\]

\[
\leq \frac{\beta_1}{r} \int_{R^N} |\nabla u|^{m-1} |u|^{p-1} \phi^{1-1/} \, dx
\]

(2.63)

\[
\leq \frac{\beta_1}{r} \int_{R^N} \left( |\nabla u|^m |u|^{p-2} \phi + |u|^{p+m-2} \phi^{1-m/} \right) \, dx.
\]

Therefore, if \( r \geq 2\beta_1 / (p - 1) \),

\[
D_1 \geq \frac{p-1}{2} \int_{R^N} |\nabla u|^m |u|^{p-2} \phi \, dx - \frac{\beta_1}{r} \int_{R^N} |u|^{p+m-2} \phi^{1-m/} \, dx.
\]

(2.64)

Further, we estimate the first term of the right-hand side in (2.64). Since

\[
\frac{\partial}{\partial x_i} \left( |u \phi^{1/p}|^T u \phi^{1/p} \right) = (\tau + 1)|u|^\tau \phi^{\tau/p} \left( \phi^{1/p} \frac{\partial u}{\partial x_i} + \frac{u}{p} \frac{\partial \phi}{\partial x_i} \phi^{1-p-1} \right), \quad i = 1, 2, \ldots, N,
\]

\[
|\nabla \left( |u \phi^{1/p}|^T u \phi^{1/p} \right)|^2 = (\tau + 1)^2 |u|^2 \phi^{2\tau/p} \left( |\nabla u|^2 \phi^{2/p} + \frac{u^2}{p^2} |\nabla \phi|^2 \phi^{2/p-2} + \frac{2u}{p} \phi^{2/p-1} \nabla u \nabla \phi \right),
\]

(2.65)

we have

\[
D_3 = \left| \nabla \left( |u \phi^{1/p}|^T u \phi^{1/p} \right) \right|^m = \left[ \left| \nabla \left( |u \phi^{1/p}|^T u \phi^{1/p} \right) \right|^2 \right]^{m/2}
\]

\[
\leq \lambda_0 \left( |u|^m \nabla u |\nabla u|^m \phi^{m(\tau_2) + 1} + |u|^m \nabla \phi |\nabla \phi|^m \phi^{m(\tau_2-1)} + |u|^m \nabla u |\nabla \phi|^m \phi^{m(\tau_2-m/2)} \right),
\]

(2.66)

where \( \tau_2 = \tau_0 / p \), \( \tau_0 = 1 + \tau = (p - 2 + m) / m \) and with some constant \( \lambda_0 > 0 \). The second term of (2.66) is

\[
(2.66)_2 \leq \frac{\beta_1^{m}}{r^m} |u|^{p-2+m} \phi^{1+(m-2)/p-m/} \leq \frac{C_1}{r} |u|^{p-2+m} \phi^{1+(m-2)/p-m/}, \quad r \geq 1,
\]

(2.67)
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and the third term of (2.66) is

\[
(2.66)_3 \leq \frac{C_1}{r} |u|^{p-2+m/2} |\nabla u|^{m/2} \phi^{1+(m-2)/p-m/2k} \\
\leq \frac{C_1}{r} \left( |u|^{p-2} |\nabla u|^m \phi + |u|^{p+m-2} \phi^{1+(m-4)/p-m/k} \right), \quad r \geq 1
\]

(2.68)

with some \( C_1 > 0 \). Thus, we let \( k > pm/(2m-4) \) and have

\[
D_3 \leq C_1 \left( |u|^{p-2} |\nabla u|^m \phi + r^{-1} |u|^{p+m-2} \phi^{1+(m-2)/p-m/2k} \right)
\]

(2.69)

or

\[
|u|^{p-2} |\nabla u|^m \phi \geq C_1 \left( |u\phi^{1/p}|^T u\phi^{1/p} \right)^m - r^{-1} |u|^{p+m-2} \phi^{1+(m-2)/p-m/2k}.
\]

(2.70)

This implies that

\[
\int_{\mathbb{R}^N} |u|^{p-2} |\nabla u|^m \phi \, dx \geq C_1^{-1} \left\| \nabla \left( |u\phi^{1/p}|^T u\phi^{1/p} \right) \right\|^m_m - r^{-1} \int_{\mathbb{R}^N} |u|^{p+m-2} \phi^{1+(m-2)/p-m/2k} \, dx
\]

(2.71)

and for \( r \geq 1 \),

\[
D_1 \geq C_1^{-1} \left\| \nabla \left( |u\phi^{1/p}|^T u\phi^{1/p} \right) \right\|^m_m - C_pr^{-1} \int_{\mathbb{R}^N} |u|^{p+m-2} \left( \phi^{1+(m-2)/p-m/2k} + \phi^{1-m/k} \right) \, dx.
\]

(2.72)

On the other hand, we obtain by Lemma 2.9 that

\[
\left\| u(t)\phi^{1/p} \right\|_m \leq \left\| u(t) \right\|_m \leq M_1, \quad t \geq T_0,
\]

(2.73)

and then for \( t \geq T_0 \),

\[
\left\| \nabla \left( |u\phi^{1/p}|^T u\phi^{1/p} \right) \right\|^m_m \geq C_0 \left\| u\phi^{1/p} \right\|_p^{(m+mr)/\theta_1} \left\| u\phi^{1/p} \right\|_m^\tau_1 \geq C_0 M_1^\tau_1 \left\| u\phi^{1/p} \right\|_p^{(m+mr)/\theta_1},
\]

(2.74)

where \( \tau_1 \) and \( \theta_1 \) are determined by (2.54). Hence we get from (2.61)–(2.74) that

\[
\frac{1}{p} \frac{d}{dt} \left\| u(t)\phi^{1/p} \right\|_p^p + C_0 M_1^\tau_1 \left\| u(t)\phi^{1/p} \right\|_p^{p(1+\alpha)} \\
\leq C_p \left( \left\| h \right\|_{p,r}^{\beta} (B_r^c) + \left\| g \right\|_{\alpha_r}^{\alpha_r} (B_r^c) + r^{-1} \left\| u(t) \right\|_{p+m-2}(B_r^c) \right), \quad t \geq T_0, \quad r \geq 1.
\]

(2.75)

By Lemma 2.11, we know that there exist \( \exists T_1 > T_0 \) and \( M_{p+m-2} > 0 \), such that

\[
\left\| u(t) \right\|_{p+m-2} \leq M_{p+m-2}, \quad t \geq T_1.
\]

(2.76)
Then we obtain
\[ \int_{\mathbb{R}^N} |u|^p \phi \, dx \leq \left( H(r, t) \left( M_1 r^r C_0^{-1} \right)^{1/(1+\alpha_0)} + (C_0 M_1 r^r a_0 (t - T_1))^{-1/\alpha_0} \right), \quad t > T_1, \quad (2.77) \]
where
\[ H(r, t) = C_p \left( \| h \|_{B_r^c} + \| S \|_{B_r^c} + r^{-1} M_1^{p+m-2} \right), \quad t > T_0, \quad r \geq 1, \quad (2.78) \]
and \( H(r, t) \to 0 \) as \( r \to \infty \). Then (2.77) implies (2.58) and the proof of Lemma 2.12 is completed.

**Remark 2.13.** In fact, we see from the proof of Lemma 2.12 that if (2.73) and (2.76) are satisfied, then (2.77) and (2.58) hold.

**Remark 2.14.** In a similar argument, we can prove Lemmas 2.10–2.12 under the assumptions in Theorem 2.8.

### 3. Global Attractor in \( \mathbb{R}^N \)

In this section, we will prove the existence of the global \((L^2, L^p)\)-attractor for problem (1.1)-(1.2). To this end, we first give the definition about the bi-spaces global attractor, then, prove the asymptotic compactness of \( \{ S(t) \}_{t \geq 0} \) in \( L^p \) and the existence of the global \((L^2, L^p)\)-attractor by a priori estimates established in Section 2.

**Definition 3.1 ([2, 3, 13, 14]).** A set \( \mathcal{A}_p \subset L^p \) is called a global \((L^2, L^p)\)-attractor of the semigroup \( \{ S(t) \}_{t \geq 0} \) generated by the solution of problem (1.1)-(1.2) with initial data \( u_0 \in L^2 \) if it has the following properties:

1. \( \mathcal{A}_p \) is invariant in \( L^p \), that is, \( S(t) \mathcal{A}_p = \mathcal{A}_p \) for every \( t \geq 0 \);
2. \( \mathcal{A}_p \) is compact in \( L^p \);
3. \( \mathcal{A}_p \) attracts every bounded subset \( B \) of \( L^2 \) in the topology of \( L^p \), that is,
\[ \text{dist}(S(t)B, \mathcal{A}_p) = \sup_{v \in B} \inf_{u \in \mathcal{A}_p} \| S(t)v - u \|_p \to 0 \quad \text{as} \quad t \to +\infty. \quad (3.1) \]

Now we can prove the main result.

**Theorem 3.2.** Assume that all assumptions in Theorem 2.5 (Theorem 2.7) are satisfied. Then the semigroup \( \{ S(t) \}_{t \geq 0} \) generated by the solutions of the problem (1.1)-(1.2) with \( u_0 \in L^2 \) has a global \((L^2, L^p)\)-attractor \( \mathcal{A}_p \) for any \( p > m \).

**Proof.** We only consider the case in Theorem 2.5 and the other is similar and omitted. Define
\[ \mathcal{A}_p = \bigcap_{\tau \geq 0} \mathcal{A}(\tau), \quad \mathcal{A}(\tau) = \left[ \bigcup_{t \geq \tau} S(t)\mathcal{D} \right]_{L^p}, \quad (3.2) \]
where \( \mathcal{D} \) is defined in (2.45) and \( [E]_{L^p} \) is the closure of \( E \) in \( L^p \).
Obviously, $\mathcal{A}(\tau)$ is closed and nonempty and $\mathcal{A}(\tau_1) \subset \mathcal{A}(\tau_2)$ if $\tau_1 \geq \tau_2$. Thus, $\mathcal{A}_p$ is nonempty. We now prove that $\mathcal{A}_p$ is a global $(L^2, L^p)$-attractor for (1.1)-(1.2).

We first prove $\mathcal{A}_p$ is invariant in $L^p$. Let $\phi \in \mathcal{A}_p$. Then, $\exists t_n \to +\infty$ and $\theta_n \in \mathcal{D}$ such that $S(t_n)\theta_n \to \phi$ in $L^p$. Since $S(t)$ is continuous from $L^p \to L^p$ by Lemma 2.10, we obtain $S(t + t_n)\theta_n = S(t)(S(t_n)\theta_n) \to S(t)\phi$ in $L^p$. Note that

$$S(t + t_n)\theta_n \in \bigcup_{\tau \geq t} S(\tau)\mathcal{D} \implies S(t)\phi \in \mathcal{A}(\tau) \implies S(t)\phi \in \bigcap_{\tau \geq 0} \mathcal{A}(\tau). \quad (3.3)$$

That is, $S(t)\phi \in \mathcal{A}_p$ and $S(t)\mathcal{A}_p \subset \mathcal{A}_p$.

On the other hand, let $\phi \in \mathcal{A}_p$. Suppose $t_n \to +\infty$ and $\theta_n \in \mathcal{D}$ such that $S(t_n)\theta_n \to \phi$ in $L^p$. We claim that there exists $\psi \in \mathcal{A}_p$ such that $S(t)\psi = \phi$. This implies $\mathcal{A}_p \subset S(t)\mathcal{A}_p$.

First, since $\{\theta_n\}$ is bounded in $W^{1,m}$ by Lemma 2.9, so is $\{S(t_n - t)\theta_n\}$ by Theorem 2.7. That is, $\exists n_0 > 1, T_0 > 0, M_3 > 0$, such that

$$\|u_n\|_m \leq M_3, \quad \|\nabla u_n\|_m \leq M_3 \quad \text{for } n \geq n_0, t_n - t \geq T_0,$$

with $u_n(x) = S(t_n - t)\theta_n(x)$. Then,

$$\|u_n\|_{W^{1,m}(B_{r_0})} = \|\nabla u_n\|_{m}(B_{r_0}) + \|u_n\|_{m}(B_{r_0}) \leq h(r_0, M_3), \quad n \geq n_0, \quad (3.5)$$

where the constant $h(r_0, M_3)$ depends on $r_0, M_3$, and $r_0$ is from Lemma 2.12. By the compact embedding theorem, $\exists \{u_{n_k}\} \subset \{u_n\}$ such that $u_{n_k} \to \psi$ in $L^p(B_{r_0})$ if $2 \leq p < m^*$. We extend $\psi(x)$ as zero when $|x| > r_0$. Then $u_{n_k} \to \psi$ in $L^p$, and $\psi \in \mathcal{A}(\tau), \psi \in \mathcal{A}_p$. By the continuity of $S(t)$ in $L^p$, we have

$$S(t_n)\theta_n = S(t)(S(t_n - t)\theta_n) \to S(t)\psi = \phi = S(t)\phi \quad \text{in } L^p. \quad (3.6)$$

So, $\mathcal{A}_p \subset S(t)\mathcal{A}_p$ and $\mathcal{A}_p$ is invariant in $L^p$ for every $t \geq 0$.

For the case $p \geq m^*$, we take $\mu \in (m, m^*)$ and $u_{n_0} \to \psi$ in $L^\mu$ as the above proof. Thus $\{u_{n_k}\}$ is a Cauchy sequence in $L^\mu$. We claim that $\{u_{n_k}\}$ is also a Cauchy sequence in $L^p$.

In fact, it follows from Lemma 2.11 that $\exists M_p$ and $n_0$ such that if $n \geq n_0$, then $t_n - t \geq T_0$ and

$$\|u_n\|_\rho \leq M_p, \quad \rho = \frac{(p - 1)\mu}{\mu - 1}. \quad (3.7)$$

Notice that

$$\int_{\mathbb{R}^N} \left| u_{n_i} - u_{n_j} \right|^p \, dx \leq \left\| u_{n_i} - u_{n_j} \right\|_\mu \left\| u_{n_i} - u_{n_j} \right\|_\rho^{p - 1} \leq (2M_p)^{p - 1} \left\| u_{n_i} - u_{n_j} \right\|_\mu \quad (3.8)$$

for $i, j \geq n_0$. This gives our claim. Therefore, $\exists \psi \in L^p$ such that $u_{n_k} = S(t_n - t)\theta_n \to \psi$ in $L^p$ and $\phi = S(t)\psi$. Hence $\mathcal{A}_p \subset S(t)\mathcal{A}_p$ and $S(t)\mathcal{A}_p = \mathcal{A}_p$.

We now consider the compactness of $\mathcal{A}_p$ in $L^p$. In fact, from the proof of $\mathcal{A}_p \subset S(t)\mathcal{A}_p$, we know that $\{\bigcup_{\tau \geq t} S(\tau)\mathcal{D}\}_{L^p}$ is compact in $L^p$, so is $\mathcal{A}_p$. 


For claim (3), we argue by contradiction and assume that for some bounded set $B_0$ of $L^2$, $\text{dist}_{L^p}(S(t)B_0, \mathcal{A}_p)$ does not tend to 0 as $t \to +\infty$. Thus, there exists $\delta > 0$ and a sequence $t_n \to \infty$ such that

$$\text{dist}_{L^p}(S(t_n)B_0, \mathcal{A}_p) \geq \frac{\delta}{2} > 0, \quad \text{for } n = 1, 2, \ldots. \quad (3.9)$$

For every $n = 1, 2, \ldots$, there exists $\theta_n \in B_0$ such that

$$\text{dist}_{L^p}(S(t_n)\theta_n, \mathcal{A}_p) \geq \frac{\delta}{2} > 0. \quad (3.10)$$

By Lemma 2.9, $\mathfrak{D}$ is an absorbing set, and $S(t_n)\theta_n \subset \mathfrak{D}$ if $t_n \geq T_0$. By the aforementioned proof, we know that $\exists \phi \in L^p$ and a subsequence $\{S(t_{n_k})\theta_{n_k}\}$ of $\{s(t_n)\theta_n\}$ such that

$$\phi = \lim_{k \to \infty} S(t_{n_k})\theta_{n_k} = \lim_{k \to \infty} S(t_{n_k} - T_0)(S(T_0)\theta_{n_k}), \quad \text{in } L^p. \quad (3.11)$$

When $\theta_{n_k} \in B_0$ and $T_0$ is large, we have from Lemma 2.9 that $S(T_0)\theta_{n_k} \in \mathfrak{D}$ and

$$S(t_{n_k} - T_0)(S(T_0)\theta_{n_k}) \in \bigcup_{t \geq T_0} S(t)\mathfrak{D}. \quad (3.12)$$

Thus, $\phi \in \mathcal{A}_p$ which contradicts (3.10). Then the proof of Theorem 3.2 is completed. \hfill \Box

Remark 3.3. Let $p = m^* = mN/(N - m)$. Theorem 3.2 gives the results in [2, Theorem 2] for the case $N > m > 2$ and improve the corresponding results in [3]. The attractor $\mathcal{A}_p$ in Theorem 3.2 is independent of the order of $u$ on $f(x, u)$.

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