Research Article

On Boundedness of Solutions of the Difference Equation \( x_{n+1} = \frac{px_n + qx_{n-1}}{1 + x_n} \) for \( q > 1 + p > 1 \)

Hongjian Xi, 1, 2 Taixiang Sun, 1 Weiyong Yu, 1 and Jinfeng Zhao 1

1 Department of Mathematics, Guangxi University, Nanning, Guangxi 530004, China
2 Department of Mathematics, Guangxi College of Finance and Economics, Nanning, Guangxi 530003, China

Correspondence should be addressed to Taixiang Sun, stx1963@163.com

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We study the boundedness of the difference equation \( x_{n+1} = \frac{px_n + qx_{n-1}}{1 + x_n} \) for \( q > 1 + p > 1 \) and the initial values \( x_{-1}, x_0 \in (0, +\infty) \). We show that the solution \( \{x_n\}_{n=-1}^\infty \) of this equation converges to \( \mathcal{X} = q + p - 1 \) if \( x_n \geq \mathcal{X} \) or \( x_n \leq \mathcal{X} \) for all \( n \geq -1 \); otherwise \( \{x_n\}_{n=-1}^\infty \) is unbounded. Besides, we obtain the set of all initial values \( (x_{-1}, x_0) \in (0, +\infty) \times (0, +\infty) \) such that the positive solutions \( \{x_n\}_{n=-1}^\infty \) of this equation are bounded, which answers the open problem 6.10.12 proposed by Kulenović and Ladas (2002). 洞

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1. Introduction

In this paper, we study the following difference equation:

\[ x_{n+1} = \frac{px_n + qx_{n-1}}{1 + x_n}, \quad n = 0, 1, \ldots, \]

where \( p, q \in (0, +\infty) \) with \( q > 1 + p \) and the initial values \( x_{-1}, x_0 \in (0, +\infty) \).

The global behavior of (1.1) for the case \( p + q < 1 \) is certainly folklore. It can be found, for example, in [1] (see also a precise result in [2]).

The global stability of (1.1) for the case \( p + q = 1 \) follows from the main result in [3] (see also Lemma 1 in Stević’s paper [4]). Some generalizations of Copson’s result can be found, for example, in papers [5–8]. Some more sophisticated results, such as finding the asymptotic behavior of solutions of (1.1) for the case \( p + q = 1 \) (even when \( p = 0 \)) can be found, for
In this section, let $q = 1 + p$ was treated for the first time by Stević’s in paper [12]. The main trick from [12] has been later used with a success for many times; see, for example, [13–15]. Some existing results for (1.1) are summarized as follows[16].

**Theorem A.** (1) If $p + q \leq 1$, then the zero equilibrium of (1.1) is globally asymptotically stable.

(2) If $q = 1$, then the equilibrium $\bar{x} = p$ of (1.1) is globally asymptotically stable.

(3) If $1 < q < 1 + p$, then every positive solution of (1.1) converges to the positive equilibrium $\bar{x} = p + q - 1$.

(4) If $q = 1 + p$, then every positive solution of (1.1) converges to a period-two solution.

(5) If $q > 1 + p$, then (1.1) has unbounded solutions.

In [16], Kulenović and Ladas proposed the following open problem.

**Open problem B (see Open problem 6.10.12of [16])**

Assume that $q \in (1, +\infty)$.

(a) Find the set $B$ of all initial conditions $(x_{-1}, x_0) \in (0, +\infty) \times (0, +\infty)$ such that the solutions $\{x_n\}_{n=-1}^{\infty}$ of (1.1) are bounded.

(b) Let $(x_{-1}, x_0) \in B$. Investigate the asymptotic behavior of $\{x_n\}_{n=-1}^{\infty}$.

In this paper, we will obtain the following results: let $p, q \in (0, +\infty)$ with $q > 1 + p$, and let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of (1.1) with the initial values $(x_{-1}, x_0) \in (0, +\infty) \times (0, +\infty)$. If $x_n \geq \bar{x}$ for all $n \geq -1$ (or $x_n \leq \bar{x}$ for all $n \geq -1$), then $\{x_n\}_{n=-1}^{\infty}$ converges to $\bar{x} = q + p - 1$. Otherwise $\{x_n\}_{n=-1}^{\infty}$ is unbounded.

For closely related results see [17–34].

### 2. Some Definitions and Lemmas

In this section, let $q > 1 + p > 1$ and $\bar{x} = q + p - 1$ be the positive equilibrium of (1.1). Write $D = (0, +\infty) \times (0, +\infty)$ and define $f : D \to D$ by, for all $(x, y) \in D$,

$$f(x, y) = \left(y, \frac{py + qx}{1 + y}\right).$$

(2.1)

It is easy to see that if $\{x_n\}_{n=-1}^{\infty}$ is a solution of (1.1), then $f^n(x_{-1}, x_0) = (x_{n-1}, x_n)$ for any $n \geq 0$. Let

$$A_1 = (0, \bar{x}) \times (0, \bar{x}), \quad A_2 = (\bar{x}, +\infty) \times (\bar{x}, +\infty),$$

$$A_3 = (0, \bar{x}) \times (\bar{x}, +\infty), \quad A_4 = (\bar{x}, +\infty) \times (0, \bar{x}),$$

$$R_0 = [\bar{x}] \times (0, \bar{x}), \quad L_0 = [\bar{x}] \times (\bar{x}, +\infty),$$

$$R_1 = (0, \bar{x}) \times [\bar{x}], \quad L_1 = (\bar{x}, +\infty) \times [\bar{x}].$$

(2.2)
Lemma 2.1. The following statements are true.

1. \( f : D \to f(D) \) is a homeomorphism.
2. \( f(L_1) = L_0 \) and \( f(L_0) \subset A_4 \).
3. \( f(R_1) = \{x\} \times (p, +\infty) \) and \( f(R_0) \subset A_3 \).
4. \( f(A_3) \subset A_4 \) and \( f(A_4) \subset A_3 \).
5. \( A_2 \cup L_1 \subset f(A_2) \subset A_2 \cup L_1 \cup A_4 \) and \( A_1 \cup R_1 \subset f(A_1) \subset A_1 \cup R_1 \cup A_3 \).

Lemma 2.2. Let \( q > 1 + p > 1 \), and let \( \{x_n\}_{n=1}^\infty \) be a positive solution of (1.1).

1. If \( \lim_{n \to +\infty} x_{2n} = a \in (0, +\infty) \) and \( a \neq p \), then \( \lim_{n \to +\infty} x_{2n+1} = a = \bar{x} \).
2. If \( \lim_{n \to +\infty} x_{2n-1} = b \in (0, +\infty) \) and \( b \neq p \), then \( \lim_{n \to +\infty} x_{2n} = b = \bar{x} \).

Proof. We show only (1) because the proof of (2) follows from (1) by using the change \( y_n = x_{n-1} \) and the fact that (1) is autonomous. Since \( \lim_{n \to +\infty} x_{2n} = a \in (0, +\infty) \) and \( a \neq p \), by (1.1) we have

\[
\lim_{n \to +\infty} x_{2n+1} = \lim_{n \to +\infty} \frac{q x_{2n} - x_{2n+2}}{x_{2n+2} - p} = \frac{(q-1)a}{a-p}.
\]

Also it follows from (1.1) that

\[
a = \lim_{n \to +\infty} x_{2n} = \lim_{n \to +\infty} \frac{q x_{2n-1} - x_{2n+1}}{x_{2n+1} - p} = \frac{(q-1)^2 a}{(q-1)a - p(a-p)},
\]

from which we have \( a = \bar{x} \) and \( \lim_{n \to +\infty} x_{2n+1} = a = \bar{x} \). This completes the proof. \( \Box \)

Lemma 2.3. Let \( q > 1 + p > 1 \), and let \( \{x_n\}_{n=1}^\infty \) be a positive solution of (1.1) with the initial values \((x_{-1}, x_0) \in A_4\). If there exists some \( n \geq 0 \) such that \( x_{2n-1} \geq x_{2n+1} \), then \( x_{2n} \geq x_{2n+2} \).

Proof. Since \((x_{-1}, x_0) \in A_4\), it follows from Lemma 2.1 that \( (x_{2n-1}, x_{2n}) \in A_4 \) for any \( n \geq 0 \). Without loss of generality we may assume that \( n = 0 \), that is, \( x_{-1} \geq x_1 \). Now we show \( x_0 \geq x_2 \).

Suppose for the sake of contradiction that \( x_0 < x_2 \), then

\[
x_{-1} \geq x_1 = \frac{px_0 + qx_{-1}}{1 + x_0},
\]

\[
x_0 < x_2 = \frac{px_1 + qx_0}{1 + x_1}.
\]

By (2.5) we have

\[
x_0 \geq \frac{x_{-1}(q-1)}{x_{-1} - p},
\]
and by (2.6) we get

\[(q - 1 - p)x_0^2 + (p^2 + q - 1 - qx_{-1})x_0 + pqx_{-1} > 0. \tag{2.8}\]

**Claim 1.** If $x_{-1} \geq \bar{x}$, then

\[
(p^2 + q - 1 - qx_{-1})^2 - 4(q - 1 - p)pqx_{-1} \geq 0. \tag{2.9}
\]

**Proof of Claim 1**

Let $g(x) = (p^2 + q - 1 - qx)^2 - 4(q - 1 - p)pqx \ (x \geq \bar{x})$, then we have

\[
g'(x) = 2q\left(1 + qx - p^2 - q\right) - 4pq(q - 1 - p)
\geq 2q\left[(q - 1)^2 + p^2 + p(1 - q) + p\right]
= 2q\left[(q - 1)(q - p - 1) + p^2 + p\right]
> 0.
\]

Since $x_{-1} \geq \bar{x}$, it follows

\[
(p^2 + q - 1 - qx_{-1})^2 - 4(q - 1 - p)pqx_{-1}
\geq (q^2 + qp - 2q + 1 - p^2)^2 - 4(q - 1 - p)qp(q + p - 1)
= (q^2 - 2q + 1 - p^2)^2 + 2qp\left(q^2 - 2q + 1 - p^2\right)
+ (qp)^2 - 4\left(q^2 - 2q + 1 - p^2\right)pq
= (q^2 - 2q + 1 - p^2 - pq)^2
\geq 0.
\]

This completes the proof of Claim 1.

By (2.8), we have

\[
x_0 > \lambda_1 = \frac{(1 + qx_{-1} - p^2 - q) + \sqrt{(1 + qx_{-1} - p^2 - q)^2 - 4pq(q - 1 - p)x_{-1}}}{2(q - 1 - p)}. \tag{2.12}
\]
or

\[ x_0 < \lambda_2 = \frac{(1 + qx_{-1} - p^2 - q) - \sqrt{(1 + qx_{-1} - p^2 - q)^2 - 4pq(q - 1 - p)x_{-1}}}{2(q - 1 - p)}. \]  

(2.13)

Claim 2. We have

\[ \lambda_1 \geq \overline{x}, \]  

(2.14)

\[ \lambda_2 \leq \frac{x_{-1}(q - 1)}{x_{-1} - p}. \]  

(2.15)

Proof of Claim 2

Since

\[
\sqrt{[1 + q(q + p - 1) - p^2 - q]^2 - 4pq(q - 1 - p)(p + q - 1)}
\]

\[ = q^2 - p^2 - 2q + 1 - qp \]  

(2.16)

\[ = 2(q + p - 1)(q - 1 - p) - [1 + q(q + p - 1) - p^2 - q], \]

we have

\[
\lambda_1 = \frac{(1 + qx_{-1} - p^2 - q) + \sqrt{(1 + qx_{-1} - p^2 - q)^2 - 4pq(q - 1 - p)x_{-1}}}{2(q - 1 - p)}
\]

\[
\geq \frac{(1 + q\overline{x} - p^2 - q) + \sqrt{(1 + q\overline{x} - p^2 - q)^2 - 4pq(q - 1 - p)\overline{x}}}{2(q - 1 - p)}
\]

\[ = \frac{[1 + q(q + p - 1) - p^2 - q] + \sqrt{[1 + q(q + p - 1) - p^2 - q]^2 - 4pq(q - 1 - p)(p + q - 1)}}{2(q - 1 - p)}
\]

\[ \geq (q + p - 1) = \overline{x}. \]  

(2.17)

The proof of (2.14) is completed.

Now we show (2.15). Let

\[ h(x) = pq(x - p)^2 - (x - p)(q - 1)(1 + qx - p^2 - q) + (q - 1)^2(q - 1 - p)x. \]  

(2.18)
Note that $2pq - 2q(q - 1) < 0$; it follows that if $x \geq \bar{x}$, then

$$H'(x) = 2pq(x - p) - \left[(q - 1)(1 + qx - p^2 - q) + q(q - 1)(x - p) - (q - 1)^2(q - 1 - p)\right]$$

$$\leq 2pq(q - 1) - \left[(q - 1)(2pq - q - p^2 + q^2 - p)\right]$$

$$= (q - 1)(q + p)(p + 1 - q),$$

(2.19)

which implies that $h(x)$ is decreasing for $x \geq \bar{x}$. Since $x_{-1} \geq \bar{x}$ and

$$h(\bar{x}) = pq(q - 1)^2 - (q - 1)(q - 1)\left[1 + q(q + p - 1) - p^2 - q\right]$$

$$+ (q - 1)^2(q - 1 - p)(q + p - 1) = 0,$$

(2.20)

it follows that

$$h(x_{-1}) = pq(x_{-1} - p)^2 - (x_{-1} - p)(q - 1)\left(1 + qx_{-1} - p^2 - q\right)$$

$$+ (q - 1)^2(q - 1 - p)x_{-1} \leq h(\bar{x}) = 0.$$

Thus

$$\begin{align*}
(q - 1)^2\left[1 + qx_{-1} - p^2 - q\right]^2 - 4pq(q - 1 - p)x_{-1} \\
\geq 4p^2q^2(x_{-1} - p)^2 - 4pq(x_{-1} - p)(q - 1)\left(1 + qx_{-1} - p^2 - q\right) \\
+ (q - 1)^2\left(1 + qx_{-1} - p^2 - q\right)^2.
\end{align*}$$

(2.22)

This implies that

$$\begin{align*}
(q - 1)\sqrt{(1 + qx_{-1} - p^2 - q)^2 - 4pq(q - 1 - p)x_{-1}} \\
\geq 2pq(x_{-1} - p) - (q - 1)\left(1 + qx_{-1} - p^2 - q\right).
\end{align*}$$

(2.23)

Finally we have

$$\begin{align*}
\frac{x_{-1}(q - 1)}{x_{-1} - p} &\geq \frac{4(q - 1 - p)pqx_{-1}}{2(q - 1 - p)\left[1 + qx_{-1} - p^2 - q\right] + \sqrt{(1 + qx_{-1} - p^2 - q)^2 - 4pq(q - 1 - p)x_{-1}}} \\
&= \frac{1 + qx_{-1} - p^2 - q - \sqrt{(1 + qx_{-1} - p^2 - q)^2 - 4pq(q - 1 - p)x_{-1}}}{2(q - 1 - p)} = \lambda_2.
\end{align*}$$

(2.24)

The proof of (2.15) is completed.
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Note that $x_0 < \bar{x}$ since $(x_-, x_0) \in A_1$. By (2.12), (2.13), (2.14), and (2.15), we see $x_0 < x_-(q-1)/(x_- - p)$, which contradicts to (2.7). The proof of Lemma 2.3 is completed. □

3. Main Results

In this section, we investigate the boundedness of solutions of (1.1). Let $q > 1 + p > 1$, and let $\{x_n\}_{n=1}^{\infty}$ be a positive solution of (1.1) with the initial values $(x_-, x_0) \in (0, +\infty) \times (0, +\infty)$, then we see that $(x_{n+1} - \bar{x})(x_n - \bar{x}) < 0$ for some $n \geq -1$ or $x_n \geq \bar{x}$ for all $n \geq -1$ or $x_n \leq \bar{x}$ for all $n \geq -1$.

**Theorem 3.1.** Let $q > 1 + p > 1$, and let $\{x_n\}_{n=1}^{\infty}$ be a positive solution of (1.1) such that $x_n \geq \bar{x}$ for all $n \geq -1$ or $x_n \leq \bar{x}$ for all $n \geq -1$, then $\{x_n\}_{n=1}^{\infty}$ converges to $\bar{x} = q + p - 1$.

**Proof.**

**Case 1.** $0 < x_n \leq \bar{x}$ for any $n \geq -1$. If $0 < x_{2n} \leq q - 1$ for some $n$, then

$$x_{2n+1} - x_{2n-1} = \frac{px_{2n} + qx_{2n-1} - x_{2n-1} - x_{2n-1}x_{2n}}{1 + x_{2n}} > 0. \quad (3.1)$$

If $q - 1 < x_{2n} \leq \bar{x}$ for some $n$, then

$$\frac{px_{2n}}{x_{2n} - q + 1} \geq \frac{p\bar{x}}{\bar{x} - q + 1} = \frac{\bar{x}}{x_{2n-1}}, \quad (3.2)$$

which implies that $px_{2n} \geq x_{2n-1}(x_{2n} - q + 1)$ and

$$x_{2n+1} - x_{2n-1} = \frac{px_{2n} + qx_{2n-1} - x_{2n-1} - x_{2n-1}x_{2n}}{1 + x_{2n}} \geq 0. \quad (3.3)$$

Thus $\bar{x} \geq x_{2n+1} \geq x_{2n-1}$ for any $n \geq 0$. In similar fashion, we can show $\bar{x} \geq x_{2n+2} \geq x_{2n}$ for any $n \geq 0$. Let $\lim_{n \to +\infty} x_{2n+1} = a$ and $\lim_{n \to +\infty} x_{2n} = b$, then

$$a = \frac{pb + qa}{1 + b}, \quad b = \frac{pa + qb}{1 + a}, \quad (3.4)$$

which implies $a = b = \bar{x}$. 
Case 2. \( x_n \geq \overline{x} = p + q - 1 \) for any \( n \geq -1 \). Since \( f(x, y) = (py + qx)/(1 + y) \) \((x > p/q)\) is decreasing in \( y \), it follows that for any \( n \geq -1 \),

\[
x_{n+2} = \frac{px_{n+1} + qx_n}{1 + x_{n+1}} \leq \frac{px + qx_n}{1 + \overline{x}} \leq x_n.
\]

(3.5)

In similar fashion, we can show that \( \lim_{n \to +\infty} x_{2n+1} = \lim_{n \to +\infty} x_{2n} = \overline{x} \). This completes the proof.

\[\square\]

**Lemma 3.2** (see [20, Theorem 5]). Let \( I \) be a set, and let \( F : I \times I \to I \) be a function \( F(u, v) \) which decreases in \( u \) and increases in \( v \), then for every positive solution \( \{x_n\}_{n=-1}^{+\infty} \) of equation \( x_{n+1} = F(x_n, x_{n-1}) \), \( \{x_{2n}\}_{n=0}^{\infty} \) and \( \{x_{2n-1}\}_{n=0}^{\infty} \) do exactly one of the following.

1. They are both monotonically increasing.
2. They are both monotonically decreasing.
3. Eventually, one of them is monotonically increasing, and the other is monotonically decreasing.

**Remark 3.3.** Using arguments similar to ones in the proof of Lemma 3.2, Stević proved Theorem 2 in [25]. Beside this, this trick have been used by Stević in [18, 28, 29].

**Theorem 3.4.** Let \( q > 1 + p > 1 \), and let \( \{x_n\}_{n=-1}^{\infty} \) be a positive solution of (1.1) such that \((x_{n+1} - \overline{x})(x_n - \overline{x}) < 0 \) for some \( n \geq -1 \), then \( \{x_n\}_{n=-1}^{\infty} \) is unbounded.

**Proof.** We may assume without loss of generality that \((x_0 - \overline{x})(x_1 - \overline{x}) < 0 \) and \((x_1, x_0) \in A_4 \) (the proof for \((x_1, x_0) \in A_3 \) is similar). From Lemma 2.1 we see \((x_{2n-1}, x_{2n}) \in A_4 \) for all \( n \geq 0 \). If \( \{x_{2n}\}_{n=0}^{\infty} \) is eventually increasing, then it follows from Lemma 2.3 that \( \{x_{2n-1}\}_{n=0}^{\infty} \) is eventually increasing. Thus \( \lim_{n \to +\infty} x_{2n-1} = b > \overline{x} \) and \( \lim_{n \to +\infty} x_{2n} = a \leq \overline{x} \), it follows from Lemma 2.2 that \( b = \infty \).

If \( \{x_{2n}\}_{n=0}^{\infty} \) is not eventually increasing, then there exists some \( N \geq 0 \) such that

\[
x_{2N} \geq x_{2N+2} = \frac{px_{2N+1} + qx_{2N}}{1 + x_{2N+1}},
\]

from which we obtain \( x_{2N} \geq px_{2N+1}/(1 + x_{2N+1} - q) \geq p \), since \( x_{2N+1} \geq \overline{x} = p + q - 1 \) and \( q > 1 \).

Since \( f(y, x) = (py + qx)/(1 + y) = p + (qx - p)/(1 + y) \) \((x \geq p, y \geq p)\) is increasing in \( x \) and is decreasing in \( y \), we have that \( x_{2n} \geq p \) for any \( n \geq N \). It follows from Lemma 3.2 that \( \{x_{2n}\}_{n=0}^{\infty} \) is eventually decreasing. Thus \( \lim_{n \to +\infty} x_{2n} = a < \overline{x} \) and \( \lim_{n \to +\infty} x_{2n-1} = b \geq \overline{x} \). It follows from Lemma 2.2 that \( b = \infty \). This completes the proof.

\[\square\]

**By Theorems 3.1 and 3.4 we have the following.**

**Corollary 3.5.** Let \( q > 1 + p > 1 \), and let \( \{x_n\}_{n=-1}^{\infty} \) be a positive bounded solution of (1.1), then \( x_{n-1} \geq x_n \geq \overline{x} \) for all \( n \geq 0 \) or \( \overline{x} \geq x_n \geq x_{n-1} \) for all \( n \geq 0 \).
Now one can find out the set of all initial values \((x_{-1}, x_0) \in (0, +\infty) \times (0, +\infty)\) such that the positive solutions of (1.1) are bounded. Let \(P_0 = \overline{A_2}, Q_0 = \overline{A_1}\). For any \(n \geq 1\), let

\[
P_n = f^{-1}(P_{n-1}), \quad Q_n = f^{-1}(Q_{n-1}).
\]  

(3.7)

It follows from Lemma 2.1 that \(P_1 = f^{-1}(P_0) \subset P_0, \; Q_1 = f^{-1}(Q_0) \subset Q_0\), which implies

\[
P_n \subset P_{n-1}, \quad Q_n \subset Q_{n-1}
\]  

(3.8)

for any \(n \geq 1\).

Let \(S\) be the set of all initial values \((x_{-1}, x_0) \in (0, +\infty) \times (0, +\infty)\) such that the positive solutions \(\{x_n\}_{n=1}^{\infty}\) of (1.1) are bounded. Then we have the following theorem.

**Theorem 3.6.** \(S = \left[\bigcap_{n=0}^{\infty} Q_n \right] \cup \left[\bigcap_{n=0}^{\infty} P_n \right] \subset A_1 \cup A_2 \cup \{(\overline{x}, \overline{x})\} \).

**Proof.** Let \(\{x_n\}_{n=1}^{\infty}\) be a positive solution of (1.1) with the initial values \((x_{-1}, x_0) \in S\).

If \((x_{-1}, x_0) \in \bigcap_{n=0}^{\infty} Q_n\), then \(f^n(x_{-1}, x_0) = (x_{n-1}, x_n) \in \overline{A_1}\) for any \(n \geq 0\), which implies \(x_n \leq \overline{x}\) for any \(n \geq -1\). It follows from Theorem 3.1 that \(\lim_{n \to \infty} x_n = \overline{x}\).

If \((x_{-1}, x_0) \in \bigcap_{n=0}^{\infty} P_n\), then \(f^n(x_{-1}, x_0) = (x_{n-1}, x_n) \in \overline{A_2}\), which implies \(x_n \geq \overline{x}\) for any \(n \geq -1\). It follows from Theorem 3.1 that \(\lim_{n \to \infty} x_n = \overline{x}\).

Now assume that \(\{x_n\}_{n=1}^{\infty}\) is a positive solution of (1.1) with the initial values \((x_{-1}, x_0) \in D - S\).

If \((x_{-1}, x_0) \in A_3 \cup A_4 \cup L_0 \cup L_1 \cup R_0 \cup R_1\), then it follows from Lemma 2.1 that \(f^2(x_{-1}, x_0) = (x_1, x_2) \in \{(x, y) : (x - \overline{x})(y - \overline{x}) < 0\}\), which along with Theorem 3.4 implies that \(\{x_n\}\) is unbounded.

If \((x_{-1}, x_0) \in \overline{A_2} - \bigcap_{n=0}^{\infty} P_n\), then there exists \(n \geq 0\) such that \((x_{-1}, x_0) \in P_n - P_{n+1} = f^{-n}(A_2) - f^{-n-1}(A_2)\). Thus \(f^n(x_{-1}, x_0) = (x_{n-1}, x_n) \in \overline{A_2} - f^{-1}(A_2)\). By Lemma 2.1, we obtain \(f^{n+1}(x_{-1}, x_0) \in L_1 \cup A_4\) and \(f^{n+1}(x_{-1}, x_0) = (x_{n+2}, x_{n+3}) \in A_4\), which along with Theorem 3.4 implies that \(\{x_n\}\) is unbounded.

If \((x_{-1}, x_0) \in \overline{A_1} - \bigcap_{n=1}^{\infty} Q_n\), then there exists \(n \geq 0\) such that \((x_{-1}, x_0) \in Q_n - Q_{n+1} = Q_n - f^{-1}(Q_n)\) and \(f^n(x_{-1}, x_0) = (x_{n-1}, x_n) \in \overline{A_1} - f^{-1}(A_1)\). Again by Lemma 2.1 and Theorem 3.4, we have that \(\{x_n\}\) is unbounded. This completes the proof. \(\Box\)

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