Research Article

The Central Extension Defining the Super Matrix Generalization of $W_{1+\infty}$

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We prove that the Lie superalgebra of regular differential operators on the superspace $C^{M|N}[t,t^{-1}]$ has an essentially unique non-trivial central extension.

1. Introduction

The $W$ infinity algebras naturally arise in various physical systems, such as two-dimensional quantum gravity and the quantum Hall effects (see the review [1, 2] and references therein). The most fundamental one is the $W_{1+\infty}$ which is the central extension of the Lie algebra of regular differential operators on the circle [1–5], and it contains the $W_\infty$ algebra as a subalgebra. Various extensions where constructed: super extension $(W_{1+\infty}^{|1})$ [6, 7], $u(M)$ matrix version of $W_{1+\infty}(W_1^M)$ [8], and the most general super matrix generalization $W_{1+\infty}^{M|N}$ presented in [1, 2, 9]. It seems difficult to decide where and when the first definition of a (version of) super-$W$ algebra appeared, but a book by Guieu and Roger [10] has a good historical and bibliographic base, including the pioneering papers of Radul where the superanalogues of the Bott-Virasoro cocycles were introduced (see [11]). The original $W_{1+\infty}$ corresponds to $M = 1$, $N = 0$. The general study of representation theory of $W$ infinity algebras started in the remarkable work [4] by Kac and Radul and continued in several works (some of them are [6, 12–14]). Matrix generalizations are deeply related to the main examples of infinite rank conformal algebras (see [15–17]).

The super matrix generalization $W_{1+\infty}^{M|N}$ is defined as a specific central extension of the Lie superalgebra of regular differential operators on the superspace $C^{M|N}[t,t^{-1}]$. Only in the special case of $W_{1+\infty}$ (i.e., $M = 1, N = 0$) was it proved that the 2-cocycle defining this central extension is unique up to coboundary [18]. The main goal of the present work is to extend
2. Basic Definitions and Main Result

Let \( L \) and \( \hat{L} \) be two Lie superalgebras over \( \mathbb{C} \). The Lie superalgebra \( \hat{L} \) is said to be a one-dimensional central extension of \( L \) if \( \hat{L} \) is the direct sum of \( L \) and \( \mathbb{C}C \) as vector spaces and the Lie superBracket in \( \hat{L} \) is given by

\[
[a, b]^- = [a, b] + \Psi(a, b)C, \quad [a, c]^- = 0,
\]

for all \( a, b \in L \), where \([\cdot, \cdot]^-\) is the Lie Bracket in \( L \) and \( \Psi: L \times L \rightarrow \mathbb{C} \) is a 2-cocycle on \( L \), that is, a bilinear \( \mathbb{C} \)-valued form satisfying the following conditions for all homogeneous elements \( a, b, c \in L \):

\[
\begin{align*}
(1) & \quad \Psi(a, b) = -(-1)^{|a||b|}\Psi(b, a), \\
(2) & \quad \Psi([a, b], c) = \Psi(a, [b, c]) - (-1)^{|a||b|}\Psi(b, [a, c]),
\end{align*}
\]

where \(|a|\) denote the parity of \( a \). A central extension is trivial if \( \hat{L} \) is the direct sum of a subalgebra \( M \) and \( \mathbb{C}C \) as Lie algebras, where \( M \) is isomorphic to \( L \). A 2-cocycle corresponding to a trivial central extension is called a 2-coboundary, and it is given by an \( f \in L^* \) as follows:

\[
a_f(a, b) = f([a, b]),
\]

for \( a, b \in L \). It is easy to check that \( a_f \) is a 2-cocycle. We say that the 2-cocycles \( \Psi, \phi \) are equivalent if \( \phi - \Psi \) is a 2-coboundary. The second cohomology group of \( L \) with coefficients in \( \mathbb{C} \) is the set of equivalent classes of 2-cocycles, and it will be denoted by \( H^2(L, \mathbb{C}) \). If \( \dim H^2(L, \mathbb{C}) = 1 \), we say that \( L \) has an essentially unique nontrivial one-dimensional central extension.

Now, we will introduce the Lie superalgebra that will be considered in this work. Let us denote by \( \text{Mat}(M | N) \) the associative superalgebra of linear transformations on the complex \((M | N)\)-dimensional superspace \( \mathbb{C}^{M|N} \). Namely, we consider the set of all \((M + N) \times (M + N)\) matrices of the form

\[
A = \begin{pmatrix} A^0 & A^+ \\ A^- & A^1 \end{pmatrix},
\]

where \( A^0, A^+, A^-, A^1 \) are \( M \times M, M \times N, N \times M, N \times N \) matrices, respectively, with complex entries. The \( \mathbb{Z}_2 \)-gradation is defined by declaring that matrices of the form (2.4) with \( A^+ = A^- = 0 \) are even, and those with \( A^0 = A^1 = 0 \) are odd. We denote by \(|A|\) the degree of \( A \) with respect to this \( \mathbb{Z}_2 \)-gradation. The supertrace is defined by

\[
\text{Str}(A) = \text{tr}(A^0) - \text{tr}(A^1),
\]

and it satisfies \( \text{Str}(AB) = (-1)^{|A||B|}\text{Str}(BA) \).
Let $\mathfrak{D}_{as}$ be the associative algebra of regular differential operators on the circle, that is, the operators on $\mathbb{C}[t, t^{-1}]$ of the form

$$E = e_k(t)\partial_t^k + e_{k-1}(t)\partial_t^{k-1} + \cdots + e_0(t), \quad \text{where } e_i(t) \in \mathbb{C}[t, t^{-1}],$$

(2.6)

The elements

$$J^l_k = -t^{l+k}(\partial_t)^l \quad (l \in \mathbb{Z}_+, k \in \mathbb{Z})$$

(2.7)

form its basis, where $\partial_t$ denotes $d/dt$. Another basis of $\mathfrak{D}_{as}$ is

$$L^l_k = -t^k D^l \quad (l \in \mathbb{Z}_+, k \in \mathbb{Z}),$$

(2.8)

where $D = t\partial_t$. It is easy to see that

$$J^l_k = -t^k [D]^l.$$  

(2.9)

Here and further we use the notation

$$[x]_l = x(x-1) \ldots (x-l+1).$$

(2.10)

Denote by $S\mathfrak{D}_{as}^{M|N}$ the associative superalgebra of $(M + N) \times (M + N)$ (super)matrices with entries in $\mathfrak{D}_{as}$. The $\mathbb{Z}_2$-gradation is the one inherited by the corresponding $\mathbb{Z}_2$-gradation in $\text{Mat}(M | N)$. By taking the usual superbracket we make $S\mathfrak{D}_{as}^{M|N}$ into a Lie superalgebra, which is denoted by $S\mathfrak{D}^{M|N}$. A set of generators is given by $\{t^s f(D)A : s \in \mathbb{Z}, f \in \mathbb{C}[x], A \in \text{Mat}(M | N)\}$.

Let $W_{1+\infty}^{M,N} = S\mathfrak{D}^{M|N} \otimes \mathbb{C}C$ be the central extension of $S\mathfrak{D}^{M|N}$ by a one-dimensional vector space with a specified generator $C$, whose commutation relation for homogeneous elements is given by

$$[t^f f(D)A, t^s g(D)B] = t^{s+f} f(D+s)g(D)AB - (-1)^{|A||B|} t^{s+f} f(D)g(D+r)BA$$

$$+ \Psi(t^f f(D)A, t^s g(D)B)C,$$

(2.11)

where the 2-cocycle $\Psi$ is given by

$$\Psi(t^f f(D)A, t^s g(D)B) = \begin{cases} \left( \sum_{-s \leq j \leq -1} f(j)g(j+r) \right) \text{Str}(AB) & \text{if } r = -s \geq 0, \\ 0, & \text{if } r + s \neq 0. \end{cases}$$

(2.12)

Now, we are in condition to state our main result.

**Theorem 2.1.** One has the following: $\dim H^2(S\mathfrak{D}^{M|N}, \mathbb{C}) = 1.$
3. Proof of Theorem 2.1

We will need the explicit expression of the bracket of basis elements of type (2.9) in $\mathfrak{D}^{|M|N}$:

\[
[t^m[D]_jE_{ij}, t^n[D]_kE_{rs}] = t^{m+n}\left( [D + n]_j[D]_k \delta_{jr} E_{is} - (-1)^{|E_{is}|} [D]_j[D + m]_k \delta_{is} E_{rj} \right).
\]

(3.1)

In particular, we have

\[
\begin{align*}
[t^{-1}DE_{ii}, t^m[D]_jE_{ii}] &= (l + m)t^{m-1}[D]_jE_{ii}, \\
[t^{-1-1}[D]_jE_{ii}, DE_{ii}] &= (l + 1)t^{l-1-1}[D]_jE_{ii}, \\
[E_{ii}, t^m[D]_jE_{ij}] &= t^m[D]_jE_{ij}, \quad i \neq j.
\end{align*}
\]

(3.2)

Let $\beta$ be a 2-cocycle on $\mathfrak{D}^{|M|N}$. We consider the linear functional in $\mathfrak{D}^{|M|N}$ defined by

\[
\begin{align*}
f_{\beta}(t^{m-1}[D]_jE_{ii}) &= \frac{1}{l + m} \beta(t^{-1}DE_{ii}, t^m[D]_jE_{ii}), \quad l \neq -m, \\
f_{\beta}(t^{-1-1}[D]_jE_{ii}, DE_{ii}) &= \frac{1}{l + 1} \beta(t^{-1-1}[D]_jE_{ii}, DE_{ii}), \\
f_{\beta}(t^m[D]_jE_{ij}) &= \beta(E_{ii}, t^m[D]_jE_{ij}), \quad i \neq j.
\end{align*}
\]

(3.3)

Then $\beta_1 = \beta - \alpha_{f_{\beta}}$ is a 2-cocycle on $\mathfrak{D}^{|M|N}$ that is equivalent to $\beta$, and using (3.3), we obtain

\[
\begin{align*}
\beta_1(t^{-1}DE_{ii}, t^m[D]_jE_{ii}) &= 0, \quad l \neq -m, \\
\beta_1(t^{-1-1}[D]_jE_{ii}, DE_{ii}) &= 0, \\
\beta_1(E_{ii}, t^m[D]_jE_{ij}) &= 0, \quad i \neq j.
\end{align*}
\]

(3.4)

In order to complete the proof we need to show that $\Psi = a\beta_1$ for some $a \in \mathbb{C}$. By observing the supertrace that appears in the expression of $\Psi$ in (2.12), we immediately obtain that for any $f, g \in D_{as}$

\[
\Psi(fE_{ij}, gE_{sk}) = 0 \quad \text{if } i \neq k \text{ or } j \neq s.
\]

(3.5)

In Lemmas 3.1 and 3.2, we will show that $\beta_1$ also satisfies (3.5).
Lemma 3.1. For any \( f, g \in D \), \( \beta_1(fE_{ii}, gE_{s}) = 0 \) if \( i \neq j \) or \( i \neq s \).

Proof. Case \( j = i \) and \( s \neq i \).

Using that \( E_{ii} \) is even, \( i \neq s \), and (2.2), we obtain that

\[
\beta_1(fE_{ii}, gE_{s}) = \beta_1(fE_{ii}, [E_{ss}, gE_{s}]) = -\beta_1([E_{ss}, gE_{s}], fE_{ii})
\]

\[
= -\beta_1(E_{ss}, [gE_{sir}, fE_{ii}]) + \beta_1(gE_{sir}, [E_{ss}, fE_{ii}])
\]

\[
= -\beta_1(E_{ss}, (g \circ f)E_{si}) = 0, \quad \text{(using \( i \neq s \) and (3.4)),}
\]

where \( g \circ f \) is the product in \( D \).

Case \( j \neq i \) and \( s = i \).

In this case we have

\[
\beta_1(fE_{ii}, gE_{ij}) = \beta_1(fE_{ii}, [gE_{ij}, E_{jj}]) = -\beta_1([gE_{ij}, E_{jj}], fE_{ii})
\]

\[
= -\beta_1(gE_{ij}, [E_{jj}, fE_{ii}]) + \beta_1(E_{jj}, [gE_{ij}, fE_{ii}]) \quad \text{(by (2.2))}
\]

\[
= \beta_1(E_{jj}, (f \circ g)E_{ij}) = 0 \quad \text{(using \( i \neq j \) and (3.6)).}
\]

Case \( j \neq i \) and \( s \neq i \).

By taking the usual bracket, we make the associative algebra \( D \) into a Lie algebra which is denoted by \( D \). Observe that

\[
D = SD^{10}.
\]

It is easy to show that \([D, D] = D\); therefore, for any \( f \in D \), we have

\[
f = \sum_l [f_l, h_l], \quad f_l, h_l \in D.
\]

Thus, if \( j \neq i \) and \( s \neq i \), using (2.2),

\[
\beta_1(fE_{ii}, gE_{sij}) = \beta_1\left(\sum_l [f_lE_{ii}, h_lE_{ii}], gE_{sij}\right)
\]

\[
= \sum_l \beta_1(f_lE_{ii}, h_lE_{ii}, gE_{sij}) \quad \text{by (2.2)}
\]

\[
= \sum_l \beta_1(h_lE_{ii}, [f_lE_{ii}, gE_{sij}]) = 0.
\]

The proof is finished. \( \square \)
Lemma 3.2. For any \( f, g \in \mathfrak{D}_n \) and \( i \neq j, s \neq k \), \( \beta_1(fE_{ij}, gE_{sk}) = 0 \) when \( i \neq k \) or \( j \neq s \).

Proof. If \( i \neq j \) and \( k \neq i \), we have

\[
\beta_1(fE_{ij}, gE_{sk}) = \beta_1([E_{ii}, fE_{ij}], gE_{sk})
= \beta_1(E_{ii}, [fE_{ij}, gE_{sk}]) - \beta_1(fE_{ij}, [E_{ii}, gE_{sk}])
= \delta_{j,s}\beta_1(E_{ii}, (f \circ g)E_{ik}) - \delta_{i,s}\beta_1(fE_{ij}, gE_{ik})
= -\delta_{i,s}\beta_1(fE_{ij}, gE_{ik}) \quad \text{(using (3.4))}.
\]

Hence we have \( \beta_1(fE_{ij}, gE_{sk}) = 0 \).

Finally, using skew-symmetry and the previous case, if \( i \neq j \), \( s \neq k \), and \( s \neq j \), we have that \( \beta_1(fE_{ij}, gE_{sk}) = 0 \).

Now, it remains to consider the expression \( \beta_1(fE_{ij}, gE_{ji}) \). In order to do it, consider again the Lie algebra \( \mathfrak{D} = \mathfrak{S}\mathfrak{D}_{10}^{10} \) (see (3.8)) and denote by \( \varphi_{\mathfrak{D}} \) the 2-cocycle \( \Psi \) defined in (2.12) with \( M = 1 \) and \( N = 0 \).

In fact, from the expression of \( \Psi \), we have

\[
\Psi(fA, gB) = \varphi_{\mathfrak{D}}(f, g) \text{Str}(AB).
\]

Lemma 3.3. There exist \( a_i \in \mathbb{C} \) such that for all \( f, g \in \mathfrak{D}_n \)

\[
\beta_1(fE_{ii}, gE_{ii}) = a_i \varphi_{\mathfrak{D}}(f, g).
\]

Moreover, the constants \( a_i \) satisfy \( a_i = (-1)^{|E_{ii}|}a_j \) for all \( i \neq j \).

Proof. Let \( \gamma_i : \mathfrak{D} \times \mathfrak{D} \to \mathbb{C} \) be the bilinear map defined by (\( i = 1, \ldots, M + N \))

\[
\gamma_i(f, g) = \beta_1(fE_{ii}, gE_{ii}).
\]

Since \( E_{ii} \) is even, we have that \( \gamma_i \) is a 2-cocycle in \( \mathfrak{D} \).

The following statement was proved in [18] (see Proof of Theorem 2.1 in page 74 and (3.2) and (3.3) in this work) if a 2-cocycle \( \beta_1 \) in \( \mathfrak{D} \) satisfies (\( l \in \mathbb{Z}, m \in \mathbb{Z} \))

\[
\beta_1(t^m[D]_l, t^{-1}D) = 0,
\]

\[
\beta_1(t^{-1}[D]_l, D) = 0.
\]

Then \( \beta_1 = a \varphi_{\mathfrak{D}} \) for some \( a \in \mathbb{C} \). Now, using (3.4), we have that \( \gamma_i \) satisfies (3.15); thus, we get \( \gamma_i = a_ia \varphi_{\mathfrak{D}} \) for some \( a_i \in \mathbb{C} \), proving the first part of this lemma.
In order to prove the second part, consider $i \neq j$. Then

\[
\beta_1 \left( tE_{ii}, t^{-1}E_{ii} \right) = \beta_1 \left( t \left( E_{ii} - (-1)^{|E_i||E_j|} E_{jj} \right), t^{-1}E_{ii} \right) \quad \text{(by Lemma 3.1)} \\
= \beta_1 \left( [E_{ii}, tE_{jj}], t^{-1}E_{ii} \right) \\
= \beta_1 \left( [E_{ii}, tE_{jj}], t^{-1}E_{ii} \right) - (-1)^{|E_i||E_j|} \beta_1 \left( tE_{jj}, \left[ E_{jj}, t^{-1}E_{ii} \right] \right) \\
= \beta_1 \left( [E_{ii}, tE_{jj}], t^{-1}E_{ii} \right) + (-1)^{|E_i||E_j|} \beta_1 \left( tE_{jj}, t^{-1}E_{ii} \right).
\]

Similarly,

\[
\beta_1 \left( tE_{jj}, t^{-1}E_{jj} \right) = \beta_1 \left( tE_{jj}, t^{-1} \left( E_{jj} - (-1)^{|E_j||E_i|} E_{ii} \right) \right) \quad \text{(by Lemma 3.1)} \\
= \beta_1 \left( tE_{jj}, \left[ E_{jj}, t^{-1}E_{ii} \right] \right) \\
= \beta_1 \left( tE_{jj}, \left[ E_{jj}, t^{-1}E_{ii} \right] \right) + \beta_1 \left( \left[ E_{jj}, tE_{ii} \right], t^{-1}E_{jj} \right) \\
= \beta_1 \left( tE_{jj}, t^{-1}E_{jj} \right) - \beta_1 \left( E_{jj}, E_{jj} \right) \\
= \beta_1 \left( tE_{jj}, t^{-1}E_{jj} \right) + (-1)^{|E_i||E_j|} \beta_1 \left( E_{jj}, E_{jj} \right).
\]

Therefore, $\beta_1 \left( tE_{ii}, t^{-1}E_{ii} \right) = (-1)^{|E_i||E_j|} \beta_1 \left( tE_{jj}, t^{-1}E_{jj} \right)$, which means that, $a_i = (-1)^{|E_i||E_j|} a_j$ for all $i \neq j$, finishing the proof. \qed

**Lemma 3.4.** $\beta_1 (E_{ij}, gE_{ji}) = \beta_1 (gE_{ij}, E_{ji})$ for $i \neq j$ and $g \in \mathcal{D}_g$.

**Proof.** Since $i \neq j$,

\[
\beta_1 (E_{ij}, gE_{ji}) = \beta_1 (E_{ij}, \left[ E_{ji}, gE_{ii} \right]) \\
= \beta_1 (E_{ij}, \left[ E_{ji}, gE_{ii} \right]) - (-1)^{|E_i||E_j|} \beta_1 (E_{ji}, \left[ E_{ij}, gE_{ii} \right]) \\
= \beta_1 (E_{ii}, gE_{ii}) - (-1)^{|E_i||E_j|} \beta_1 (E_{jj}, gE_{ii}) - (-1)^{|E_i||E_j|} \beta_1 (E_{ji}, gE_{ij}) \\
= a_i \varphi_g (1, g) + \beta_1 (gE_{ij}, E_{ji}), \quad \text{(by Lemmas 3.3 and 3.1)} \\
= \beta_1 \left( gE_{ij}, E_{ji} \right) \quad \text{(by definition of } \varphi_g) \quad \text{\qed}
\]
Lemma 3.5. \( \beta_1(fE_{ij}, gE_{ji}) = \beta_1(fE_{ii}, gE_{ii}) \) for \( i \neq j \) and any \( f, g \in D \).

Proof. Observe that

\[
\beta_1(fE_{ii}, gE_{ii}) = \beta_1([fE_{ij}, E_{ji}], gE_{ii}) \quad \text{(by Lemma 3.1)}
\]

\[
= \beta_1(fE_{ij}, [E_{ji}, gE_{ii}]) - (-1)^{|fE_{ij}||E_{ji}|} \beta_1(E_{ji}, [fE_{ij}, gE_{ii}])
\]

\[
= \beta_1(fE_{ij}, gE_{ji}) + (-1)^{|E_{ii}|} \beta_1(E_{ji}, (g \circ f)E_{ij})
\]

\[
= \beta_1(fE_{ij}, gE_{ji}) - \beta_1((g \circ f)E_{ij}, E_{ji}).
\]

Similarly,

\[
\beta_1(fE_{ii}, gE_{ii}) = (-1)^{|E_{ii}|} \beta_1(fE_{jj}, gE_{jj}) \quad \text{(by Lemma 3.3)}
\]

\[
= (-1)^{|E_{ii}|} \beta_1(fE_{jj}, [gE_{jj}, E_{ij}]) \quad \text{(by Lemma 3.1)}
\]

\[
= (-1)^{|E_{ii}|} \beta_1([fE_{jj}, gE_{jj}], E_{ij}) + (-1)^{|E_{ii}|} \beta_1(gE_{jj}, [fE_{jj}, E_{ij}])
\]

\[
= (-1)^{|E_{ii}|} \beta_1((f \circ g)E_{ij}, E_{jj}) + \beta_1(fE_{jj}, gE_{jj})
\]

\[
= -\beta_1((f \circ g)E_{ij}, E_{jj}) + \beta_1(fE_{jj}, gE_{jj}) \quad \text{(by Lemma 3.4)}.
\]

Hence, from (3.19) and (3.20), we obtain

\[
\beta_1([f, g]E_{ij}, E_{ji}) = 0. \quad \text{(3.21)}
\]

Since \([D, D] = D\), we have \( \beta_1(D_{ij}, E_{ji}) = 0 \). Therefore, (3.19) becomes the statement of this lemma. \( \square \)

Proof of Theorem 2.1. From the previous lemmas, one can easily see that \( \beta_1 = a_1 \Psi \), by observing that the relation between the \( a_i \)'s in Lemma 3.3 is essentially the supertrace term in expression (2.12) of \( \Psi \). \( \square \)

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