Research Article

Constancy of $\bar{\phi}$-Holomorphic Sectional Curvature for an Indefinite Generalized $g \cdot f \cdot f$-Space Form

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1. Introduction

For an almost Hermitian manifold $(M^{2n}, g, J)$ with $\dim(M) = 2n > 4$, Tanno [1] has proved the following.

Theorem 1.1. Let $\dim(M) = 2n > 4$, and assume that almost Hermitian manifold $(M^{2n}, g, J)$ satisfies

$$R(JX, JY, JZ, JX) = R(X, Y, Z, X)$$  \hspace{1cm} (1.1)

for every tangent vector $X, Y,$ and $Z$. Then $(M^{2n}, g, J)$ has a constant holomorphic sectional curvature at $x$ if and only if

$$R(X, JX)X \text{ is proportional to } JX$$  \hspace{1cm} (1.2)

for every tangent vector $X$ at $x \in M$.

Tanno [1] has also proved an analogous theorem for Sasakian manifolds as follows.
Theorem 1.2. A Sasakian manifold $\geq 5$ has a constant $\phi$-sectional curvature if and only if

$$R(X, \phi X)X \text{ is proportional to } \phi X$$  \hspace{1cm} (1.3)

for every tangent vector $X$ such that $g(X, \xi) = 0$.

Nagaich [2] has proved the generalized version of Theorem 1.1, for indefinite almost Hermitian manifolds as follows.

Theorem 1.3. Let $(M^{2n}, g, J)$ $(n > 2)$ be an indefinite almost Hermitian manifold that satisfies (1.1), then $(M^{2n}, g, J)$ has a constant holomorphic sectional curvature at $x$ if and only if

$$R(X, JX)X \text{ is proportional to } JX$$  \hspace{1cm} (1.4)

for every tangent vector $X$ at $x \in M$.

Bonome et al. [3] generalized Theorem 1.2 for an indefinite Sasakian manifold as follows.

Theorem 1.4. Let $(M^{2n+1}, \phi, \eta, \xi, g)$ $(n \geq 2)$ be an indefinite Sasakian manifold. Then $M^{2n+1}$ has a constant $\phi$-sectional curvature if and only if

$$R(X, \phi X)X \text{ is proportional to } \phi X$$  \hspace{1cm} (1.5)

for every vector field $X$ such that $g(X, \xi) = 0$.

In this paper, we generalize Theorem 1.4 for an indefinite generalized $g \cdot f \cdot f$-space form by proving the following.

Theorem 1.5. Let $(\overline{M}^{2n+r}, F_1, F_2, \overline{\phi})$ be an indefinite generalized $g \cdot f \cdot f$-space form. Then $\overline{M}^{2n+r}$ is of constant $\phi$-sectional curvature if and only if

$$\overline{R}(X, \overline{\phi}X)X \text{ is proportional to } \overline{\phi}X$$  \hspace{1cm} (1.6)

for every vector field $X$ such that $\overline{g}(X, \overline{\xi}_\alpha) = 0$, for any $\alpha \in \{1, \ldots, r\}$.

2. Preliminaries

A manifold $\overline{M}$ is called a globally framed $f$-manifold (or $g \cdot f \cdot f$-manifold) if it is endowed with a nonnull $(1,1)$-tensor field $\overline{\phi}$ of constant rank, such that ker $\overline{\phi}$ is parallelizable; that is, there exist global vector fields $\overline{\xi}_\alpha$, $\alpha \in \{1, \ldots, r\}$, with their dual 1-forms $\overline{\eta}^\alpha$, satisfying $\overline{\phi}^2 = -I + \sum_{\alpha=1}^{r} \overline{\eta}^\alpha \otimes \overline{\xi}_\alpha$ and $\overline{\eta}^\alpha(\overline{\xi}_\beta) = \delta^\alpha_\beta$. 

The $g \cdot f \cdot f$-manifold $(\mathbb{M}^{2n+r}, \Phi, \xi_1, \ldots, \xi_r, \eta^a)$, $a \in \{1, \ldots, r\}$, is said to be an indefinite metric $g \cdot f \cdot f$-manifold if $\Phi$ is a semi-Riemannian metric with index $\nu$ $(0 < \nu < 2n + r)$ satisfying the following compatibility condition:

$$
\tilde{g}(\tilde{\phi}X, \tilde{\phi}Y) = \tilde{g}(X, Y) - \sum_{a=1}^r \epsilon_a \tilde{\eta}^a(X) \tilde{\eta}^a(Y),
$$

(2.1)

for any $X, Y \in \Gamma(\mathbb{T}\mathbb{M})$, being $\epsilon_a = \pm 1$ according to whether $\xi_a$ is spacelike or timelike. Then, for any $a \in \{1, \ldots, r\}$, one has $\tilde{\eta}^a(X) = \epsilon_a \tilde{g}(X, \xi_a)$. Following the notations in [4, 5], we adopt the curvature tensor $\tilde{R}$, and thus we have $\tilde{R}(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$ and $\tilde{R}(X, Y, Z, W) = \tilde{g}(\tilde{R}(Z, W, Y), X)$, for any $X, Y, Z, W \in \Gamma(\mathbb{T}\mathbb{M})$.

We recall that, as proved in [6], the Levi-Civita connection $\nabla$ of an indefinite $g \cdot f \cdot f$-manifold satisfies the following formula:

$$
2\tilde{g}\left(\left(\nabla_X \tilde{\phi}\right)Y, Z\right) = 3d\Phi\left(X, \tilde{\phi}Y, \tilde{\phi}Z\right) - 3d\Phi(X, Y, Z)
$$

$$
+ \tilde{g}\left(N(Y, Z, \tilde{\phi}X) + \epsilon_a N_a^\tilde{\phi}(Y, Z)\tilde{\eta}^a(X)\right)
$$

$$
+ 2\epsilon_a d\tilde{\eta}^a\left(\tilde{\phi}Y, X\right)\tilde{\eta}^a(Z) - 2\epsilon_a d\tilde{\eta}^a\left(\tilde{\phi}Z, X\right)\tilde{\eta}^a(Y),
$$

(2.2)

where $N_a$ is given by $N_a^\tilde{\phi}(X, Y) = 2d\tilde{\eta}^a(\tilde{\phi}X, Y) - 2d\tilde{\eta}^a(\tilde{\phi}Y, X)$.

An indefinite metric $g \cdot f \cdot f$-manifold is called an indefinite $S$-manifold if it is normal and $d\tilde{\eta}^a = \Phi$, for any $a \in \{1, \ldots, r\}$, where $\Phi(X, Y) = \tilde{g}(X, \tilde{\phi}Y)$ for any $X, Y \in \Gamma(\mathbb{T}\mathbb{M})$. The normality condition is expressed by the vanishing of the tensor field $N = N_a^\tilde{\phi} + \sum_{a=1}^r 2d\tilde{\eta}^a \otimes \tilde{\eta}^a$, $N_a^\tilde{\phi}$ being the Nijenhuis torsion of $\tilde{\phi}$.

Furthermore, the Levi-Civita connection of an indefinite $S$-manifold satisfies

$$
\left(\nabla_X \tilde{\phi}\right)Y = \tilde{g}\left(\tilde{\phi}X, \tilde{\phi}Y\right)\tilde{\xi} + \tilde{\eta}(Y)\tilde{\phi}^2(X),
$$

(2.3)

where $\tilde{\xi} = \sum_{a=1}^r \tilde{\xi}_a$ and $\tilde{\eta} = \sum_{a=1}^r \epsilon_a \tilde{\eta}^a$. We recall that $\nabla_X \tilde{\xi}_a = -\epsilon_a \tilde{\phi}X$ and ker $\tilde{\phi}$ is an integrable flat distribution since $\nabla_X \tilde{\xi}_a = 0$ (see more details in [6]).

A plane section in $T_p\mathbb{M}$ is a $\tilde{\phi}$-holomorphic section if there exists a vector $X \in T_p\mathbb{M}$ orthogonal to $\tilde{\xi}_1, \ldots, \tilde{\xi}_r$ such that $\{X, \tilde{\phi}X\}$ span the section. The sectional curvature of a $\tilde{\phi}$-holomorphic section, denoted by $c(X) = R(X, \tilde{\phi}X, \tilde{\phi}X, X)$, is called a $\tilde{\phi}$-holomorphic sectional curvature.
Proposition 2.1 (see [7]). An indefinite Sasakian manifold \((\overline{M}^{2n+1}, \overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})\) has \(\overline{\phi}\)-sectional curvature \(c\) if and only if its curvature tensor verifies
\[
\overline{R}(X, Y) Z = \frac{(c + 3)}{4} \left[ \overline{g}(Y, Z) X - \overline{g}(X, Z) Y \right] \\
+ \frac{(c - 1)}{4} \left\{ \Phi(X, Z) \overline{\phi} Y - \Phi(Y, Z) \overline{\phi} X + 2\Phi(X, Y) \overline{\phi} Z \right\} \\
- \overline{g}(Z, Y) \overline{\eta}(X) \overline{\xi} + \overline{g}(Z, X) \overline{\eta}(Y) \overline{\xi} - \overline{\eta}(Y) \overline{\eta}(Z) X + \overline{\eta}(Z) \overline{\eta}(X) Y \right\}
\]
(2.4)

for any vector fields \(X, Y, Z, W \in \Gamma(\overline{T\overline{M}})\).

A Sasakian manifold \(\overline{M}^{2n+1}\) with constant \(\overline{\phi}\)-sectional curvature \(c \in \mathbb{R}\) is called a Sasakian space form, denoted by \(\overline{M}^{2n+1}(c)\).

Definition 2.2. An almost contact metric manifold \((\overline{M}^{2n+1}, \overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})\) is an indefinite generalized Sasakian space form, denoted by \(\overline{M}^{2n+1}(f_1, f_2, f_3)\), if it admits three smooth functions \(f_1, f_2, f_3\) such that its curvature tensor field verifies
\[
\overline{R}(X, Y) Z = f_1 \left[ \overline{g}(Y, Z) X - \overline{g}(X, Z) Y \right] \\
+ f_2 \left\{ \Phi(X, Z) \overline{\phi} Y - \Phi(Y, Z) \overline{\phi} X + 2\Phi(X, Y) \overline{\phi} Z \right\} \\
+ f_3 \left\{ -\overline{g}(Z, Y) \overline{\eta}(X) \overline{\xi} + \overline{g}(Z, X) \overline{\eta}(Y) \overline{\xi} \right. \\
- \left. \overline{\eta}(Y) \overline{\eta}(Z) X + \overline{\eta}(Z) \overline{\eta}(X) Y \right\}
\]
(2.5)

for any vector fields \(X, Y, Z, W \in \Gamma(\overline{T\overline{M}})\).

Remark 2.3. Any indefinite generalized Sasakian space form has \(\overline{\phi}\)-sectional curvature \(c = f_1 + 3f_2\). Indeed, \(f_1 = (c + 3)/4\) and \(f_2 = f_3 = (c - 1)/4\).

Proposition 2.4 (see [6]). An indefinite \(S\)-manifold \(\overline{M}^{2n+1}\) has \(\overline{\phi}\)-sectional curvature \(c\) if and only if its curvature tensor verifies
\[
\overline{R}(X, Y) Z = \frac{(c + 3e)}{4} \left\{ \overline{g}(\overline{\phi} X, \overline{\phi} Z) \overline{\phi}^2 Y - \overline{g}(\overline{\phi} Y, \overline{\phi} Z) \overline{\phi}^2 X \right\} \\
+ \frac{(c - e)}{4} \left\{ \Phi(Z, Y) \overline{\phi} X - \Phi(Z, X) \overline{\phi} Y + 2\Phi(X, Y) \overline{\phi} Z \right\} \\
+ \left\{ \overline{\eta}(Z) \overline{\eta}(X) \overline{\phi}^2 Y - \overline{\eta}(Y) \overline{\eta}(Z) \overline{\phi}^2 X + \overline{g}(\overline{\phi} Z, \overline{\phi} Y) \overline{\eta}(X) \overline{\xi} - \overline{g}(\overline{\phi} Z, \overline{\phi} X) \overline{\eta}(Y) \overline{\xi} \right\}
\]
(2.6)

for any vector fields \(X, Y, Z, W \in \Gamma(\overline{T\overline{M}})\) and \(e = \sum \epsilon_a\).
An indefinite $S$-manifold $\mathbf{M}^{2n+r}$ with constant $\phi$-sectional curvature $c \in \mathbb{R}$ is called a $S$-space form, denoted by $\mathbf{M}^{2n+r}(c)$. One remarks that for $r = 1$ (2.6) reduces to (2.4).

3. An Indefinite Generalized $g \cdot f \cdot f$-Manifold

Let $\mathcal{F}$ denote any set of smooth functions $F_{ij}$ on $\mathbf{M}^{2n+r}$ such that $F_{ij} = F_{ji}$ for any $i, j \in \{1, \ldots, r\}$.

Definition 3.1. An indefinite generalized $g \cdot f \cdot f$-space-form, denoted by $(\mathbf{M}^{2n+r}, F_1, F_2, \mathcal{F})$, is an indefinite $g \cdot f \cdot f$-manifold $(\mathbf{M}^{2n+r}, \phi, s, \eta^a, g)$ which admits smooth function $F_1, F_2, \mathcal{F}$ such that its curvature tensor field verifies

\[
\overline{\mathcal{R}}(X,Y)Z = F_1 \left\{ \overline{g}(\phi X, \phi Z) \phi^2 Y - \overline{g}(\phi Y, \phi Z) \phi^2 X \right\}
+ F_2 \left\{ \Phi(Z,Y)\phi X - \Phi(Z,X)\phi Y + 2\Phi(X,Y)\phi Z \right\}
+ \sum_{\alpha, \beta=1}^r F_{\alpha \beta} \left\{ \eta^\alpha(X) \eta^\beta(Y) Y - \eta^\beta(Y) \eta^\alpha(Z) Y - \overline{g}(\phi Z, \phi Y) \eta^\alpha(X) \phi \eta^\beta(Y) \phi - \overline{g}(\phi Z, \phi X) \eta^\alpha(Y) \phi \eta^\beta(Y) \phi \right\}
\]  

(3.1)

for any vector fields $X, Y, Z, W \in \Gamma(T\mathbf{M})$.

For $r = 1$, we obtain an indefinite Sasakian space form $\mathbf{M}^{2n+1}(f_1, f_2, f_3)$ with $f_1 = F_1$, $f_2 = F_2$, and $f_3 = F_1 - F_{11}$. In particular, if the given structure is Sasakian, (3.1) holds with $F_{11} = 1, F_1 = (c + 3)/4, F_2 = (c - 1)/4$, and $f_3 = F_1 - F_{11} = (c - 1)/4 = f_2$.

Theorem 3.2. Let $(\mathbf{M}^{2n+r}, F_1, F_2, \mathcal{F})$ be an indefinite generalized $g \cdot f \cdot f$-space form. Then $\mathbf{M}^{2n+r}$ is of constant $\phi$-sectional curvature if and only if

\[
\overline{\mathcal{R}}(X, \phi X) X = \text{proportional to } \phi X
\]  

(3.2)

for every vector field $X$ such that $\overline{g}(X, \zeta_a) = 0$, for any $a \in \{1, \ldots, r\}$.

Proof. Let $(\mathbf{M}^{2n+r}, F_1, F_2, \mathcal{F})$ be an indefinite generalized $g \cdot f \cdot f$-space form. To prove the theorem for $n \geq 2$, we will consider cases when $n = 2$ and when $n > 2$, that is, when $n \geq 3$.

Case 1 ($\overline{g}(X, X) = \overline{g}(Y, Y)$). The proof is similar as given by Lee and Jin [8], so we drop the proof.

Case 2 ($\overline{g}(X, X) = -\overline{g}(Y, Y)$). Here, if $X$ is spacelike, then $Y$ is timelike or vice versa. First of all, assume that $\mathbf{M}$ is of constant $\phi$-holomorphic sectional curvature. Then (3.1) gives

\[
\overline{\mathcal{R}}(X, \phi X) = \{F_1 + 3F_2\} \phi X = c\phi X.
\]  

(3.3)
Conversely, let \( \{X, Y\} \) be an orthonormal pair of tangent vectors such that \( \overline{g} (\phi X, Y) = \overline{g} (X, Y) = \overline{g} (Y, X_{\alpha}) = 0, \alpha \in \{1, \ldots, r\} \), and \( n \geq 3 \). Then \( X = (X+iY)/\sqrt{2} \) and \( Y = (i\bar{\phi} X + \phi Y)/\sqrt{2} \) also form an orthonormal pair of tangent vectors such that \( \overline{g} (\bar{\phi} X, Y) = 0 \). Then (3.1) and curvature properties give

\[
0 = \overline{R} \left( X, \bar{\phi} X, \bar{\phi} Y, \bar{\phi} X \right) \\
= \overline{g} \left( \overline{R} (X, \bar{\phi} X) X, \bar{\phi} X \right) - \overline{g} \left( \overline{R} (Y, \bar{\phi} Y) Y, \bar{\phi} Y \right) \\
- 2\overline{g} \left( \overline{R} (X, \bar{\phi} Y) Y, \bar{\phi} Y \right) + 2\overline{g} \left( \overline{R} (X, \bar{\phi} X) Y, \bar{\phi} X \right).
\] 

(3.4)

From the assumption, we see that the last two terms of the right-hand side vanish. Therefore, we get \( c(X) = c(Y) \).

Now, if \( \text{span} \{U, V\} \) is \( \bar{\phi} \)-holomorphic, then for \( \bar{\phi} U = aU + bV \), where \( a \) and \( b \) are constant, we have

\[
\text{span} \left\{ U, \bar{\phi} U \right\} = \text{span} \{U, aU + bV\} = \text{span} \{U, V\}.
\] 

(3.5)

Similarly,

\[
\text{span} \left\{ V, \bar{\phi} V \right\} = \text{span} \{U, V\}, \quad \text{span} \left\{ U, \bar{\phi} U \right\} = \text{span} \{V, \bar{\phi} V\}.
\] 

(3.6)

These imply

\[
\overline{R} \left( U, \bar{\phi} U, U, \bar{\phi} U \right) = \overline{R} \left( V, \bar{\phi} V, V, \bar{\phi} V \right), \quad \text{or} \quad c(U) = c(V).
\] 

(3.7)

If \( \text{span} \{U, V\} \) is not \( \bar{\phi} \)-holomorphic section, then we can choose unit vectors \( X \in \text{span} \{U, \bar{\phi} U\} \) and \( Y \in \text{span} \{V, \bar{\phi} V\} \) such that \( \text{span} \{X, Y\} \) is \( \bar{\phi} \)-holomorphic. Thus we get

\[
c(U) = c(X) = c(Y) = c(V),
\] 

(3.8)

which shows that any \( \bar{\phi} \)-holomorphic section has the same \( \bar{\phi} \)-holomorphic sectional curvature.
Now, let \( n = 2 \), and let \( \{X, Y\} \) be a set of orthonormal vectors such that \( \overline{g}(X, X) = -\overline{g}(Y, Y) \) and \( \overline{g}(X, \phi X) = 0 \), and we have \( c(X) = c(Y) \) as before. Using the property (3.2), we get

\[
\overline{R}(X, \phi X)X = -\{F_1 + 3F_2\} \phi X = -c(X)\phi X, \\
\overline{R}(X, \phi X)Y = -2F_2\phi Y, \\
\overline{R}(X, \phi Y)X = -F_1\phi Y, \\
\overline{R}(X, \phi Y)Y = F_2\phi X, \\
\overline{R}(Y, \phi X)Y = F_1\phi X, \\
\overline{R}(Y, \phi Y)X = -F_2\phi Y, \\
\overline{R}(Y, \phi Y)Y = 2F_2\phi X, \\
\overline{R}(Y, \phi Y)Y = \{F_1 + 3F_2\} \phi = c(Y)\phi Y = c(X)\phi Y. 
\]

Now, define \( \tilde{X} = aX + bY \) such that \( a^2 - b^2 = 1 \) and \( a^2 \neq b^2 \). Using the above relations, we get

\[
R\left(\tilde{X}, \phi \tilde{X}\right)\tilde{X} = C_1\phi X + C_2\phi Y. \tag{3.10}
\]

Therefore, we have

\[
C_1 = -a^3 c(X) + ab^2 c(X), \\
C_2 = b^3 c(X) - a^2 bc(X). \tag{3.11}
\]

On the other hand,

\[
\overline{R}(\tilde{X}, \phi \tilde{X})\tilde{X} = c\left(\tilde{X}\right)\phi \tilde{X} = c\left(\tilde{X}\right)\left\{a\phi X + b\phi Y\right\}. \tag{3.12}
\]

Comparing (3.11) and (3.12), we get

\[
-a^2 c(X) + b^2 c(X) = c\left(\tilde{X}\right), \\
b^2 c(X) - a^2 c(X) = c\left(\tilde{X}\right). \tag{3.13}
\]
On solving (3.13), we have

\[ c(X) = c(\tilde{X}). \]  

(3.14)

Similarly, we can prove

\[ c(Y) = c(\tilde{Y}). \]  

(3.15)

Therefore, $\overline{M}$ has constant $\overline{\phi}$-holomorphic sectional curvature.

**Case 3 ($\overline{g}(\mathcal{U}, \mathcal{U}) = 0$).** It is enough to show a sufficient condition. Let $Y_{\alpha}$ be a unit vector tangent to $\xi_{\alpha}$, for any $\alpha \in \{1, \ldots, r\}$, such that $\overline{g}(Y_{\alpha}, Y_{\alpha}) = -\overline{g}(\xi_{\alpha}, \xi_{\alpha}) = -\epsilon_{\alpha}$, and consider the null vector $U_{\alpha} = \xi_{\alpha} + Y$. From (3.2),

\[
c(U_{\alpha}) \overline{\phi} U_{\alpha} = c(U_{\alpha}) \overline{\phi}(\xi_{\alpha} + Y_{\alpha})
= \overline{R}(\xi_{\alpha} + Y_{\alpha}, \overline{\phi}(\xi_{\alpha} + Y_{\alpha}))(\xi_{\alpha} + Y_{\alpha}).
\]

(3.16)

Therefore,

\[
c(U_{\alpha}) = \overline{\phi}(U_{\alpha}) c(U_{\alpha}) (\xi_{\alpha} + Y_{\alpha}),
= \epsilon_{\alpha} \overline{g}(\overline{\phi}(Y_{\alpha}), Y_{\alpha}).
\]

(3.17)

From Cases 1 and 2, depending on the sign of $\epsilon_{\alpha}$,$\overline{g}(\overline{R}(Y_{\alpha}, \overline{\phi}Y_{\alpha})Y_{\alpha}, \overline{\phi}Y_{\alpha}) = \epsilon_{\alpha} c(Y_{\alpha})$ is constant, and hence $c(U_{\alpha}) = c(Y_{\alpha})$ is constant.
Theorem 3.3 (see [9]). Let \((\overline{M}^{2n+r},\overline{\phi},\overline{\eta},\overline{\xi},\overline{\xi})(n \geq 2)\) be an indefinite \(S\)-manifold. Then \(M^{2n+r}\) is of constant \(\phi\)-sectional curvature if and only if
\[
R\left(X,\overline{\phi}X\right)X \text{ is proportional to } \overline{\phi}X
\] (3.18)
for every vector field \(X\) such that \(g(X,\overline{\xi}_\alpha) = 0\), for any \(\alpha \in \{1,\ldots,r\}\).

Proof. An \(S\)-space form is a special case of \(g \cdot f \cdot f\)-space form, and hence the proof follows from Theorem 3.2 and (2.6).

Theorem 3.4 (cf. Bonome et al. [3]). Let \((M^{2n+1},\phi,\eta,\xi,g)(n \geq 2)\) be an indefinite Sasakian manifold. Then \(M^{2n+1}\) is of constant \(\phi\)-sectional curvature if and only if
\[
R\left(X,\phi X\right)X \text{ is proportional to } \phi X
\] (3.19)
for every vector field \(X\) such that \(g(X,\xi) = 0\).

Proof. When \(r = 1\), an indefinite \(S\)-space form \(M^{2n+1}(c)\) reduces to a Sasakian space form. The proof follows from (2.4) and Theorem 3.3.

References

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