Research Article

Results and Conjectures about Order $q$ Lyness’ Difference Equation $u_{n+q}u_n = a + u_{n+q-1} + \cdots + u_{n+1}$ in $\mathbb{R}^+*$, with a Particular Study of the Case $q = 3$

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We study order $q$ Lyness’ difference equation in $\mathbb{R}^*_+$: $u_{n+q}u_n = a + u_{n+q-1} + \cdots + u_{n+1}$, with $a > 0$ and the associated dynamical system $F_a$ in $\mathbb{R}^*_+$. We study its solutions (divergence, permanency, local stability of the equilibrium). We prove some results, about the first three invariant functions and the topological nature of the corresponding invariant sets, about the differential at the equilibrium, about the role of 2-periodic points when $q$ is odd, about the nonexistence of some minimal periods, and so forth and discuss some problems, related to the search of common period to all solutions, or to the second and third invariants. We look at the case $q = 3$ with new methods using new invariants for the map $F_a^3$ and state some conjectures on the associated dynamical system in $\mathbb{R}^*_+^3$ in more general cases.

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1. Introduction

We will study the global behavior of solutions of the order $q$ Lyness’ difference equation for $q \geq 3$ and $a > 0$:

$$u_{n+q} = \frac{a + u_{n+q-1} + \cdots + u_{n+1}}{u_n}, \quad u_0, u_1, \ldots, u_{q-1} > 0. \quad (1.1)$$

The associated dynamical system in $\mathbb{R}^*_+^q$ is given by the map $F_a$:

$$F_a : x = (x_1, \ldots, x_q) \mapsto F_a(x) = \left(\frac{a + x_1 + \cdots + x_{q-1}}{x_q}, x_1, x_2, \ldots, x_{q-1}\right). \quad (1.2)$$
The orbit of \( M_0 = (u_{q-1}, \ldots, u_1, u_0) \) under \( F_a \) is \( \{ M_0, M_1, \ldots, M_n, \ldots \} \) where the point \( M_n \) is \( M_n = (u_{n+q-1}, \ldots, u_{n+1}, u_n) \).

For \( q = 2 \), we have so the ordinary order 2 Lyness’ difference equation, which is studied in some previous papers; see, for example, [1, 7]. This equation was introduced in 1942 by Lyness, for the case \( a = 1 \), while he was working on a problem in number theory. This equation is also used in geometry and frieze patterns (see [3] for references). When \( q = 3 \), it is “Todd’s equation,” whose deep study for \( a \neq 1 \) is made in [4].

In Section 2 we study the elementary properties of the solutions of (1.1): divergence, permanent character, and local stability of the equilibrium. Then we look at the first invariant function for order \( q \) Lyness’ equation, and prove that the invariant manifolds associated to it are homeomorphic to the sphere \( S^{q-1} \).

In Section 3 we look at the second invariant for order \( q \) Lyness’ equation \((q \geq 3)\), prove results about it, and make some conjectures, that we prove for \( q = 3, q = 4, \) and \( q = 5 \).

In Sections 5 and 6 we study the eigenvalues and the global nature of the differential \( dF_a \) at the equilibrium, give some elementary results about periods of solutions of (1.1), and study the possible common periods to all solutions of (1.1).

In Section 7 we study the particular case \( q = 3 \). We use another approach than in [4] (which solves almost completely the case \( q = 3 \)), using new invariants of the map \( F_2^a \).

In Section 8 we make some general conjectures for the behaviour of the solutions of order \( q \) Lyness’ difference equation.

We have recently received the nice preprint [6], which studies also order \( q \) Lyness’ equation, with new and very interesting tools. Some points are common to this paper and to our one.

2. Elementary Properties, First Invariant Function, and Invariant Manifolds

We start with a classical result (see [3, 7, 8]).

**Lemma 2.1.** The function on \( \mathbb{R}_{+}^{q} \)

\[
G_a(x_1, x_2, \ldots, x_q) := \left(1 + \frac{1}{x_1}\right)\left(1 + \frac{1}{x_2}\right) \cdots \left(1 + \frac{1}{x_q}\right)(a + x_1 + x_2 + \cdots + x_q)
\]  

is invariant under the action of \( F_a \). So, the sequence

\[
G_a(M_n) = \left(1 + \frac{1}{u_{n+q-1}}\right)\left(1 + \frac{1}{u_{n+q-2}}\right) \cdots \left(1 + \frac{1}{u_n}\right)(a + u_{n+q-1} + u_{n+q-2} + \cdots + u_n)
\]

is constant.

Some properties of function \( G_a \) will be useful.
Lemma 2.2. (1) The quantity $G_a(x)$ tends to infinity when $x$ tends to the point at infinity of $\mathbb{R}^q_+$; that is, the sets $\{ x \mid G_a(x) \leq K \}$ are compact.

(2) $G_a$ has a unique critical point, at its absolute minimum, attained at the equilibrium of (1.1) $L := (\ell, \ell, \ldots, \ell)$, where $\ell$ is the unique positive solution of the equation

$$X^2 - (q - 1)X - a = 0.$$  

(2.3)

One has $\ell > q - 1$, and the minimum of $G_a$ is

$$K(a) := \left(1 + \frac{1}{\ell}\right)^q (a + q\ell) = \frac{(\ell + 1)^{q+1}}{\ell^{q-1}} > (q + \sqrt{a})^2.$$  

(2.4)

Proof. (1) On the closed sets $\{ x \mid G_a(x) \leq K \}$ of $\mathbb{R}^q_+$ we have $\sum_i x_i \leq K - a$ and $\forall i \ 1 + 1/x_i \leq K/a$, and these estimates define a compact set in $\mathbb{R}^q_+$.

(2) Subtracting the two equations $\partial G_a/\partial x_i = 0$ and $\partial G_a/\partial x_j = 0$ we have $(x_i - x_j)(x_i + x_j + 1) = 0$, and so all the $x_i$ are equal at a critical point. Hence, the equation $\partial G_a/\partial x_1 = 0$ gives $x_1^2 - (q - 1)x_1 - a = 0$, and then for all $i$ $x_i = \ell$. So point $L$ is the unique critical point of $G_a$, and then $G_a$ attains only its absolute minimum $K(a)$ at this point. Now, evaluation of this minimum is obvious.

So we can apply [17, Proposition 2.1], to give a synthesis of new and older results about the dynamical system $(F_a, \mathbb{R}^q_+)$ (see also [8]).

Proposition 2.3. (1) For $K > K(a)$, the sets

$$\Sigma_a(K) := \{ x \mid G_a(x) = K \}, \quad \tilde{\Sigma}_a(K) := \{ x \mid G_a(x) \leq K \}$$  

(2.5)

are compact in $\mathbb{R}^q_+$ and invariant by $F_a$. The sets $\Sigma_a(K)$ are the boundaries of the open sets $\{ x \mid G_a(x) < K \}$, which are connected and constitute a fundamental system of neighborhoods of $L$.

If $G_a(M_0) = K$, then the orbit of $M_0$ remains on $\Sigma_a(K)$ and so is bounded.

(2) If $M_0 \neq L$, then the sequence $(u_n)$ diverges, but for all $n$ $a/(K-a) \leq u_n \leq K - aK = G_a(M_0)$, $(u_n)$ is permanent.

(3) The equilibrium $L$ of $F_a$ is locally stable.

For $q = 2$, one knows that $\Sigma_a(K)$ is homeomorphic to a circle (see [1]). For $q \geq 3$ one can generalize this fact. First one proves a local result.

Proposition 2.4. If $K > K(a)$ is sufficiently near to $K(a)$, then the set $\tilde{\Sigma}_a(K)$ is starlike with respect to the point $L$, and a ray from $L$ cuts $\Sigma_a(K)$ at exactly one point. Hence $\tilde{\Sigma}_a(K)$ is homeomorphic to an euclidean ball, and $\Sigma_a(K)$ is homeomorphic to a sphere, if $K$ is sufficiently near to $K(a)$.
Proof. Put, for $\rho > 0$, $\rho < \ell$, and $\overrightarrow{u}$ an unitary vector of $\mathbb{R}^q$, $\delta \overrightarrow{u}(\rho) := \ln G_\alpha(L + \rho \overrightarrow{u})$. It is sufficient to prove the relation $\delta' \overrightarrow{u}(\rho) = \rho C(\overrightarrow{u}) + o(\rho)$, uniformly with respect to $\overrightarrow{u}$, with $C(\overrightarrow{u}) \geq D > 0$. In fact, this assertion will prove that $\delta' \overrightarrow{u}(\rho) > 0$ if $\rho$ is sufficiently small, with $\delta' \overrightarrow{u}(0) = 0$. Hence $\delta$ will be increasing on a neighborhood of 0 independent of $\overrightarrow{u}$, and the equation $\ln G_\alpha(L + \rho \overrightarrow{u}) = \ln K$ will have a unique solution $\rho$ if $K$ is sufficiently near to $K(\alpha)$, because the sets $\{x \mid G_\alpha(x) < K\}$ are fundamental neighborhoods of $L$.

But easy calculations give

$$
\delta' \overrightarrow{u}(\rho) = \sum_j u_j \left\{ \frac{1}{a + q\ell + \rho \sum_i u_i} - \frac{1}{(\ell + \rho u_j)^2 + \ell + \rho u_j} \right\}
$$

$$
= \frac{1}{a + q\ell} \sum_j u_j \left\{ \left( 1 - \frac{\rho \sum_i u_i}{a + q\ell} + o(\rho) \right) - \left( 1 - \frac{\rho(2\ell + 1)u_j}{a + q\ell} + o(\rho) \right) \right\}
$$

$$
= \frac{1}{(a + q\ell)^2} \left( 2\ell + 1 - \left( \sum_i u_i \right)^2 \right) \rho + o(\rho),
$$

(2.6)

where it is easy to see that $o(\rho)$ is uniform with respect to $\overrightarrow{u}$. The vector $\overrightarrow{u}$ is unitary, so $|\sum_i u_i| \leq \sqrt{q}$, and then $2\ell + 1 - (\sum_i u_i)^2 \geq q - 1 > 0$, and we can take $D = (q - 1)/(a + q\ell)^2$. Hence this proves the proposition.

In fact, the homeomorphism of $\Sigma_\alpha(K)$ with a sphere is true for every value of $K > K(\alpha)$.

**Theorem 2.5.** If $K > K(\alpha)$, then the invariant set $\Sigma_\alpha(K)$ is homeomorphic to the $(q - 1)$-dimensional sphere $S^{q-1}$.

**Proof.** First, we remark that the set $\Sigma_\alpha(K)$ is a compact $(q - 1)$-manifold in $\mathbb{R}^*_+^q$, because the function $G_\alpha$ has no critical point in $\mathbb{R}^*_+^q \setminus \{L\}$.

Now consider the function $s(x) := \sum_{j=1}^q x_j$. It is a $C^\infty$ function, and has exactly 2 critical points on $\Sigma_\alpha(K)$. In fact, if we write the equation $ds = \lambda dG_\alpha$, we find, with $p := \prod_{j=1}^q (1 + 1/x_j)$, the relations $x_j^2 + x_j = (a + s(x))/(-1 - p)$. By substracting, we get $(x_i - x_j)(x_i + x_j + 1) = 0$, and so at a critical point we would have for all $i$ $x_i = t$, for some $t > 0$. Of course, at a point $(t, t, \ldots, t)$ on $\Sigma_\alpha(K)$ one has, by symmetry, $ds = \lambda dG_\alpha$. So we have to find the intersections of $\Sigma_\alpha(K)$ with the diagonal, id est to solve the equation $\phi_\alpha(t) := (1 + 1/t)^q(a + q\ell t) = K$. But we have $\phi_\alpha'(t) = (q/t^2)(1 + 1/t)^{q-1}[t^2 - (q - 1)t - a]$. So $\phi_\alpha$ is decreasing on $[0, \ell]$ from $\infty$ to $K(\alpha)$ and increasing on $[\ell, \infty]$ from $K(\alpha)$ to $\infty$. Hence, for $K > K(\alpha)$, there are exactly two numbers $t_1$ and $t_2$, with $0 < t_1 < t_2$, such that $\phi_\alpha(t_i) = K$. Then, the two points $T_1 = (t_1, t_1, \ldots, t_1)$ and $T_2 = (t_2, t_2, \ldots, t_2)$ are the two critical points of the function $s$ on $\Sigma_\alpha(K)$.

Now, a classical result of differential topology due to G. Reeb says that if on a compact manifold there is a smooth function which has exactly two critical points, then this manifold is homeomorphic to a sphere (see [10, page 116], or [2, page 25]), and so $\Sigma_\alpha(K) \approx S^{q-1}$.

□
3. The Second Invariant of Order $q$ Lyness’ Equation

3.1. General Results

In [5] the authors found new invariants for order $q$ Lyness’ equation.

**Proposition 3.1** ([5]). If $q \geq 3$, the following function $H_a$ on $\mathbb{R}_+^q$,

$$H_a(x) = \frac{(1 + x_1 + x_2)(1 + x_2 + x_3) \cdots (1 + x_{q-1} + x_q)(a + x_1x_q + x_1 + x_2 + \cdots + x_q)}{x_1 \cdots x_q}$$

(3.1)

is invariant under the action of $F_a$.

It is proved in [5] that $H_a$ and $G_a$ are independent.

**Proposition 3.2.** (1) The quantity $H_a(x)$ tends to infinity when $x$ tends to the point at infinity of $\mathbb{R}_+^q$; that is, the sets $\{H_a(x) \leq M\}$ are compact.

(2) $H_a$ has a strict minimum at the point $L = (\ell, \ldots, \ell)$, whose value is

$$M(a) := (2\ell + 1)^q e^{1-q}.$$  

(3.2)

**Proof.** (1) We have to prove that for $M > 0$ the closed set $\bar{S}_a(M) := \{H_a(x) \leq M\}$ is compact. Let $h_a$ be the numerator in $H_a$. Every product $x_1x_2 \cdots x_{i-1}x_i^2x_{i+1} \cdots x_q$ for $i = 1, 2, \ldots, q$, appears in the left-hand member of relation $h_a(x_1, \ldots, x_q) \leq Mx_1 \cdots x_q$. So we have $x_1 \leq M$, and $\bar{S}_a(M)$ is bounded. One see that every product of $q - 1$ variables $x_i$ appears in the left-hand member of the inequality $h_a(x) \leq Mx_1 \cdots x_q$, with coefficient $a$. So we have $x_i \geq a/M$, and these inequalities imply the compactness of $\bar{S}_a(M)$.

(2) We have not succeed in proving that the only critical point of $H_a$ is $L$ (see Conjecture 1), which would give an easy proof of the second point of Proposition 3.2 (as for $G_a$), and so we give here a direct proof of this point, with inequalities.

(i) First, we have the inequality, with equality only if $x_1 + x_2 = x_2 + x_3 = \cdots = x_{q-1} + x_q$:

$$\prod_{1}^{q-1} (1 + x_j + x_{j+1}) \geq \left\{ \prod_{1}^{q-1} (x_j + x_{j+1})^{1/(q-1)} + 1 \right\}^{q-1}. $$  

(3.3)

This results from the inequality $\prod_{1}^{q}(u_j + 1)^{1/n} \geq (\prod_{1}^{q} u_j)^{1/n+1}$, with equality only when the $u_j > 0$ are all equal. This comes from the strict convexity of the function $v \mapsto \ln (e^v + 1)$.

(ii) Now we write $a + x_1x_q + x_1 + \cdots + x_q = a + x_1x_q + (x_1 + x_q)/2 + (1/2) \sum_{j=1}^{q-1} (x_j + x_{j+1})$. So we have

$$a + x_1x_q + x_1 + \cdots + x_q \geq a + x_1x_q + \frac{x_1 + x_q}{2} + \frac{q - 1}{2} \left[ \prod_{1}^{q-1} (x_j + x_{j+1}) \right]^{1/(q-1)}$$

(3.4)

with equality only if $x_1 + x_2 = \cdots = x_{q-1} + x_q$ (arithmetic-geometric inequality).
(iii) But for each \( j \in \{1, 2, \ldots, q-1\} \), we have \( x_j + x_{j+1} \geq 2(x_j \, x_{j+1})^{1/2} \) with equality only if \( x_j = x_{j+1} \). So relations (3.3) and (3.4) give easily

\[
\prod_{1}^{q-1}(1 + x_j + x_{j+1}) \geq \left( \frac{2^{q-1}}{\sqrt{x_1x_q}} \prod_{1}^{q} x_j \right)^{1/(q-1)} + 1 \]

(3.5)

with equality only if all the \( x_j \) are equal, and

\[
a + x_1x_q + x_1 + \cdots + x_q \geq a + x_1x_q + \sqrt{x_1x_q} + \frac{q-1}{2} \left[ \frac{2^{q-1}}{\sqrt{x_1x_q}} \prod_{1}^{q} x_j \right]^{1/(q-1)}
\]

(3.6)

with equality only if all the \( x_j \) are equal.

(iv) Now we put \( \sqrt{x_1x_q} = u \) and \( x_2x_3 \cdots x_{q-1} = t^{q-1} \). From (3.5) and (3.6) we obtain

\[
H_a(x_1, \ldots, x_q) \geq \psi(t, u) := \left( 2t \, u^{1/(q-1)} + 1 \right)^{q-1} \left( u^2 + u + a + (q-1)t \, u^{1/(q-1)} \right)^{1-q} u^{-2}
\]

(3.7)

with equality only if all \( x_j \) are equal.

(v) We have to find the minimum of \( \psi(t, u) \) on \( \mathbb{R}_+^2 \). We put \( s = t \, u^{1/(q-1)} \) and have

\[
\psi(t, u) = \tilde{\psi}(s, u) := \left( 2 + \frac{1}{s} \right)^{q-1} \left( u + \frac{a + (q-1)s}{u} + 1 \right).
\]

(3.8)

First, it is easy to see that if, for \( C > 0 \), \( \tilde{\psi}(s, u) \leq C \), then one has \( 0 < \alpha \leq s \leq \alpha' \) and \( 0 < \beta \leq u \leq \beta' \) for some positive constants \( \tilde{\psi} \) tends to infinity at the infinity point of \( \mathbb{R}_+^2 \). So it has a minimum, which is a critical point. At such a point, we have

\[
u^2 = a + (q-1)s, \quad s(2s+1) - \left( u^2 + u + a + (q-1)s \right) = 0.
\]

(3.9)

So we obtain \( u(2u+1) = s(2s+1) \), and then \( u = s \) and \( u^2 - (q-1)u - a = 0 \), that is, \( u = s = \ell \). At this point, we have \( \tilde{\psi}(s, u) = (2\ell + 1)^{q}/\ell^{q-1} = H_a(\ell, \ldots, \ell) \). This value is then the minimum of \( H_a \), and it can be attained only when the \( x_j \) are all equal, and then all equal to number \( \ell \).

\[\square\]

### 3.2. Conjectures about the Second Invariant

As we noticed before, the proof of Proposition 3.2 will be dramatically simpler if the following conjecture would be true.

**Conjecture 1.** The point \( L \) is the only critical point for \( H_a \).

We have a partial result in this direction, for a general type of invariants.
**Lemma 3.5.** A differentiable numerical function $R$ on $\mathbb{R}^{q\times q}$, invariant under the action of $F_a$, is said “diagonal” if it has the two properties:

(i) every critical point of $R$ with form $(t, t, t, \ldots, t)$ is the point $L$;
(ii) if $q$ is odd, every critical point of $R$ with form $(t, u, t, u, \ldots, u, t)$ is the point $L$.

**Proposition 3.4.** Let $R$ be a differentiable diagonal invariant, and suppose $q \geq 3$.

(a) If it exists some $j \in \{1, \ldots, q - 1\}$ such that every critical point of $R$ satisfies $x_j = x_{j+1}$, then $R$ has a unique critical point which is the point $L$.

(b) If the equalities $x_1 = x_q$ and $x_2 = x_{q-1}$ are true for every critical point of $R$, then $R$ has only one critical point which is the point $L$.

**Proof.** We will use the following easy lemma.

**Lemma 3.5.** If $M$ is a critical point of a differentiable invariant $R$, all the points $F^p_a(M)$ are also critical points of $R$ for $p \in \mathbb{Z}$.

This lemma results from the relations $R[F_a(M)] = R(M)$ and from the fact that $dF_a(M)$ and $dF^{-1}_a(M)$ are invertible, which is easy to see from (1.2).

Now, the maps $F_a$ and $F^{-1}_a$ act on last or first coordinates of $M$ as right or left shift ($j - 1$ time for $F_a$ and $q - j - 1$ times for $F^{-1}_a$), and the equality $x_j = x_{j+1}$ spreads right and left. Then, such a critical point is $(t, t, \ldots, t)$ for some $t > 0$. Because $R$ is diagonal, this point is $L$; this gives assertion (a) of the proposition.

In order to prove assertion (b), we suppose that for every critical point $M = (x_1, \ldots, x_q)$ we have $x_1 = x_q$ and $x_2 = x_{q-1}$.

We write $F_a(M) = M' = (x'_1, \ldots, x'_q) = (t + a_1 + \cdots + a_{q-1})/x_q, x_1, x_2, \ldots, x_{q-2}, x_{q-1})$.

$$F^{-1}_a(M) = M'' = (x''_1, \ldots, x''_q) = (x_2, x_3, \ldots, x_{q-1}, a + x_2 + \cdots + x_q)/x_1.$$ (3.10)

So, by Lemma 3.5 we have the implications, if $q \geq 4$, $\{x'_2 = x'_{q-1}\} \Rightarrow \{x_1 = x_{q-2}\}, \{x'_2 = x'_q\} \Rightarrow \{x_3 = x_q\}$, then, $\{x''_2 = x''_q\} \Rightarrow \{x_3 = x_{q-1}\}$. And then, $\{x''_1 = x''_{q-2}\} \Rightarrow \{x_2 = x_{q-1}\}$; also $\{x''_1 = x''_{q-2}\} \Rightarrow \{x_2 = x_{q-1}\}$.

At this stage, we have $x_1 = x_3 = x_{q-2} = x_q$ and $x_2 = x_4 = x_{q-3} = x_{q-1}$. If $q = 4$, all the $x_i$ have the same value. If $q > 4$, we can continue in this way and obtain by induction the following:

- (**) if $q$ is even, for all $i$, $x_{2i+1} = t$ and for all $j$, $x_{2j} = u$ for some positive $t$ and $u$.

If $q = 3$, the hypothesis said already that the critical point has form $(t, u, t)$.

In all cases, the diagonal character of $R$ gives the result.

Of course, for proving Conjecture 1 from Proposition 3.4 it suffices to see that $H_a$ is diagonal, and to prove the hypothesis $x_j = x_{j+1}$ or $\{x_1 = x_q$ and $x_2 = x_{q-1}\}$ for one critical point from the relations $\partial H_a/\partial x_k = 0$ only, because every critical point satisfies these equalities. In fact, this will be useful for all usual invariants.
Lemma 3.6. Invariants $G_a$, $H_a$ (for $q \geq 3$), and $J_a$ (for odd $q \geq 5$) are diagonal.

The third invariant $J_a$ is defined in Section 4.

Proof. It is very easy for $G_a$, and we will not use this result. For $H_a$, notice that for the critical point $(t, t, \ldots, t)$ the relation $\partial H_a/\partial x_1 = 0$ gives the equation $t(t+1)(2t+1) = (a+t^2+qt)(t+1)$, that is, $t = \ell$: this is property (i). If the critical point has the form $(t, u, t, u, \ldots, u, t)$, the same relation gives $(1+u)/(1+t+u) = (1+t)/(a+t^2+(m+1)t+mu)$, where $q = 2m+1$. But the relation $\partial H_a/\partial x_2 = 0$ gives $2/(1+t+u) - 1/u + 1/(a+t^2+(m+1)t+mu) = 0$. One deduce from these two relations the equality $u(1+u)/(1+t) = 1+t-u$, so we have $t = u$ and we apply property (i).

In order to prove (i) for $J_a$, the relation $\partial J_a/\partial x_1 = 0$ is $t/(1+t)(a+(q-1)t) = 1/(1+t)$, which gives $t = \ell$. Now suppose that a point $(t, u, t, u, \ldots, u, t)$ is critical. When we write that the two first partial derivatives are zero, we obtain $t/(1+t)(a+m(t+u)) = [u(1+u)]^m/[t(1+t)^{m+1}]$ and $(1+u)/u(a+1+(m+1)(t+u)) = [u(1+u)]^m/[t(1+t)^{m+1}]$. So we have $a+m(t+u) = tu$, and by putting this equality in the first previous relation, we get $(1+u)^m u^{m+1} = (1+t)^m u^{m+1}$, that is, $t = u$: we can apply property (i).

Conjecture 1 would have many consequences.

Definition 3.7. We put, for $M > M(a)$,

$$S_a(M) := \{x \mid H_a(x) = M\}, \quad \tilde{S}_a(M) := \{x \mid H_a(x) \leq M\}. \tag{3.11}$$

Obviously, these sets are compact and invariant by the action of $F_a$. From [9, Proposition 2.1], Conjecture 1 would imply other properties, which are assertions of Proposition 2.3 for the first invariant $G_a$ and are still conjectures for $H_a$.

Conjecture 2. The sets $S_a(M)$ are the boundaries of the open sets $\{x \mid H_a(x) < M\}$, which are connected.

Conjecture 1 would imply an important property of $S_a(M)$, but we have no proof of it.

Conjecture 3. The sets $S_a(M)$ are, for $M > M(a)$, smooth manifolds of $\mathbb{R}^q$.

Thus, we think that Theorem 2.5 is valid for $S_a$.

Conjecture 4. The invariant sets $S_a(M)$ are, for $M > M(a)$, homeomorphic to the $(q-1)$-dimensional sphere $\mathbb{S}^{q-1}$.

In fact, Conjecture 4 will be a consequence of Conjecture 1 and of a Conjecture 6 which will be introduced later. First, we prove a result similar to Proposition 2.4.

Proposition 3.8. Suppose $q = 3$. If $M > M(a)$ is sufficiently near to $M(a)$, then the set $\tilde{S}_a(M)$ is starlike with respect to the point $L$, and a ray from $L$ cuts $S_a(K)$ at exactly one point. Moreover, $S_a(M)$ is the boundary of $\tilde{S}_a(M)$ and of the set $\{H_a < M\}$ Hence $\tilde{S}_a(M)$ is homeomorphic to an euclidean ball, and $S_a(M)$ is homeomorphic to a sphere, if $M$ is sufficiently near to $M(a)$.

Proof. We start the proof for $q \geq 3$. Put, for $\rho > 0$, $\rho < \ell$, and $\vec{u}$ any unitary vector of $\mathbb{R}^q$, $\delta_\ell(\rho) := \text{Inf} H_a(L + \rho \vec{u})$. As in the proof of Proposition 2.4, it is sufficient to prove the relation $\lim_{\rho \to 0}(\delta_\ell(\rho)/\rho) \geq D > 0$, uniformly with respect to $\vec{u}$, in order to get the starlikeness of
(\tilde{S}_\varepsilon(M), and the fact that a ray from L cuts \( S_\varepsilon(M) \) at exactly a point. So it is easy to see that \( S_\varepsilon(M) \) is the boundary of the two sets given in the proposition, and then the existence of the announced homeomorphisms is easy to prove.

Easy calculations give for the value of \( \delta'_{n}(\rho) \): \( (2\rho u_1 u_q + \ell(u_1 + u_q) + \sum_{i=1}^{q} u_i) / (2\ell^2 + \ell + \rho(e^2 + \ell + u_1 + u_q) + \rho^2 u_1 u_q + \sum_{j=1}^{q-1} (u_j + u_{j+1}) / (1 + 2\ell + \rho(u_j + u_{j+1})) - \sum_{i=1}^{q} u_i / (\ell + \rho u_i) \). The asymptotic expansion of \( \delta'_{n}(\rho) \) at 0 has the form \( \rho A(\tilde{u}) + o(\rho) \), where \( o(\rho) \) is uniform with respect to \( \tilde{u} \), and where one has, with the relation \( \sum_{i=1}^{q} u_i^2 = 1 \),

\[
Q := \ell^2(2\ell + 1)^2 A = (2\ell + 1)^2 + 2\ell(2\ell + 1)u_1 u_q - \left[ \ell(u_1 + u_q) + \sum_{i=1}^{q} u_i \right]^2 - \ell^2 \sum_{j=1}^{q-1} (u_j + u_{j+1})^2
\]

\[
= 2\ell^2 + 4\ell - \left[ 2 \sum_{2 \leq i < k \leq q-1} u_i u_k + 2l(u_1 + u_q) \sum_{i=2}^{q-1} u_i + 2 \left\{ \ell^2 \left( u_1^2 + u_q^2 \right) - (\ell^2 - \ell - 1) u_1 u_q \right\} \right]
\]

\[
+ 2\ell^2 \sum_{j=1}^{q-1} u_j u_{j+1} \right].
\]

(3.12)

Now we suppose that \( q = 3 \). We have to find an upper bound on the unit sphere of the quantity in the square brackets. It is a quadratic form \( B \), and the eigenvalues of its matrix are \( \lambda_3 \). Hence the proposition results from the following easy lemma.

So it is clear from this proof that Proposition 3.8 would be true for every \( q \geq 4 \) if the following conjecture was true.

**Lemma 3.9.** If \( \ell \geq 2 \), one has

\[
2\ell^2 + 4\ell - \frac{-\ell^2 + 5\ell + 1 + \sqrt{(\ell^2 - 5\ell - 1)^2 + 32(\ell^2 + \ell + 1)^2}}{4} \geq 2
\]

(3.13)

**Conjecture 5.** For \( \ell > q - 1 \geq 3 \) the quadratic form on \( \mathbb{R}^q \)

\[
\left( 2\ell^2 + 4\ell \right) \sum_{i=1}^{q} x_i^2 - 2 \left[ \sum_{2 \leq i < k \leq q-1} u_i u_k + \ell(u_1 + u_q) \sum_{i=2}^{q-1} u_i \right]
\]

\[
+ \ell(u_1^2 + u_q^2) - (\ell^2 - \ell - 1) u_1 u_q + \ell^2 \sum_{j=1}^{q-1} u_j u_{j+1} \right]
\]

(3.14)

is positively defined.
Corollary 3.10. Conjectures 1 and 5 imply Conjecture 4 (and then all the four conjectures).

Proof. It will be an easy consequence of the classical following result in differential geometry (see [10, Theorem 50, pages 109-110]).

\textbf{Fact 1.} Let $V$ be a differential manifold, and $f : V \rightarrow \mathbb{R}$ a smooth function, such that

(i) $f$ has a unique critical point $a$, at its absolute minimum $m = f(a)$;

(ii) for every $\lambda, \mu$ such that $m \leq \lambda < \mu$, the subset $V_m^\lambda := \{x \mid \lambda \leq f(x) \leq \mu\}$ is compact.

Then there is a diffeomorphism of $V$ which maps $V_m^\lambda$ onto $V_m^\mu$.

We apply this fact with $V = \mathbb{R}^3_+$, $f = H_a$, $a = L_0$ and $m = M(a)$. If $M$ is sufficiently near to $M(a)$, $\overline{S}_a(M)$ is homeomorphic to the ball, and for $M' > M$ the set $\{M \leq H_a \leq M'\}$ is compact. So for every $M' > M$ the set $\overline{S}_a(M')$ is homeomorphic to a ball, and the set $S_a(M)$ is homeomorphic to the unit sphere $\mathbb{S}^{q-1}$; this is Conjecture 4.

Remark 3.11. Of course the previous corollary is true for the invariant $G_x$, and then gives an other proof of Theorem 2.5 which does not use Reeb’s theorem. The analogue of Conjecture 6 is in the proof of Proposition 2.4.

3.3. The Truth of the Four Conjectures for $q = 3$

Theorem 3.12. If $q = 3$, Conjectures 1, 2, 3, and 4 are true.

Proof. As we saw before, it is sufficient to prove Conjecture 1 (Conjecture 6 is true for $q = 3$). We have

$$H_a(x, y, z) = \frac{(1 + x + y)(1 + y + z)(a + xz + x + y + z)}{xyz}. \quad (3.15)$$

We write the two equalities $(H_a)_x = 0, (H_a)_y = 0$. Easy calculations give first the relations $x(z+1)(x+y+1) = (y+1)(a+zx+x+z+y)$ and $z(x+1)(z+y+1) = (y+1)(a+zx+x+z+y)$. So we get $x = z$. Now the third equation $(H_a)_z = 0$ is written as

$$2y(x + x^2 + 2x + y) = (x + y + 1)(a + x^2 + 2x), \quad (3.16)$$

and the two first equations reduced to the only one

$$x(x+1)(x+y+1) = (y+1)(a+x^2+2x+y). \quad (3.17)$$

We write (3.16) in the form $(y-x)(y+2x+a+1) - (x+1)(x^2-2x-a) = 0$ and (3.17) in the form $(y-x)(x^2+4x+2y+a) + (x^2-2x-a) = 0$. These two relations imply the equalities $x = y$ and $x^2-2x-a = 0$. So we obtain $x = y = z = \ell$.

Corollary 3.13. Let $q = 3$. If $G_x(M_0) = K$ and $H_x(M_0) = M$, then the orbit of $M_0$ remains on the invariant compact curve (part in $\mathbb{R}^3_+$ of the intersection of two regular surfaces)

$$C^+_a(K, M) := \Sigma_a(K) \cap S_a(M). \quad (3.18)$$
3.4. The Truth of the First Three Conjectures for $q = 4$

**Theorem 3.14.** If $q = 4$, then Conjectures 1, 2, and 3 are true.

**Proof.** It is sufficient to prove Conjecture 1. We have now

$$H_a(x, y, z, t) = \frac{(1 + x + y)(1 + y + z)(1 + z + t)(a + xt + x + y + z + t)}{xyzt}.$$ (3.19)

We will use Proposition 3.4; so it suffices to see that a critical point $(x, y, z, t)$ satisfies $y = z$ and $x = t$. We put $D := a + xt + x + y + z + t$. The logarithmic derivation gives for a critical point of $H_a$ the four equations (they exchange if we exchange $x$ and $t$, and $y$ and $z$):

$$\frac{\partial H_a}{\partial x} = 0 \quad \text{or} \quad \frac{1}{D} = \frac{1 + y}{x(1 + x + y)(1 + t)},$$ (3.20)

$$\frac{\partial H_a}{\partial t} = 0 \quad \text{or} \quad \frac{1}{D} = \frac{1 + z}{t(1 + t + z)(1 + x)},$$ (3.21)

$$\frac{\partial H_a}{\partial y} = 0 \quad \text{or} \quad \frac{1}{D} = \frac{(1 + x)(1 + z) - y^2}{y(1 + x + y)(1 + y + z)},$$ (3.22)

$$\frac{\partial H_a}{\partial z} = 0 \quad \text{or} \quad \frac{1}{D} = \frac{(1 + t)(1 + y) - z^2}{z(1 + t + z)(1 + y + z)}.$$ (3.23)

From these equations, we will prove two fundamental relations:

$$xz(z + 1) = ty(y + 1),$$ (3.24)

$$t^2(x + 1) = (z + 1)(a + x + y + z).$$ (3.25)

In order to prove (3.24) we make the ratio of relations (3.20) and (3.21), subtract (3.23) from (3.22), and make the ratio of the two results. The second one is a consequence of Lemma 3.5: the point $F_a(x, y, z, t) = ((a + x + y + z)/t, x, y, z)$ is a critical point, so its coordinates satisfy relation (3.24): $((a + x + y + z)/t)y(y + 1) = zx(x + 1)$. We use once more (3.24) for $(x, y, z, t)$ and obtain (3.25). Now we put $A := a + y + z$. Relation (3.25) becomes $t^2(x + 1) = (z + 1)(A + x)$. By exchange of $y$ and $z$ and of $x$ and $t$ we obtain also $x^2(t + 1) = (y + 1)(A + t)$. We substract these two relations and obtain

$$(t - x)(x + 1)(t + 1) = A(z - y) + xz - ty.$$ (3.26)

But we have $t - x = t(1 - x/t) = t(1 - y(y + 1)/z(z + 1)) = t(z - y)(y + z + 1)/z(z + 1)$, from relation (3.24), and $xz = ty(y + 1)/(z + 1)$, and then $xz - ty = (ty/(z + 1))(y - z)$. We put these relations in
(3.26) and obtain the factorization
\[
\frac{(z-y)}{z(z+1)} \left[ t(x+1)(t+1)(y+z+1)+tyz-Az(z+1) \right] = 0.
\]
But from the value of \( A \) and relation (3.25) we have \( A(z+1) = t^2(x+1) - x(z+1) \). We put these value of \( A(z+1) \) in the previous formula and obtain
\[
\frac{(z-y)}{z(z+1)} \left[ t(x+1)(t+1)(y+z+1)+tyz+xyz(z+1) - zt^2(x+1) \right] = 0. \tag{3.27}
\]

The development of the first term in square brackets in formula (3.27) contains the last term \( zt^2(x+1) \), and so the factor of \( z-y \) is strictly positive. So we obtain \( y = z \), but the relation which follows formula (3.26) gives \( x = t \), and this proves the proposition. \( \blacksquare \)

### 3.5. The Truth of the First Three Conjectures for \( q = 5 \)

In this section we prove the following result.

**Theorem 3.15.** If \( q = 5 \), then Conjectures 1, 2, and 3 are true.

**Proof.** As usual, it suffices to prove Conjecture 1. The proof follows the same line as for \( q = 4 \).

First we put \( D := a + x_1x_5 + x_1 + x_2 + x_3 + x_4 + x_5 \), and we write the five equations \( \partial H_a / \partial x_j = 0 \):

\[
\frac{1}{1 + x_1 + x_2} - \frac{1}{x_1} + \frac{1 + x_5}{D} = 0, \tag{3.28}
\]

\[
\frac{1}{1 + x_1 + x_2} + \frac{1}{1 + x_2 + x_3} - \frac{1}{x_2} + \frac{1}{D} = 0, \tag{3.29}
\]

\[
\frac{1}{1 + x_2 + x_3} + \frac{1}{1 + x_3 + x_4} - \frac{1}{x_3} + \frac{1}{D} = 0, \tag{3.30}
\]

\[
\frac{1}{1 + x_3 + x_4} + \frac{1}{1 + x_4 + x_5} - \frac{1}{x_4} + \frac{1}{D} = 0, \tag{3.31}
\]

\[
\frac{1}{1 + x_4 + x_5} - \frac{1}{x_5} + \frac{1 + x_1}{D} = 0. \tag{3.32}
\]

The difference of (3.28) and (3.29) gives \( 1/x_1 - x_5/D - (1 + x_3)/x_2(1 + x_2 + x_3) = 0 \), that is, with \( S := \sum_{j=1}^{5} x_j, (a + S)/D = x_1(1 + x_3)/x_2(1 + x_2 + x_3) \). But the left-hand member is invariant by exchange of \( x_1 \) with \( x_3 \) and of \( x_2 \) with \( x_4 \), and so we have \( x_1(1 + x_3)/x_2(1 + x_2 + x_3) = x_5(1 + x_3)/x_4(1 + x_4 + x_3) \). So we obtain the fundamental relation, analogous to (3.24) in case \( q = 4 \),

\[
x_1x_4(1 + x_3 + x_4) = x_5x_2(1 + x_3 + x_2). \tag{3.33}
\]

We have three useful consequences of (3.33). By writing from (3.33) the ratio \( x_2x_5/x_1x_4 \), we obtain

\[
x_1x_4 - x_2x_5 = (x_2 - x_4) \frac{x_1x_4}{1 + x_2 + x_3}. \tag{3.34}
\]
By writing \( x_1 - x_5 = x_1(1 - x_5/x_1) \) and using (3.33) for \( x_5/x_1 \), we have also
\[
x_1 - x_5 = (x_2 - x_4) \frac{x_1(1 + x_2 + x_3 + x_4)}{x_2(1 + x_2 + x_3)}.
\] (3.35)

Finally, we write \( x_2x_1 - x_4x_5 = x_2x_1(1 - (x_4/x_2)(x_5/x_1)) \) and put in this relation the ratio \( x_5/x_1 \) from relation (3.33); we obtain after easy calculations
\[
x_2x_1 - x_4x_5 = (x_2 - x_4) \frac{x_1}{x_2} \frac{(1 + x_3)(x_2 + x_4) + x_2^2 + x_2x_4 + x_4^2}{1 + x_2 + x_3}.
\] (3.36)

The essential fact in formulas (3.34), (3.35), and (3.36) is that the three factors of \( x_2 - x_4 \) are positive. Now (3.28) gives \( D = x_1(1 + x_5)(1 + x_1 + x_2)/(1 + x_2) \), and from invariance of the left-hand member by exchange of \( x_1 \) with \( x_5 \) and of \( x_2 \) with \( x_4 \) we get
\[
x_1(1 + x_4)(1 + x_3)(1 + x_1 + x_2) - x_5(1 + x_2)(1 + x_1)(1 + x_5 + x_4) = 0.
\] (3.37)

The goal is to make appear the three first member of (3.34), (3.35), and (3.36) in (3.37). We obtain \( (1 + x_1)(1 + x_3)\left([x_1 - x_3] + [x_1x_4 - x_2x_3]\right) + x_1x_2(1 + x_4)(1 + x_5) - x_5x_4(1 + x_1)(1 + x_2) = 0 \), or \( (1 + x_1)(1 + x_3)\left([x_1 - x_3] + [x_1x_4 - x_2x_3]\right) + [x_2x_1 - x_4x_5] + x_2x_4[x_1 - x_3] + x_1x_5(x_2 - x_4) = 0 \). From formulas (3.34), (3.35), and (3.36) we obtain in fine a relation \( (x_2 - x_4)A = 0 \), where \( A > 0 \). So if \( x \) is a critical point we have \( x_2 = x_4 \), and formula (3.35) gives \( x_1 = x_3 \).

Then it results from Proposition 3.4(b) that the only critical point is \( L \).

\[ \square \]

4. The Third Invariant of Order \( q \) Lyness’ Equation When \( q = 2m + 1 \geq 5 \) is Odd

In [5], the authors give a simple third invariant of order \( q \) Lyness’ equation for odd \( q \geq 5 \).

**Proposition 4.1** ([11]). When \( q = 2m + 1 \geq 5 \) is odd, the function
\[
J_a(x_1, x_2, \ldots, x_q) := \frac{a + \sum_{i=1}^{q} x_i}{\prod_{j=1}^{m+j=1}^q x_{2j} + \prod_{j=0}^{m} x_{2j+1}(1 + x_{2j+1})} \text{ (4.1)}
\]
is invariant under the action of \( F_a \).

See Remark 7.2 of Section 7.1 for an easy proof of Proposition 4.1. It is to be noticed that \( J_a \) is invariant under a permutation of variables of odd ranks, and the same fact holds for even ranks. Then the analogue of Propositions 2.3 and 3.2 is the following.

**Proposition 4.2.** (1) The quantity \( J_a(x) \) tends to infinity when \( x \) tends to the point at infinity of \( \mathbb{R}^{+q} \) \( (q = 2m + 1 \geq 5) \).

(2) \( J_a \) has a strict minimum at the point \( L = (\ell^p, \ldots, \ell^p) \), whose value is

\[
N(a) = 2(\ell + 1)^{m+1} \ell^{-m}. \text{ (4.2)}
\]
(3) \( J_a \) has only one critical point, its minimum \( L = (\ell_j, \ldots, \ell) \).

(4) For \( N > N(a) \) the sets

\[
\Delta_a(N) := \{ x \mid J_a(x) = N \}
\]

are compact \((q - 1)\)-dimensional smooth manifolds in \( \mathbb{R}_+^{4q} \) which are invariant under the action of \( F_a \). They are the boundaries of the open sets \( \{ x \mid J_a(x) < N \} \), which are connected and form a fundamental system of neighborhoods of point \( L \).

**Proof.** (i) It is to see that the closed set \( \bar{\Delta}_a(N) := \{ x \mid J_a(x) \leq N \} \) is compact. We denote by \( \Pi_{\text{od}} \) the product of the \( x_{2j+1} \), by \( \Pi_{\text{ev}} \) the analogue for even ranks, and by \( \Pi \) the product of all the \( x_i \).

From inequality \( J_a(x) \leq N \), we deduce \( N \geq (x_{ji}, \Pi_{\text{ev}} + \Pi_{\text{od}})/\Pi \geq 2(\sqrt{a_j}/\sqrt{\Pi_{\text{ev}}} \times \Pi_{\text{od}})/\Pi \) and so all the \( x_{ji} \) are bounded by some \( a > 0 \) on \( \bar{\Delta}_a(N) \). But we have also \( N \geq (a \Pi_{\text{ev}} + \Pi_{\text{od}})/\Pi \geq 2\sqrt{a}/\sqrt{\Pi} \), and so \( \Pi \geq \beta \) for some \( \beta > 0 \). We have then \( \Pi \geq \beta \) and so \( x_{ji} \geq \gamma > 0 \). These inequalities prove the compacity of \( \bar{\Delta}_a(N) \).

(ii) Now it suffices to prove point (3) of the proposition: of course, we deduce of it that \( J_a \) has a strict minimum at \( L \), whose value \( N(a) \) is easy to calculate. Then point (3) and [17, Proposition 2.1] together imply point (4). In fine, the fact that \( \bar{\Delta}_a(N) \) is a manifold comes from point (3).

(iii) **Proof of point (3).** We will prove that at a critical point \((x_1, x_2, \ldots, x_{2m+1})\) we have \( x_1 = x_{2m+1} \) and \( x_2 = x_{2m} \). Then, Proposition 3.4 (b) and Lemma 3.6 will give the result.

We have, with evident denotations, \( J_a(x) = (a + \Sigma)\Pi_{\text{ev}}(1 + x_j)/\Pi_{\text{od}}x_j + \Pi_{\text{od}}(1 + x_j)\). We write the relation \( \partial J_a/\partial x_1 = 0 \), which by an easy calculation becomes

\[
\frac{x_1}{(1 + x_1)(a + x_2 + x_3 + \cdots + x_{2m+1})} = \frac{\Pi_{\text{ev}}x_j(1 + x_j)}{\Pi_{\text{od}}x_j(1 + x_j)}.
\]

But the right member of (4.4) is invariant under the exchange of \( x_1 \) and \( x_{2m+1} \), and so the relation \( \partial J_a/\partial x_{2m+1} = 0 \) is

\[
\frac{x_{2m+1}}{(1 + x_{2m+1})(a + x_1 + x_2 + \cdots + x_{2m})} = \frac{\Pi_{\text{ev}}x_j(1 + x_j)}{\Pi_{\text{od}}x_j(1 + x_j)}.
\]

If we put \( A = a + x_2 + x_3 + \cdots + x_{2m} \), (4.4) and (4.5) give \( x_1(A + x_1)/(1 + x_1) = x_{2m+1}(A + x_{2m+1})/(1 + x_{2m+1}) \). The strict monotonicity of the function \( s \mapsto s(A + s)/(1 + s) \) on \( \mathbb{R}_+^* \), if \( A > 0 \), gives the equality \( x_1 = x_{2m+1} \). By similar calculations, the relation \( \partial J_a/\partial x_2 = 0 \) gives by using symmetry

\[
\frac{\Pi_{\text{ev}}x_j(1 + x_j)}{\Pi_{\text{od}}x_j(1 + x_j)} = \frac{1 + x_2}{x_2} = \frac{1 + x_{2m}}{x_{2m}} = \frac{1}{1 + a + x_2 + \sum_{i=1}^{q} x_i}.
\]

If we put \( B = 1 + a + \sum_{i \neq 2, i \neq 2m} x_i > 0 \), we obtain \( (1 + 1/x_2)/(1 + 1/x_{2m}) = (2x_2 + x_m + B)/(x_2 + 2x_{2m} + B) = 1 + (x_2 - x_{2m})/(x_2 + 2x_{2m} + B) \), and so \( (1/x_2 - 1/x_{2m})/(1 + 1/x_{2m}) = (x_2 - x_{2m})/(x_2 + 2x_{2m} + B) \). Clearly this relation implies the equality \( x_2 = x_{2m} \). \( \square \)
As for invariant \( H_a \) we make the following conjecture.

**Conjecture 6.** For \( N > N(a) \), the invariant compact \((q - 1)\)-dimensional manifold \( \Delta_a(N) = \{ x \mid J_a(x) = N \} \) is homeomorphic to the sphere \( S^{q-1} \).

We can reduce this conjecture to the following, which can be tested on a computer.

**Conjecture 7.** For integer \( m \geq 2 \), real numbers \( a > 0 \) and \( N > 0 \) the two curves in the positive plane \( \mathbb{R}^*_+ \)

\[
\begin{aligned}
    u(1 + t) + t(1 + u) &= Nu^{m+1}(1 + t)^{-m}, \\
    a + (m + 1)t + mu &= t^{m+2}(1 + t)^m u^{-(m+1)}(1 + u)^{1-m}
\end{aligned}
\]

have at most two common points \((t_1, u_1)\) and \((t_2, u_2)\).

Tests with a graphic computer give a great evidence to Conjecture 7, but its proof seems not easy.

**Proof of Conjecture 7⇒Conjecture 6.** As for invariant \( G_a \) we will use a smooth function which will have, by Conjecture 7, exactly two critical points on manifold \( \Delta_a(N) \); so the theorem of Reeb will give us the result similarly to the proof of Theorem 2.5.

Let \( f(x) = \sum_{i=1}^q x_i \). We search critical point \( x \) of \( f \) on \( \Delta_a(N) \). The relation \( df = \lambda dJ_a \) means that all the partial derivatives of \( J_a \) are equal. We write with evident denotations \( J_a(x) = (a + \Sigma)(1 + x_j)/(1 + x_j) \) and put \( A := a + \Sigma, B := \prod_{i=1}^q (1 + x_j)/\prod_{i=1}^q x_j \) and \( C := \prod_{i=1}^q (1 + x_j)/\prod_{i=1}^q x_j \). The equality of \( \partial J_a/\partial x_{2i+1} \) with \( \partial J_a/\partial x_{2j} \) is \( B - AB/x_{2i+1} + C/(1 + x_{2i+1}) = B + AB/(1 + x_{2j}) - C/x_{2j} \), that is

\[
\frac{AB}{C} = \frac{x_{2i+1}(1 + x_{2j})}{x_{2j}(1 + x_{2i+1})}.
\]

By writing (4.9) for \( 2i+1 \) and \( 2j' \), we obtain \( x_{2j} = x_{2j'} \), and in the same manner we have \( x_{2j+1} = x_{2j+1} \). So all the odd coordinates of a critical point have the same value \( t \), and all the even coordinates of a critical point have the same value \( u \).

Then relation (4.9) becomes \( (a + (m + 1)t + mu)((1 + u)^m)/(1 + t)^{m+1}) = \frac{1}{u^m} (1 + u)/(1 + t)^{m+1} \), that is, relation (4.8) of Conjecture 7. The relation \((t, u, t, \ldots, u, t) \in \Delta_a(N) \) gives

\[
N = (a + (m + 1)t + mu) \frac{(1 + u)^m}{t^{m+1}} + \frac{(1 + t)^{m+1}}{u^m}.
\]

We put the value of \( a + (m + 1)t + mu \) from (4.8) in this relation and obtain relation (4.7) of Conjecture 7. This conjecture implies that there is at most two critical points \((t_1, u_1, \ldots, t_1)\) and \((t_2, u_2, \ldots, t_2)\) of function \( f \) on \( \Delta_a(N) \). But there is at least two such critical points: the maximum and minimum of \( f \) on the manifold; these two numbers are different if \( N > N(a) \), because a compact \((q - 1)\)-manifold in \( \mathbb{R}^q \) cannot be included in an hyperplane. So we can conclude by Reeb’s theorem.

\( \square \)
5. Nature of the Differential $dF_a(L)$ at the Equilibrium

Now we look at the eigenvectors of $dF_a(L)$. The matrix of the differential $dF_a(L)$ is

$$A = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 & -1 \\
\ell & \ell & \ell & \ldots & \ell & 0 \\
1 & 0 & 0 & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 1 & 0
\end{pmatrix}. \tag{5.1}
$$

So we search a vector $x = (x_1, \ldots, x_q) \neq 0$ such that $Ax = \lambda x$ for some $\lambda \in \mathbb{C}$. We get easily the relations

$$\left[ \lambda^q + 1 - \frac{1}{\ell} \left( \lambda + \lambda^2 + \cdots + \lambda^{q-1} \right) \right] x_q = 0, \quad x_i = \lambda^{q-i} x_q \quad \text{for } i = 1, 2, \ldots, q - 1. \tag{5.2}
$$

So $x \neq 0 \iff x_q \neq 0$, and we have the equation of the eigenvalues

$$\lambda^q + 1 - \frac{1}{\ell} \left( \lambda + \lambda^2 + \cdots + \lambda^{q-1} \right) = 0, \tag{5.3}
$$

and, for each such eigenvalue $\lambda$, a 1-dimensional eigenspace generated by the eigenvector $V_\lambda = (\lambda^{q-1}, \lambda^{q-2}, \ldots, \lambda, 1)$.

So we have to determine the nature of solutions of (5.3). The results are not the same if $q$ is odd or if it is even. Recall that from Lemma 2.2 we have $1/\ell < 1/(q - 1)$.

5.1. The Eigenvalues When $q$ Is Even

First we study the case where $q = 2m$ is even. We use the following result.

Lemma 5.1. If $0 < z \leq 1/(2m - 1)$, the polynomial

$$P_{2m}(\lambda) := \lambda^{2m} + 1 - z \left( \lambda + \lambda^2 + \cdots + \lambda^{2m-1} \right) \tag{5.4}
$$

has $2m$ distinct roots, with modulus 1, different from 1 and $-1$, hence of the form $e^{\pm \theta_j}$, $j = 1, 2, \ldots, m$, with $\theta_j \neq \theta_k$ if $j \neq k$, and $0 < \theta_j < \pi$ for all $j$.

Proof. First it is obvious that 0, 1, $-1$ are not roots of $P_{2m}$. Then the proof has three steps.

First step. $P_{2m}$ has no real roots.

Because $P_{2m}$ is reciprocal, it suffices to show that there is no real root $\lambda$ with $0 < |\lambda| < 1$. If such a root exists, then one has $1 < \lambda^{2m} + 1 = \lambda z (1 + \lambda + \lambda^2 + \cdots + \lambda^{2m-2}) < z (2m - 1)$, and so $z > 1/(2m - 1)$, which is false.
Theorem 6.1. The number \( z = (\lambda^{2m+1} + 1)/(\lambda - 1) \) is real, and \( z + 1 \) also. Hence we have 
\[
(\lambda^{2m+1} - 1)(\lambda^{2m-1} - 1) = (\lambda^{2m-1} - 1)(\lambda^{2m} - 1),
\]
so 
\[
(\lambda - 1)(\lambda^{2m-1} + 1 - \lambda^{2m-1} + \lambda^{2m-2} + \cdots + 1 = 0.
\]
(5.5)

But \( \lambda \neq \bar{\lambda} \); if \( \lambda \bar{\lambda} = 1 \), we have \( |\lambda| = 1 \). If not, put \( r := \lambda \bar{\lambda} > 0 \). Because \( P_{2m} \) is reciprocal, we can suppose that \( |\lambda| < 1 \). From the equality 
\[
(\lambda \bar{\lambda})^{2m-1} + (\lambda \bar{\lambda})^{2m-2} + \cdots + 1 - (\lambda^{2m-1} + \lambda^{2m-2} + \cdots + 1) = 0,
\]
(5.6)
we get \( r^{2m-1} + r^{2m-2} + \cdots + 1 \leq 2mr^{m-1/2} \), or \( r^{m-1/2} + r^{m-1/2} + \cdots + r^{1/2} + 1 \leq 2m \). But if \( r \neq 1 \), the first member is greater than \( 2m \), which gives a contradiction.

Third step. Roots of \( P_{2m} \) are all distinct.

The proof of this step will be given in common with the one of the third step in the proof of Lemma 5.3.

\[ \square \]

5.2. The Eigenvalues When \( q \) Is Odd

Now we study the case where \( q = 2m + 1 \) is odd. First we have the following result.

**Lemma 5.2.** If \( q = 2m + 1 \) is odd, then \(-1\) is an eigenvalue of \( dF_a(L) \), for the eigen vector \( V_{-1} = (1, -1, 1, \ldots, -1, 1) \), which is a vector tangent at the point \( L \) to the hyperbola \( \mathcal{H}_a \) defined in Theorem 6.1 in Section 6.1.

**Proof.** A first proof is to see that the vector \( V_{-1} \) satisfies the relation \((A + I)V_{-1} = 0\), which is obvious. One can also see that \(-1\) is a root of (5.3), which is also obvious. A deeper proof is in relation with the 2-periodic points of the hyperbola \( \mathcal{H}_a \) (see Theorem 6.1 in Section 6.1). We have, by Theorem 6.1, the relations \((F_a(h(t)) - F_a(h(\ell)))/(t - \ell) = (h(\phi_a^m(t)) - h(\phi_a^m(\ell)))/(t - \ell)\) (see the definition of \( h \) and \( \phi_a^m \) in Theorem 6.1 and formula (6.4)). But the first term tends to the vector \( dF_a(L) \circ h'(\ell) \) when \( t \to \ell \), and the second tends to the vector \((\phi_a^m)'(\ell) \circ h'(\ell) = -h'(\ell)\). So \( h'(\ell) \) is an eigenvector of \( dF_a(L) \) for the eigenvalue \(-1\). And it is obvious to see that \( h'(\ell) = (1, -1, \ldots, -1, 1) = V_{-1} \).

\[ \square \]

So (5.3) can be written \((\lambda+1)(\lambda^{2m} - \lambda^{2m-1} + \cdots - 1) = (\lambda/\ell)(1 + \lambda^3 + \cdots + \lambda^{2m-1}).\) Then we cancel the factor \( \lambda + 1 \) and add to the two members the quantity \( \lambda + \lambda^3 + \cdots + \lambda^{2m-1}. \)

In fine, we see that the eigenvalues of \( dF_a(L) \) distinct of \(-1\) are the roots of the polynomial \( P_{2m+1}(\lambda) := \lambda^{2m} + \lambda^{2m-2} + \cdots + \lambda^2 + 1 - (1 + z)(\lambda + \lambda^2 + \lambda^3 + \cdots + \lambda^{2m-1}) \), for \( z = 1/\ell \). We have now the following result, analogous to Lemma 5.1.

**Lemma 5.3.** If \( 0 < z \leq 1/2m \), the polynomial 
\[
P_{2m+1}(\lambda) = \lambda^{2m} + \lambda^{2m-2} + \cdots + \lambda^2 + 1 - (1 + z)(\lambda + \lambda^3 + \cdots + \lambda^{2m-1})
\]
(5.7)
has $2m$ distinct roots, with modulus 1, different from 0 and $-1$, hence of the form $e^{\pm i \theta_j}$, $j = 1, 2, \ldots, m$, with $\theta_j \neq \theta_k$ if $j \neq k$, and $0 < \theta_j < \pi$ for all $j$.

**Proof.** First, it is obvious that 0, 1, and $-1$ are not roots of $P_{2m+1}$. Then, the proof has three steps, as for Lemma 5.1.

**First step.** $P_{2m+1}$ has no real roots.

A real root $\lambda$ is necessarily positive; because $P_{2m+1}$ is reciprocal, we can suppose $\lambda < 1$. So we have $1 + 1/2m + 1 + z = (1 + \lambda^2 + \cdots + \lambda^{2m}) / \lambda (1 + \lambda^2 + \cdots + \lambda^{2m-2}) = 1 / \lambda + \lambda^{2m-1} / (1 + \lambda^2 + \cdots + \lambda^{2m-2}) > 1 / \lambda + \lambda^{2m-1} / m$. But the function of $\lambda$ in the right-hand member has for minimum on $]0, +\infty[$ the positive quantity $(2m / (2m - 1))((2m - 1) / m)^{1/2m} > 1 + 1/2m$, and this is a contradiction.

**Second step.** Roots of $P_{2m+1}$ have modulus 1.

Remark that $1 + z = (\lambda^{2m+2} - 1) / (\lambda^{2m} - 1)$ is real, hence equals to its conjugated. As in proof of Lemma 5.1, this gives the relation

$$
\left(\lambda - \bar{\lambda}\right)\left(\lambda \bar{\lambda} - 1\right)\left\{ \left(\lambda \bar{\lambda}\right)^{2m} + \left(\lambda \bar{\lambda}\right)^{2m-1} + \cdots + \lambda \bar{\lambda} + 1 - \left(\lambda^2 + \lambda^{2m-1} \bar{\lambda} + \cdots + \left(\bar{\lambda}\right)^2\right) \right\} = 0.
$$

(5.8)

The same method as in Lemma 5.1, with $r := \lambda \bar{\lambda}$, supposed to satisfy $r < 1$, gives the inequality $r^m + 1 / r^m + r^{m-1} + 1 / r^{m-1} + \cdots + r + 1 / r + 1 \leq 2m + 1$. But if $r \neq 1$, the first member is greater than $2m + 1$, and this is a contradiction.

**Third step.** Roots of $P_{2m+1}$ are all distinct.

The proof of this step is the same of that one of third step of the proof of Lemma 5.1 and is given below.

**Common proof of third steps of Lemmas 5.1 and 5.3.** Roots of equation $\lambda^q - z(\lambda^{q-1} + \cdots + \lambda) + 1 = 0$ are distinct if and only if it is the case for equation $\lambda^{q+1} - u \lambda^q + u \lambda - 1 = 0$, where $u = z+1$, obtained by multiplication by $(\lambda - 1)$. But a multiple root is a common root to this equation and to the derivative one $(q + 1) \lambda^q - q u \lambda^q + u = 0$. We eliminate $u$ between these two equations, and obtain $q = (1 - \lambda^q) / (q + 1 - \lambda^{q+1} (1 - \lambda^2) / q)$. So we have $\lambda^{-1} = (1 + \lambda^2 + \lambda^4 + \cdots + \lambda^{2q-2}) / q$. By steps 1 and 2 of the proofs of Lemmas 5.1 and 5.3 we know that $\lambda^{-1}$ is an extremal point of the unit disk, and so the previous barycentric relation implies that $\lambda^2 = 1$, and $\lambda = \pm 1$, which is impossible.

### 5.3. The Global Nature of $dF_{a}(L)$

From Lemmas 5.1, 5.2, and 5.3, we get the description of $dF_{a}(L)$.

**Theorem 5.4.** (1) The $q$ eigenvalues of $dF_{a}(L)$ are all distinct and have modulus 1.

(2) When $q = 2m$ is even, $dF_{a}(L)$ is linearly conjugated to a product of $m$ rotations in mutually orthogonal planes, with angles $\theta_1, \theta_2, \ldots, \theta_m \in [0, \pi[ \ {\text{all distinct}}.

(3) When $q = 2m + 1$ is odd, $dF_{a}(L)$ is linearly conjugated to a product of $m$ rotations in mutually orthogonal planes in the hyperplane $x_q = 0$, of angles $\theta_1, \theta_2, \ldots, \theta_m \in [0, \pi[ \ {\text{all distinct}},

and of an orthogonal symmetry with respect to the hyperplane $x_q = 0$. 
Proof. Eigenvalues of $dF_a(L)$ are distinct in the two cases, and so the matrix is diagonalizable in a basis of eigenvectors in $\mathbb{C}^4$. But it is classical that, with conjugated eigenvalues of modulus 1 and a real matrix for $dF_a(L)$, one can obtain the forms given in the theorem. \qed

6. Results about Possible Periods of Solutions of (1.1)

6.1. Role of 2-Periodic Points When $q = 2m + 1$ Is Odd

In [12], the authors found the existence of 2-periodic points when $q = 2m + 1$ is odd (see also [4, 13]). There is a theoretical reason to the existence of these points, from Theorem 2.5 and the following result.

**Fact 2.** Every continuous map $f : \mathbb{S}^{2m} \to \mathbb{S}^{2m}$ has a 2-periodic point.

**Proof.** It is sufficient to prove the result for smooth maps. Put $g = f \circ f$ and $id(x) = x$ on $\mathbb{S}^{2m}$. If $g$ has no fixed point, then for all $x \in \mathbb{S}^{2m}$, $g(x) \neq -(id(x))$; so $g$ and $-id$ are homotopic, and then have the same degree. But $\deg(g) = [\deg(f)]^2$ and $\deg(-id) = (-1)^{2m+1}$, and this is impossible (for elementary properties of the degree, see [10, pages 237–239]). \qed

These 2-periodic points have particular properties. The relations (6.1) of the following theorem are already in [12]. See also [9] for applications to lacunary Lyness’ difference equations.

**Theorem 6.1.** Let $q = 2m + 1$, $m \geq 1$, and $a > 0$. The locus of 2-periodic points for map $F_a$ is the branch of hyperbola $\mathcal{H}_a \subset \mathbb{R}_+^q$ whose equations are

$$
x_1 = x_3 = \cdots = x_q, \quad x_2 = x_4 = \cdots = x_{q-1} = \frac{mx_1 + a}{x_1 - m}, \quad x_1 > m.
$$

This branch of hyperbola $\mathcal{H}_a$ passes through the equilibrium $L$ and is invariant by the map $F_a$. The map

$$
h : t \mapsto \left( t, \frac{mt + a}{t - m}, \frac{mt + a}{t - m}, \ldots, \frac{mt + a}{t - m}, t \right) : ]m, +\infty[ \to ]m, +\infty[^q \subset \mathbb{R}_+^q
$$

is a parametrization of $\mathcal{H}_a$, which has the property that

$$
F_a(h(t)) = h\left( \frac{mt + a}{t - m} \right), \quad F_a\left( h\left( \frac{mt + a}{t - m} \right) \right) = h(t).
$$

Moreover, if $K > K(a)$, $\mathcal{H}_a$ cuts the manifold $\Sigma_a(K)$ at exactly two points $P^1_a(K)$ and $P^2_a(K)$ which are exchanged by $F_a$, their parameters are $t_1$ and $t_2 = (mt_1 + a) / (t_1 - m)$ with $m < t_1 < 1 < t_2$.

**Proof.** The $q$ equations satisfied by a point $M_0 = (u_{q-1}, u_{q-2}, \ldots, u_0)$ for being 2-periodic are $u_0 = u_2 = \cdots = u_{q-1}$, $u_1 = u_3 = \cdots = u_q$ and $a + u_q + \cdots + u_2 = u_1u_{q-1}$. If we put $u_0 = x_q = t$ and $u_1 = x_{q-1} = s$, we obtain $t = (a + m(t + s)) / s$, or $s = (mt + a) / (t - m)$, that is, (6.1) and the parametrization (6.2). Conversely the point $h(t)$ is obviously 2-periodic for $t > m$.\[\square\]
Now we put, for $t > m$,

$$
\phi_a^m(t) = \frac{mt + a}{t - m},
$$

(6.4)

which is an involution with fixed point $\ell$. We can write $h(t) = (t, \phi_a^m(t), t, \phi_a^m(t), \ldots, t)$, and hence we have $h(\phi_a^m(t)) = (\phi_a^m(t), t, \phi_a^m(t), \ldots, \phi_a^m(t)) = F_a(h(t))$, for $h(t)$ is 2-periodic. We have also $F_a(h(\phi_a^m(t))) = F_a \circ F_a(h(t)) = h(t)$.

At last it is to find the points where $\mathcal{A}_a$ cuts $\Sigma_a(K)$. First, we change the parameter for $\mathcal{A}_a$: we have $\ell = m + \sqrt{m^2 + a}$ and we put $b := \sqrt{m^2 + a} = \ell - m$; so we take the new parameter $v := (t - m)/b$. Hence, $v = 1 \Leftrightarrow t = \ell$, and if $q := (\phi_a^m - m)/b$, then $q(v) = 1/v$. So we put the values $t = bv + m$ for $x_1, x_2, \ldots, x_q$ and $\phi_a^m(t) = m + b/v$ for $x_2, x_4, \ldots, x_{q-1}$ in the equation $G_a(x) = K$ of $\Sigma_a(K)$. Some easy calculations (using the relation $a + m^2 = b^2$ for simplifying) give the equation

$$
1/v(\{b(m + 1)v^2 + ((m + 1)^2 + b^2)v + b(m + 1)\}^{m+1} / \{bm^2 + (m^2 + b^2)v + bm\}^m) = K.
$$

But the two quadratic polynomials in $v$ are reciprocal, and so we put $X = v + 1/v \geq 2$ (and $X = 2 \Leftrightarrow v = 1 \Leftrightarrow t = \ell$). Then we define $f : [2, +\infty [ \to \mathbb{R}$ by $f(X) := (m + 1) \ln (b(m + 1)X + (m + 1)^2 + b^2) - m \ln (bmX + m^2 + b^2)$. We have $f'(X) = b(m + 1)^2 / (b(m + 1)X + (m + 1)^2 + b^2) - bm^2 / (bmX + m^2 + b^2) = \gamma(m + 1) - \gamma(m)$, where the function $m \mapsto \gamma(m) = bm^2 / (bmX + m^2 + b^2)$ is obviously increasing. So we have $f'(X) > 0$, and $f$ is increasing, with $f(+\infty) = +\infty$. But $f(2) = \ln (b(m + 1)^2(m + 2) / (b(m + 1)^2 + b^2)) = \ln ((\ell + 1)^{m+1} / \ell^{q-1}) = \ln K(a)$, and so there is a unique solution $X > 2$ to the equation $f(X) = \ln K$ if $K > K(a)$. This solution gives two solutions $v$ to the equation $v + 1/v = X, 0 < v_1 < 1 < v_2$. And this gives two solutions $m < t_1 < \ell < t_2$, and from the invariance of $\mathcal{A}_a$ and $\Sigma_a(K)$ by $F_a$ we have $t_2 = \phi_a^m(t_1)$.

\[ 6.2. \textbf{The General Case} \quad q \geq 2 \]

We start with a formula easy to prove.

\textbf{Lemma 6.2.} If $(u_n)_n$ is a solution of (1.1) and $K = G_a(u_{q-1}, \ldots, u_1, u_0)$, one has the relation

$$
\forall n \geq 0 \sum_{i=0}^{q-1} u_{n+i} + a + 1 = K \prod_{i=1}^{q-1} \frac{u_{n+i}}{1 + u_{n+i}}.
$$

(6.5)

We deduce from (6.5) the possibility for $q + 1$ to be a period of some solutions of (1.1).

\textbf{Proposition 6.3.} The only $(q + 1)$-periodic solutions of (1.1) are the following:

(i) if $q$ is even, the equilibrium $L$;

(ii) if $q$ is odd, the 2-periodic solutions of Theorem 5.4.

\textbf{Proof.} If $(u_n)_n$ is $(q + 1)$-periodic, the left-hand member of (6.5) is constant when $n$ varies, and so $v_n := \prod_{i=1}^{q-1} (u_{n+i} / (1 + u_{n+i}))$ is also constant; so we have $v_{n+1}/v_n = 1 - (u_{n+q}/(1 + u_{n+q}))(1 + u_{n+1}/u_{n+1})$, and then for all $n u_{n+1} = u_{n+q}$, that is, $(u_n)_n$ is $(q - 1)$-periodic. If $p$ is the minimal period of the sequence $u_n, p$ divides $q + 1$ and $q - 1$, and then $p = 1$ or $p = 2$. If $p$ is odd, then $p = 1$ and the solution is the equilibrium. If $p$ is even, then $p = 2$, and $q - 1$ is a multiple of
2, and then \( q \) is odd; in this case, \( u_n \) is a 2-periodic solution of Theorem 5.4 (which may be constant).

As a consequence, we see that order 2 Lyness’ equation has no solution with minimal period 3, which was already known. For order 3 equation, consequences are in the following.

\[ \text{6.3. The Case } q = 3 \]

Proposition 6.4. (1) The only 4-periodic solutions of order 3 Lyness’ equation are 2-periodic, and given by Theorem 5.4.

(2) Order 3 Lyness’ equation has no solution with minimal period 3.

(3) Order 3 Lyness’ equation has no solution with minimal period 5.

Remark that in [13] authors find an infinity of starting points which are 5-periodic for order 3 Lyness’ equation, but in [13] this equation is studied in \( \mathbb{R} \) and not, as we make here, in \( \mathbb{R}^2 \): their 5-periodic solutions take negative values. In Section 7, we will find again points (2) and (3) of Proposition 6.4.

Proof. (1) This point is an immediate consequence of Proposition 6.3.

(2) We put \( u_0 = x, \ u_1 = y, \ u_2 = z, \) and for finding 3-periodic solutions we must write the relations \( u_3 = x, \ u_4 = y, \ u_5 = z, \) that is, \( x^2 = a + y + z, \ y^2 = a + z + x, \) and \( z^2 = a + x + y. \) By substracting we get \( (y - x)(x + y + 1) = 0, \) and then \( x = y. \) In the same way we obtain \( y = z, \) and so the solution is the equilibrium, whose minimal period is 1.

(3) We put also \( u_5 = s, \ u_4 = t. \) We have to write three conditions: \( u_5 = u_0, \ u_6 = u_1, \ u_7 = u_2, \) and the definition of \( s \) and \( t. \) We obtain (i) \( xz = a + s + t, \) (ii) \( ys = a + t + x, \) (iii) \( zt = a + x + y, \) (iv) \( sx = a + y + z, \) and (v) \( ty = a + s + z. \) From (i), (ii), and (iv) we obtain (vi) \( z \ (x^2 - y - 1) = (y + 1)(a + y) - x^2. \) Relation (ii) where \( s \) and \( t \) are replaced by their value obtained from (iv) and (v) gives (vii) \( z \ (y^2 - x - 1) = xy \ (a + x) + ax + a + y - y^2(a + y). \) In the same way, relation (i) gives (viii) \( z = (y + 1)(ax + x + a)/(x + 1)(xy - y - 1), \) for the denominator cannot be zero because the numerator is positive. The difference of equations (vi) and (vii) gives, after factorization, \( (x - y)(z^2 + y + 1) + (y + 1)(a + x + y) = 0. \) So we have \( x - y = 0. \)

In this case (viii) becomes (ix) \( z = (ax + x + a)/(x^2 - x - 1). \) The difference of (i) and (v) gives \( x(z - t) = t - z, \) that is, \( z = t. \) So (ix) becomes \( z^2 = a + 2x, \) or \( x = (z^2 - a)/2. \) We put this value of \( x \) in (ix) and obtain, after factorization, the relation (x) \( (z^2 - 2x - a)(z^2 + 2z^2 + (a - 2)(a - 1) = 0. \) But \( z^2 > a, \) so the second factor is greater than \( a(z + 2) - (a - 2)(a - 1) = 2z + 2 > 2, \) and we have \( z^2 - 2z - a = 0, \) that is, \( z = \ell. \) But the two relations \( z^2 = 2x + a \) and \( z^2 = 2z + a \) imply the relation \( x = z, \) and so \( x = y = z = \ell: \) the only 5-periodic solution is the point \( L. \)

\[ \text{6.4. Eigenvalues of } dF_a(L) \text{ and the Possible Common Periods of Solutions of (1.1)} \]

The question is for which values of \( a > 0 \) and \( q \geq 2 \) have all solutions of (1.1) a common period \( p? \) What are these common periods? In the case where such a common period exists, one say that (1.1) is “globally periodic” or “globally \( p \)-periodic”.

The following result is a special case of a deep theorem of Csörnyei and Laczkovich (see [12, 14] for an extension), which gives the answer to the question.
Proposition 6.5. There is no period $p$ common to all solutions of order $q$ Lyness’ equation (1.1) with parameter $a > 0$, except the cases $q = 2$, $a = 1$ (with $p = 5$), and $q = 3$, $a = 1$ (with $p = 8$).

The proof of [14] is difficult. For the case $a = 1$, there is in [15] a nice short proof, which uses the characteristic polynomial of $dF_1(L^*)$, with $L^* = (\ell^*, \ldots, \ell^*)$, $\ell^*$ being the negative root of $X^2 - (q-1)X - 1 = 0$. Unfortunately this proof does not work for $a \neq 1$.

For every value of $a > 0$, the existence of a common period $p$ to all positive solutions of order $q$ Lyness’ equation would imply that the characteristic polynomial $B_q(\lambda) := \lambda^q + 1 - (1/\ell^*)(\lambda^{q-1} + \cdots + \lambda + 1)$ would be a factor of the polynomial $A_p := \lambda^{p-1} + \lambda^{p-2} + \cdots + \lambda + 1$. Indeed, the relation $F_a^p = Id$ on $\mathbb{R}^q_*$; gives obviously the equality $dF_a(L)^p = Id$ on $\mathbb{R}^q$, so eigenvalues of $dF_a(L)$ are $p$th-roots of unity or equal to 1, and by Lemmas 5.1, 5.2, and 5.3, roots of $B_q$ are all distinct. So Proposition 6.5 is related to the following question (where in fact $z = 1/\ell^*$).

**Question 1.** Let integers $q \geq 2$, $p \geq 2$, and a number $z \in J0, 1/(q-1)\. Can it happen that

$$B_q(\lambda) = \lambda^q - z(\lambda^{q-1} + \cdots + \lambda) + 1$$

were a factor of the polynomial

$$A_p(\lambda) = \lambda^{p-1} + \lambda^{p-2} + \cdots + \lambda + 1?$$

Of course a negative answer to Question 1 when $q \geq 4$, or $q = 3$ and $a \neq 1$, or $q = 2$ and $a \neq 1$, would prove Proposition 6.5, because it is known that for $a = 1$ and $q = 2$ or $q = 3$ (1.1) is globally $p$-periodic, with $p = 5$ or 8.

But we can have a positive answer to Question 1: some $a$, $p$, and $q$ for which $B_q$ divides $A_p$, that is, all eigenvalues of $dF_a(L)$ are $p$th root of unity, but without (1.1) being globally $p$-periodic, such cases are given by the following proposition.

Proposition 6.6. (1) If $q = 2$, for every $p \geq 13$ there is a number (perhaps not unique) $z(p)$, $0 < z(p) < 1$ (and then an $a(p)$), such that the polynomial $B_2 = \lambda^2 - z(p)\lambda + 1$ is a factor of $A_p\ell^*$; that is, eigenvalues of $dF_a(p)(L)$ are $p$th root of unity.

(2) If $q = 3$, for every even $p \geq 20$ there is a number (perhaps not unique) $z(p)$, $0 < z(p) < 1/2$ (and so an $a(p)$), such that the polynomial $B_3 = \lambda^3 - z(p)(\lambda^2 + \lambda) + 1 = (\lambda + 1)(\lambda^2 - (z(p) + 1)\lambda + 1)$ is a factor of $A_p\ell^*$; that is, eigenvalues of $dF_a(p)(L)$ are $p$th root of unity.

**Proof of Proposition 6.6.** (1) If polynomial $\lambda^2 - z(\lambda + 1)$ divides $A_p\ell^*$, then it is $\lambda^2 - 2\cos(2k\pi/p)\lambda + 1$ for some integer $k$ satisfying $1 \leq k < p/4$. So we have $z = 2\cos(2k\pi/p)\ell^*$ and $a = (1 - z)/z^2$, with only the constraint that $0 < z < 1$ and $a > 0$, that is, $1 \leq k < p/4$ and $k > p/6$. This is possible if $p = 5, 9, 10, 11$, and if $p \geq 13$. This last case gives the first point of Proposition 6.5.

(2) If polynomial $(\lambda + 1)(\lambda^2 - (z + 1)\lambda + 1)$ divides $A_p\ell^*$, then $p = 2m$ is even, and the roots of $\lambda^2 - (z + 1)\lambda + 1$ are $p$th roots of unity, and then this polynomial has the form $\lambda^2 - 2\cos(k\pi/m)\lambda + 1$, for some integer $k$ satisfying $1 \leq k < m - 1$. So we have $z = 2\cos(k\pi/m)\ell^*$ and $a = (1 - 2z)/z^2$. The only constraints are $a > 0$ and $1 + 1/2 > z + 1 > 1$, that is, $1 \leq k < m/3$ and $k > m\theta_0$ where $\theta_0 = (1/\pi)\cos^{-1}(3/4) = 0.230$. But if $m > 1/(3 - \theta_0) = 9.68$, there is an integer $k \geq 1$ satisfying the constraints. So if $p = 2m \geq 20$, we have the point (2) of Proposition 6.6. But one can see that some $k$ works also for $p = 8, 14$, and 16 ($k = 2$), and only in these cases. \[\square\]
Example 6.7. (1) If \( q = 2 \) and \( p = 24 \), we find that for the equation \( u_{n+2} = (2 + \sqrt{3} - \sqrt{2 + \sqrt{3}} + u_{n+1})/u_n \), the eigenvalues of \( dF_a(L) \) are 24th roots of unity, but 24 is not a common period to all solutions.

(2) For \( p = 30 \), we find the same fact for (1.1) with \( q = 3 \) and \( a(30) = (3 - 4 \cos(4\pi/15))/2 \cos(4\pi/15) - 1)^2 \).

We make the following conjecture.

Conjecture 8. If \( q \geq 4 \), the polynomial \( B_q \) does not divide the polynomial \( A_p \) for \( 0 < z < 1/(q - 1) \) (and then for all \( a > 0 \) (1.1) is not globally periodic if \( q \geq 4 \)).

7. Order 3 Lyness’ Difference Equation

Before to study this case, we give general results for odd \( q \).

7.1. Invariants for \( F_a^2 \) When \( q = 2m + 1 \) Is Odd

If \( q \) is odd, we have nice invariants for \( F_a^2 \).

Proposition 7.1. When \( q = 2m + 1 \) is odd, the quantities

\[
R^0(x_1, \ldots, x_q) := \frac{(1 + x_1)(1 + x_2) \cdots (1 + x_{2m+1})}{x_2x_4 \cdots x_{2m}},
\]

\[
R_a^1(x_1, \ldots, x_q) := \frac{(1 + x_2)(1 + x_4) \cdots (1 + x_{2m})(a + x_1 + x_2 + \cdots + x_{2m} + x_{2m+1})}{x_1x_3 \cdots x_{2m+1}}
\]

are exchanged by \( F_a \):

\[
R^0 \circ F_a = R_a^1, \quad R_a^1 \circ F_a = R^0
\]

and then are invariant under the action of \( F_a^2 \).

Moreover one has the relations

\[
R^0 R_a^1 = G_a, \quad \text{if } q \geq 5 \quad R^0 + R_a^1 = J_a.
\]

Proof. Relations (7.3) come from and easy calculation, and relations (7.4) are obvious from the definitions of \( G_a \) and \( J_a \).

Remark 7.2. Relations (7.4) give simple proof of the invariance of \( G_a \) and \( J_a \) under the action of \( F_a \) and also of every symmetric function of \( R^0 \) and \( R_a^1 \).

Proposition 7.1 has significant consequences in the case \( q = 3 \).

7.2. The Case \( q = 3 \)

This case was intensively studied in the nice paper [4], where the authors proved, for the first time, that there are two curves in \( \mathbb{R}_a^3 \) invariant by the map \( F_a^2 \) and homeomorphic to two
circles, such that the restriction of \( F_a^2 \) to these curves is conjugated to rotations on the circles, and that these curves are exchanged by the map \( F_a \). The authors gave also some interesting results about possible periods for \( F_a^2 \) of some starting points for some values of parameter \( a \) and about initial points whose orbits are dense in the invariant curves.

Authors of [16] have found a remarkable property of even terms \( u_{2n} \) and odd terms \( u_{2n+1} \) of the order 3 Lyness' sequence: they are solutions of two-order 2 difference equations related to two elliptic quartics in the plane. In [4], the authors make the conjecture, in a note added in proof, that their (difficult) paper would be simpler with the aid of [16]. We will see that in fact this is right, at least partially, with the aid of invariants \( R^0 \) and \( R^1_a \) of \( F_a^2 \) (these invariants are implicit in [16] for the case \( q = 3 \)). In the case \( q = 3 \) these invariants become

\[
R^0(x, y, z) = \frac{(1 + x)(1 + z)}{y}, \quad R^1_a(x, y, z) = \frac{(1 + y)(a + x + y + z)}{xz}.
\] (7.5)

If we denote, as in [16],

\[
c_0 := R^0(M_0), \quad c_1 := R^1_a(M_0),
\] (7.6)

the orbit of the starting point \( M_0 \) under the action of \( F_a^2 \) is on the curve \( \mathfrak{D}(M_0) \) defined by the two equations

\[
(1 + x)(1 + z) - c_0 y = 0, \quad (1 + y)(a + x + y + z) - c_1 xz = 0.
\] (7.7)

These two surfaces are just quadratic surfaces (“quadrics”) \( Q_0 \) and \( Q_1 \), and then \( \mathfrak{D}(M_0) \) is merely a quartic curve. In fact, this curve is included in the curve \( C(M_0) \) given in (3.18).

**Theorem 7.3.** (1) In addition to the first relation (7.5) one has an expression of \( H_a \) in terms of \( R^0 \) and \( R^1_a \), and so one has:

\[
G_a = R^0 R^1_a, \quad H_a = R^0 R^1_a + R^0 + R^1_a + 2 - a.
\] (7.8)

(2) The polynomial equations \( g(x, y, z) = 0 \) and \( h(x, y, z) = 0 \) of \( \Sigma_a(K) \) and of \( S_a(M) \) given by formulas (2.5) and (3.11) (where \( K = G_a(M_0) \) and \( M = H_a(M_0) \)) are polynomial combinations of the two equations of the curve \( \mathfrak{D}(M_0) \) given in (7.7):

\[
2g(x, y, z) = A(x, y, z) [(1 + x)(1 + z) - c_0 y] + B(x, y, z) [(1 + y)(a + x + y + z) - c_1 xz],
\] (7.9)

where \( A(x, y, z) = c_1 xz + (1 + y)(a + x + y + z) \) and \( B(x, y, z) = (1 + x)(1 + z) + c_0 y, \) and

\[
h(x, y, z) = C(x, y, z) [(1 + x)(1 + z) - c_0 y] + D(x, y, z) [(1 + y)(a + x + y + z) - c_1 xz],
\] (7.10)

where \( C(x, y, z) = y^2 + xy + yz + zx + x + (a + 1) y + z + a \) and \( D(x, y, z) = (c_0 + 1) y. \)
(3) One has the inclusion

\[ \mathcal{D}(M_0) \cup F_a(\mathcal{D}(M_0)) \subset C, \]  

(7.11)

and the part \( \mathcal{D}^+(M_0) \) of \( \mathcal{D}(M_0) \) which is in \( \mathbb{R}^+ \) is compact.

**Proof.** (1) The first relation (7.8) is in (7.4), and an easy but tedious calculation proves the second relation.

(2) The proofs of the two announced expressions for \( 2g \) and \( h \) come from simple but also tedious calculations using the two relations \( K = c_0c_1 \) and \( M = c_0c_1 + c_0 + c_1 + 2 - a \), which come from relations (7.8) applied to the point \( M_0 \).

(3) The inclusion (7.11) and then the announced compacity are obvious consequences of point (2) and of the compacity of \( C^+(M_0) = \mathbb{R}^+ \cap C(M_0) \). \( \square \)

**Remark 7.4.** (1) The second formula (7.8) gives, for \( q = 3 \), an easy proof of the invariance of \( H_a \) under the action of \( F_a \).

(2) It is not the case for \( q \geq 5 \), because the second relation (7.8) has no corresponding one: we know from [5] that \( G_a, H_a \), and \( J_a \) are independent and so cannot be all three regular functions of \( R^0 \) and \( R_2^1 \).

We will see that significant properties of maps \( F_a^2 \) and \( F_a \) can be deduce only with the use of invariants \( R^0 \) and \( R_2^1 \). We start with the identification of \( F_a(\mathcal{D}(M_0)) \).

**Lemma 7.5.** (1) If \( \mathcal{D}(M_0) \) has equations \( R^0(x, y, z) = c_0 \) and \( R_2^1(x, y, z) = c_1 \), then the curve \( F_a(\mathcal{D}(M_0)) \) is a quartic curve whose equations are

\[ R^0(x, y, z) = c_1, \quad R_2^1(x, y, z) = c_0. \]

(7.12)

These two curves are disjoint if and only if \( M_0 \) does not belong to the surface \( \mathcal{K}_a \) of \( \mathbb{R}^+ \) with equation

\[ \omega(x, y, z) := R^0(x, y, z) - R_2^1(x, y, z) = 0, \]

(7.13)

that is,

\[ \tilde{\omega}(x, y, z) := xz(1 + x)(1 + z) - y^3 - y^2(1 + a + x + z) - y(a + x + z) = 0, \]

(7.14)

where \( \tilde{\omega}(x, y, z) = xyz \omega(x, y, z) \).

(2) The two parts of \( \mathbb{R}^+ \setminus \mathcal{K}_a \), denoted by

\[ \mathcal{K}_a^+ = \left\{ R^0 > R_2^1 \right\}, \quad \mathcal{K}_a^- = \left\{ R^0 < R_2^1 \right\}, \]

(7.15)

are two open sets whose union is dense in \( \mathbb{R}^+ \), which are exchanged by \( F_a \), and so both globally invariant under the action of \( F_a^2 \). If \( M_0 \notin \mathcal{K}_a \), one has

\[ \mathcal{D}^+(M_0) \subset \mathcal{K}_a^+ \iff c_0 > c_1 \iff F_a(\mathcal{D}^+(M_0)) \subset \mathcal{K}_a^+, \]

\[ \mathcal{D}^+(M_0) \subset \mathcal{K}_a^- \iff c_0 < c_1 \iff F_a(\mathcal{D}^+(M_0)) \subset \mathcal{K}_a^-. \]

(7.16)
Remark 7.6. In fine, $\mathcal{K}_a$ is the graph of a continuous function $\phi : (x, z) \mapsto \phi(x, z)$ on $\mathbb{R}^2$, whose epigraph is $\mathcal{W}^+$ and hypograph is $\mathcal{W}^-$, which are connected sets.

Proof. One has $M \in F_a(\mathcal{D}(M)) \iff R^i(F_a^{-1}(M)) = c_0$ and $R^j(F_a^{-1}(M)) = c_1$. With the expression $F_a^{-1}(x, y, z) = (y, z, (a + y + z)/x)$ this last condition is easily translated as $R^i(x, y, z) = c_0$ and $R^j(x, y, z) = c_1$. Now conditions (7.13) or (7.14) are obvious, and the conditions $\mathcal{D}^+(M) \subset \mathcal{K}_a^+$ and $\mathcal{D}^+(M) \subset \mathcal{K}_a^-$ also.

For the point (3), the equation $\bar{\omega}(t, t, t) = 0$ is $t^2 - 2t - a = 0$, whose only positive solution is $\ell$.

For $t > 1$ the equation $\bar{\omega}(t, (t + a)/(t - 1), t) = 0$ is $(t^2 - 2t - a)(t^4 + 3a^2 + 3a - 2)t + (a - 1)^2 = 0$. The polynomial of degree 4 in $t$ is a degree 2 polynomial in $a$, which is positive for $t > 1$ because its discriminant is $-t(t - 1)^2(3t + 4)$. So the only solution is $t = \ell$.

In fine the last assertion is obvious by the study of the function $y \mapsto \bar{\omega}(x, y, z)$.

Remark 7.6. (1) In [4] the surface $\mathcal{K}_a$ with (7.14) is obtained as part of the locus of points where the gradients of $G_a$ and $H_a$ are parallel (the other points are these ones of the hyperbola $\mathcal{A}_a$).

(2) Points (1), (2), and (3) of Lemma 7.5 are obviously true for every odd $q$, from Proposition 7.10. Some easy calculations show that the point (4) is also true for an odd dimension $q$.

(3) The curves $\mathcal{D}(M)$ and $F_a(\mathcal{D}(M))$ are not disjoint in $\mathbb{P}^3$; they have in common four points at infinity: $(0, 0, 1, 0), (0, 1, -1, 0), (1, 0, 0, 0)$, and $(1, -1, 0, 0)$.

From the fact that if $M_0 \notin \mathcal{K}_a$, $\mathcal{D}(M_0) \cap F_a(\mathcal{D}(M_0)) = \emptyset$, the following result is obvious (and has an obvious generalization to order $q$ Lyness’ equation when $q$ is odd, which is in [4]) and gives again points (2) and (3) of Proposition 6.4.

Corollary 7.7. No point of $\mathbb{R}^3 \setminus \mathcal{K}_a$ is periodic under the action of $F_a$ with an odd period.

Now we can assert the principal qualitative result on the map $F_a^2$. Via Proposition 7.13 below we find again a result of [4], but it comes in this last paper from new and interressant, but sophisticated, arguments, whereas we use here only previous results of [17].

Theorem 7.8. Let $M_0 \notin \mathcal{K}_a \cup \mathcal{A}_a$. Then the two quartics $\mathcal{D}(M_0)$ and $F_a(\mathcal{D}(M_0))$ are disjoint, exchanged by $F_a$, both globally invariant under the action of $F_a^2$. The positive and compact parts $\mathcal{D}^+(M_0)$ and $F_a(\mathcal{D}^+(M_0))$ are both homeomorphic to the circle, and the restrictions of $F_a^2$ to each of these two curves are conjugated to rotations on the circle, with the same angle.

Proof. We can suppose that $M_0 \in \mathcal{K}_a^+$, that is, $c_0 > c_1$. The proof is the same in the other case.

(1) Let $\pi$ be the projection on the $(x, z)$-plane, and put

$$E_0(M_0) := \pi(\mathcal{D}(M_0)).$$

First we remark that $\pi$ is injective on the surface $R^i(x, y, z) = c_0$, because it is the graph of the function $(x, z) \mapsto y = (1 + x)(1 + z)/c_0$. So the projection $\pi$ is a homeomorphism of the compact curve $\mathcal{D}^+(M_0)$ onto its projection $E_0^+(M_0) = E_0(M_0) \cap \mathbb{R}^2$.

(2) We search the conjugated of the restriction of $F_a^2$ to $\mathcal{D}^+(M_0)$ by the projection $\pi$. We have easily $F_a^2(x, y, z) = ((a + x + y + az + xz)/yz, (a + x + y)/z, x)$, and if $(x, y, z) \in \mathcal{D}(M_0)$,
we have \( y = (1 + x)(1 + z)/c_0 \), and so we obtain (after simplification by the factor \( 1 + z \)) that if \((x, z) \in E_0^+(M_0)\), then

\[
\pi \left[ F_\alpha^2 \left( \pi^{-1}(x, z) \right) \right] = \left( \frac{(1 + c_0 a) + (1 + c_0) x}{z(1 + x)}, x \right).
\] (7.18)

So we conclude that the restriction of \( F_\alpha^2 \) to \( \mathcal{D}^+(M_0) \) is conjugated by \( \pi \) to the restriction to \( E_0^+(M_0) \) of the map

\[
T_\alpha(c_0) : \mathbb{R}^2_+ \longrightarrow \mathbb{R}^2_+ : (x, z) \mapsto \left( \frac{(1 + c_0 a) + (1 + c_0) x}{z(1 + x)}, x \right).
\] (7.19)

(3) But this map \( T_\alpha(c_0) \) in \( \mathbb{R}^2_+ \) is studied in [17]. We know from this paper (see its introduction and part 4: “The homographic case”) that there is an invariant of this map and of the associated difference equation \( x_{n+2}x_n = ((1 + c_0 a) + (1 + c_0)x_{n+1})/(1 + x_{n+1}) \). More precisely, put

\[
p_0 = 1 + c_0 a, \quad q_0 = 1 + c_0;
\] (7.20)

then one has

\[
\frac{p_0 + q_0 x}{1 + x} = \frac{(p_0 + q_0 x)(q_0 + x)}{(1 + x)(q_0 + x)} = \frac{a' + b' x + c'x^2}{c' + d' x + x^2},
\] (7.21)

\[
a' = p_0q_0, \quad b' = p_0 + q_0^2, \quad c' = q_0, \quad d' = q_0 + 1.
\] (7.22)

Then we see in [17] that the quantity

\[
V_0(x, z) := \frac{1}{xz} \left( x^2 z^2 + a'xz(x + z) + c' \left( x^2 + z^2 \right) + b'(x + z) + a' \right)
\] (7.23)

is invariant under the action of \( T_\alpha(c_0) \). So the curves \( \Gamma_k \) with equations \( V_0(x, z) = k \) are invariant by \( T_\alpha(c_0) \), and it is proved in [17] that these curves are, if \( k > k_m \), elliptic quartics, which are homeomorphic to circles, and that the restriction of \( T_\alpha(c_0) \) to \( \Gamma_k \) is conjugated to a rotation. So the theorem will be proved if we show that the curve \( E_0(M_0) \) is a curve \( \Gamma_k \), for a \( k \) satisfying \( k > k_m \), that is, if \( \Gamma_k \) does not reduce to the fixed point of \( T_\alpha(c_0) \).

So we will determine the equation of \( E_0(M_0) \). We take \( y \) from the relation \( R_0(x, y, z) = c_0 \) and put it in the relation \( R_0(y, x, z) = c_1 \). We obtain for equation of \( E_0(M_0) \)

\[
\left( 1 + \frac{(1 + x)(1 + z)}{c_0} \right) \left( a + x + z + \frac{(1 + x)(1 + z)}{c_0} \right) = c_1 xz.
\] (7.24)
An easy calculation gives the form of the equation

\[ x^2z^2 + (2 + c_0)xz(x + z) + (c_0 + 1)(x^2 + z^2) + \left(1 + c_0a + (c_0 + 1)^2\right)(x + z) - xz\left(c_0^2c_1 - (3 + a)c_0 - 4\right) = 0 \]  

(7.25)

or, from (7.20) and (7.22), \( V_0(x, z) = k_0 \); that is, \( E_0(M_0) \) is the curve \( \Gamma_{k_0} \) where

\[ k_0 = c_0^2c_1 - (3 + a)c_0 - 4. \]

(7.26)

It remains to show that this curve \( \Gamma_{k_0} \), which passes through the point \( m_0 = (x_0, z_0) = \pi(M_0) \), does not reduce to the fixed point of \( T_a(c_0) \), which has the form \( x = z = t \) where

\[ t^3 + t^2 - (1 + c_0)t - (1 + c_0a) = 0. \]

(7.27)

If it were the case, we would have \( x_0 = z_0 = t \), and so \( c_0 = (1 + t)^2 / y_0 \). If we put this value in the equation giving \( t \), we obtain the relation \((1 + t)^2 / y_0)(a + t) = t^3 + t^2 - t - 1 \), and this gives \( y_0 = (t + a)/(t - 1) \). So the point \( M_0 = (t, (t + a)/(t - 1), t) \) would be in the hyperbola \( \mathcal{K}_a \), because it is easy to see that the solution \( t \) of (7.28) is greater than 1. But we have supposed that \( M_0 \notin \mathcal{K}_a \), and so the curve \( E_0(M_0) \) does not reduce to the fixed point of \( T_a(c_0) \) (besides, it was clear that if \( M_0 \in \mathcal{K}_a \), it would be a fixed point for \( F^2_a \)). Remark that we do not use the hypothesis \( M_0 \notin \mathcal{K}_a \), so the preceding results are true if only \( M_0 \notin \mathcal{K}_a \), but we will see in Proposition 7.11 what happens when \( M_0 \in \mathcal{K}_a \setminus \{L\} \).

(4) Then we know from [17] that there is a homeomorphism \( \psi \) of \( E_0^1(M_0) \) onto the unit circle \( T \) (built with the Weierstrass' function) such that the map \( T_a(c_0)E_0^1(M_0) \) is conjugated by \( \psi \) to a rotation on \( T \) by an angle \( 2\pi\theta_a(M_0) \) well defined, where \( \theta_a(M_0) \in [0, 1/2] \). So by the map \( \psi \circ \pi \) the restriction of \( F^2_a \) to the curve \( \mathcal{S}^*(M_0) \) is conjugated to the same rotation on \( T \). Moreover the map \( \psi \circ \pi \circ F_a^{-1} \) is an homeomorphism of \( F_a(\mathcal{S}^*(M_0)) \) onto \( T \), which conjugates \( F^2_a|F_a(\mathcal{S}^*(M_0)) \) to the same rotation on \( T \). \( \square \)

One can see in Figure 2 the arrangement of the curves and surfaces of Lemma 7.5 and Theorem 7.8.

From the parts (1) and (2) of the proof, which do not use the hypothesis \( M_0 \in \mathcal{K}_a \cup \mathcal{K}_a \), we find again the result of [16], which is proved in this paper by the “QRT method” (remark that the value of \( k_0 \) is not given in [16]).

**Corollary 7.9** ([16]). If \( (u_n) \) is a Lyness' sequence of order 3 (a solution of (1.1)), then the sequences \( x_n = u_{2n} \) and \( y_n = u_{2n+1} \) are solutions of two second order "homographic" difference equations

\[ x_{n+2}x_n = \frac{(1 + ac_0) + (1 + c_0)x_{n+1}}{1 + x_{n+1}}, \quad y_{n+2}y_n = \frac{(1 + ac_1) + (1 + c_1)x_{n+1}}{1 + x_{n+1}}, \]

(7.28)
where $c_0 = (1 + u_0)(1 + u_2)/u_1$ and $c_1 = (1 + u_1)(1 + u_3)/u_2 = (1 + u_1)(a + u_0 + u_1 + u_2)/u_2u_0$. The orbit of $(x_{n+1}, x_n)$ is on $E^+_0(M_0)$, and the orbit of $(y_{n+1}, y_n)$ is on the quartic

$$E^+_1(M_0) := E^+_0(M_1) = \pi(F_a(\mathcal{D}^+(M_0))),$$

(7.29)

and the equation of $E_1(M_0)$ is the same as this one of $E_0(M_0)$ with $c_1$ and $c_0$ exchanged.

**Proof.** The proof is obvious for $x_n = u_{2n}$, because $\pi(M_{2n}) = (u_{2n+2}, u_{2n})$ and that $\pi(M_{2n+2}) = \pi(F^2_a(M_{2n})) = T_a(\pi(M_{2n}))$. For $y_n = u_{2n+1}$ the proof is the same, starting from $M_1 = F_a(M_0) = (u_0, u_2, u_1)$, with the quartic curve $E_1(M_0) := E_0(M_1) = \pi(F_a(\mathcal{D}^+(M_0)))$, whose equation is the same as this one of $E_0(M_0)$ with $(1 + u_1)(1 + u_3)/u_2$ in place of $c_0$, and this number is $c_1 = (1 + u_1)(a + u_0 + u_1 + u_2)/u_0u_2$. The value $k_1$ in the equation $V_1(x, z) = k_1$ of $E_1(M_0)$ is, by (7.26), $c_1^2 - (3 + a)c_1 - 4$, where $c_2 = c_1(M_1) = R^1_a(M_1) = R^1_a(F_a(M_0)) = R_0(M_0) = c_0^*; so k_1 is nothing that $k_0$ where $c_0$ and $c_1$ are exchanged.

Now there are some questions. First, what does happen if $M_0 \in \mathcal{I}_a \setminus \mathcal{I}_a \setminus \{L\}$? Secondly, what does happen if $M_0 \in \mathcal{I}_a \setminus \{L\}$? Third, which information can one have about the numbers $\theta_a(M_0)$ with our method (such informations are of course in [4])? Fourth, what are exactly the relationships between the curves $\mathcal{D}(M_0)$ and $C(M_0)$?

For the first question, we know the action of $F_a$ and $F^2_a$ on $M_0$, by Theorem 5.4: if $M_0 = (t, (t + a)/(t - 1), t)$ for some $t > 1$, then $F_a(M_0) = ((t + a)/(t - 1), t, (t + a)/(t - 1))$ and $F^2_a(M_0) = M_0$. But it is not clear a priori that $\mathcal{D}^+(M_0)$ reduces to the point $M_0$. It is actually the case.

**Proposition 7.10.** If $M_0 \in \mathcal{I}_a$, then $\mathcal{D}^+(M_0) = \{M_0\}$.

**Proof.** We search the fixed point of $T_a(c_0)$ if $M_0 = (t, (t + a)/(t - 1), t)$ for some $t > 1$. Its coordinates are $x = z = X$, where $X$ is the positive solution of the equation

$$X^3 + X^2 - (1 + c_0)X - (1 + c_0a) = 0.$$ 

(7.30)

But this equation is satisfied by $X = t$; so the fixed point is $(t, t) = \pi(M_0)$, and by [17] the quartic $E^+_0(M_0) = \pi(\mathcal{D}^+(M_0))$ reduces to $\{t, t\}$, and so $\mathcal{D}^+(M_0)$ reduces to $\{M_0\}$.

For the second question, we have a partially answer.

**Proposition 7.11.** If $M_0 \in \mathcal{I}_a \setminus \{L\}$, one has the following properties:

1. the curves $\mathcal{D}(M_0)$ and $F_a(\mathcal{D}(M_0))$ coincide;
2. the solution $(u_n)$ of the order 3 Lyness' difference equation is such that the sequence $s_n := (1 + u_n)/c_0$ is a solution of an order 2 Lyness' equation with negative parameter $-1/c_0$: $s_{n+1} + s_n = s_{n+2} - s_n$.
3. the curve $\mathcal{D}^+(M_0)$ is homeomorphic to the circle $\mathbb{T}$, globally stable by $F_a$, and the restriction of $F^2_a$ to this curve is, as in Theorem 7.8, conjugated by this homeomorphism to a rotation by angle $2\pi \theta_a(M_0)$. 
Proof. First we note that if \( M_0 \in \mathcal{K}_a \setminus \{L\} \), then \( R^0(M_0) = c_0 = c_1 = R^1_a(M_0) \). So the equations of \( \mathcal{D}(M_0) \) and \( F_a(\mathcal{D}(M_0)) \) are the same, by (7.12). Moreover we have

\[
\frac{(1 + u_{2n+2})(1 + u_{2n})}{u_{2n+1}} = \frac{(1 + u_0)(1 + u_2)}{u_1} = c_0 = c_1 = \frac{(1 + u_1)(1 + u_3)}{u_2}
\]

by the invariance under the action of \( F^2_a \). So we have for all \( n \) \( (1 + u_n)(1 + u_{n+2})/u_{n+1} = c_0 \).

If we put \( s_n = (1 + u_n)/c_0 \), we obtain the announced order 2 Lyness’ difference equation \( s_{n+2}s_n = s_{n+1} - 1/c_0 \). For the final assertion on the restriction of \( F^2_a \) to \( \mathcal{D}^+(M_0) \), we note that the proof of Theorem 7.8 is valid in this case, and it uses only the hypothesis \( M_0 \notin \mathcal{K}_a \).

We have given in Theorem 7.8 and Proposition 7.11 a description of \( F^2_a|\mathcal{D}^+(M_0) \). We have also a good description of the restriction of \( F_a \) to \( \mathcal{D}^+(M_0) \cup F_a(\mathcal{D}^+(M_0)) \).

**Theorem 7.12.** Suppose \( a \neq 1 \) and \( M_0 \notin \mathcal{K}_a \cup \mathcal{K}_a \). Then the restriction of \( F_a \) to the set \( \mathcal{D}^+(M_0) \cup F_a(\mathcal{D}^+(M_0)) \) is conjugated to the following map in the set \( \{-1, 1\} \times \mathbb{T} \):

\[
(\varepsilon, z) \mapsto (-\varepsilon, \varphi_\varepsilon(z)), \quad \text{where } \varepsilon = \pm 1, \; \varphi_1 = I_d, \; \varphi_{-1}(z) = ze^{2\pi i \theta_a(M_0)}.
\]

See Figure 1 for this map.

**Proof.** It is easy from the proof of Theorem 7.8, where now we distinguish the two circles by a product by \( \{-1\} \) or \( \{1\} \).

Now we will give an answer to the question about relationship between curves \( \mathcal{D}(M_0) \) and \( C(M_0) \), which will give again, with Theorem 7.8, the principal result of [4].

**Proposition 7.13.** When \( M_0 \notin \mathcal{K}_a \cup \mathcal{K}_a \), the curve \( C(M_0) \) is the union of 8 straight lines \( d_1, \ldots, d_8 \) not included in \( \mathcal{D}(M_0) \) nor in \( F_a(\mathcal{D}(M_0)) \), and of the two disjoint quartics \( \mathcal{D}(M_0) \) and \( F_a(\mathcal{D}(M_0)) \):

\[
C(M_0) = d_1 \cup d_2 \cup \cdots \cup d_8 \cup \mathcal{D}(M_0) \cup F_a(\mathcal{D}(M_0)),
\]

(7.33)
Proof. We write the equations of \( C(M_0) \) in homogeneous coordinates, for taking into account the points at infinity:

\[
(t + x)(t + y)(t + z)(at + x + y + z) - Kxyzt = 0,
\]

\[
(t + x + y)(t + y + z)(at^2 + xz + tx + ty + tz) - Mxyzt = 0.
\]

So it is obvious that the following straight lines (the two first are at infinity)

\[
\{ t = 0, \ x = 0 \}; \quad \{ t = 0, \ z = 0 \}; \quad \{ x = 0, \ y = -1 \}; \quad \{ x = 0, \ a + y + z = 0 \}; \\
\{ z = 0, \ y = -1 \}; \quad \{ z = 0, \ a + x + y = 0 \}; \quad \{ y = 0, \ x = -1 \}; \quad \{ y = 0, \ z = -1 \}
\]

satisfy the two equations of the surfaces whose intersection is \( C(M_0) \). But the curve \( C(M_0) \) has degree 16, and so we can write it as the union of the 8 straight lines and of a degree 8 algebraic curve \( \tilde{C}(M_0) \). But we have the following result.

**Lemma 7.14.** The quartic curves \( \mathcal{D}(M_0) \) and \( F_a(\mathcal{D}(M_0)) \) do not contain any straight lines.

So, because the union of the two quartic curves has degree 8, we conclude that \( \tilde{C}(M_0) = \mathcal{D}(M_0) \cup F_a(\mathcal{D}(M_0)) \), and this proves the proposition.

Proof of Lemma 7.14. The proof is the same for the two curves. If \( \mathcal{D}(M_0) \) contains a straight line, then this line is a generatrix of the quadric \( Q_0 \) with equation \((1 + x)(1 + z) - c_0y = 0\). It is easy to see that these generatrices are the two families defined by

\[
D_\lambda: 1 + x = \lambda; \quad c_0y = \lambda(1 + z); \quad D'_\mu: 1 + z = \mu; \quad c_0y = \mu(1 + x) \quad (\lambda, \mu \in \mathbb{P}^1(\mathbb{R})).
\]

But it is obvious to see that none of these straight lines can be contained in the second quadric \( Q_1 \) with equation \((1 + y)(a + x + y + z) - c_1xz = 0\) (the conditions of inclusion are \(a = 1\) and \(c_1 = -1\), or \(M = 0\), which are impossible).

Now, we return to the curve \( E_0(M_0) \) when \( M_0 \not\in \mathcal{E}_a \), by using [17]. First, we put \( x = \mu X \) and \( z = \mu Z \), where \( \mu = \sqrt{\overline{a}} = \sqrt{p_0} = \sqrt{1 + c_0a} \). So the \((X, Z)\) equation of the curve is

\[
X^2Z^2 + dXZ(X + Z) + c\left(X^2 + Z^2\right) + b(X + Z) + 1 = K_0XZ,
\]

where one has

\[
b = \frac{p_0^2 + q_0^2}{p_0^{3/2}}, \quad c = \frac{q_0}{p_0}, \quad d = \frac{p_0 + 1}{\sqrt{p_0}}, \quad K_0 = \frac{k_0}{p_0}.
\]
It is proved in [17, Table 8.1, rows 2 and 7] that \( T_a(c_0) \) is globally 4-periodic if and only if \( b = d \) and \( c = 1 \), that is, \( p_0 = q_0 \), or \( a = 1 \). In this case, the restrictions of \( F^a_1 \) to \( \mathcal{S}^+(M_0) \) and to \( F_a(\mathcal{S}^+(M_0)) \) are 4-periodic for every \( M_0 \notin \mathcal{K}_a \), and so \( F_a \) is 8-periodic on \( \mathbb{R}^3 \), as it is well known. In fact it is easy to prove, by using \( T_a(c_0) \), that if some \( M_0 = (x_0, y_0, z_0) \) with \( x_0 \neq z_0 \) is 8-periodic, then \( a = 1 \), but this result is already in [3].

Now suppose that \( a \neq 1 \). Then \( p_0 \neq q_0 \), and so \( d \neq 1 \) and \( b \neq d \) from (7.39). Under this hypothesis it is proved in [17] that \( \lim_{k \to -\infty} \theta(k) = 1/4 \), where \( k \) is the value of the invariant function \( V_0 \) of (7.26) for the the dynamical system \( (\mathbb{R}^3, T_a(c_0)) \).

Then we can look at the question about numbers \( \theta_a(M_0) \). The first result comes from calculations which are made in [17], and that one can read again in the present context.

**Lemma 7.15.** The function \( \Theta : (a, M_0) \mapsto \theta_a(M_0) : [0, +\infty) \times (\mathbb{R}^3_+ \setminus \mathcal{K}_a) \to [0, 1/2] \) is analytical (as function of \( a, x_0, y_0, z_0 \)) on each of the sets \( [0, 1] \times (\mathbb{R}^3_+ \setminus \mathcal{K}_a) \) and \( [1, +\infty) \times (\mathbb{R}^3_+ \setminus \mathcal{K}_a) \).

**Proof.** The calculations of [17] show that \( \Theta \) is an analytical function of \( K_0 \), and in fact, if we look at the formulas of [17], of the coefficients \( b, c, d \) of (7.38), if \( a \neq 1 \). So it is an analytical function of \( p_0, q_0, k_0 \), and then of \( a \) and \( c_0, c_1 \) (see (7.21)), that is, of \( a \) and \( x_0, y_0, z_0 \).

The following result will be useful.

**Lemma 7.16.** If \( a \neq 1 \), there is an \( \alpha > 1 \) and a map \( t \mapsto M_0(t) : [\alpha, +\infty) \to \mathcal{K}_a \) such that \( \lim_{t \to +\infty} \theta_a(M_0(t)) = 1/4 \).

**Proof.** The dynamical system in the \((x, z)\)-plane \( T_a(c_0)(x, z) = ((p + qx)/z(1 + x), x) \) depends only on \( c_0 \) and so is the same for all the points of the quadric \( Q_0 \). Let \( M_0 \in \mathcal{K}_a \), with \( R^3(M_0) = c_0 < c_1 = R^3_x(M_0) \). Put \( M_0(t) := (t, (1 + t)/c_0, t) \), for \( t > 0 \). The point \( M_0(t) \) lies on a parabola \( P_0 \subset Q_0 \), and so for every \( t \) such that \( c_1(t) := R^3_x(M_0(t)) > c_0 \), the dynamical system in the \((x, z)\)-plane associated to \( M_0(t) \) is the same. Moreover the number \( k(t) := c_0^2 c_1(t) - (3 + a) c_0 - 4 \) of (7.27) has for value \((1 + (1 + t)^2)/c_0(x + 2t + (1 + t)^2)/c_0(1/t^2)\), which tends to \(+\infty\) when \( t \to +\infty \). So for \( t > \alpha \) one has \( c_1(t) > c_0 \). Now we have seen that \( \theta(k) \to 1/4 \) when \( k \to +\infty \), so \( \theta_a(M_0(t)) = \theta(k(t)) \to 1/4 \) when \( t \to +\infty \).

Now we have a dichotomy result.

**Theorem 7.17.** One has only two mutually exclusive possibilities.

1. The function \( \theta_a \) is constant on the open dense set \( \mathbb{R}^3_+ \setminus \mathcal{K}_a \), and then \( a = 1 \) and \( F_a \) is globally 8-periodic.
2. One has \( a \neq 1 \), and the interior of the image of the map \( \theta_a \) is a nonempty interval \( [\alpha, \beta] \); in this case, one has the four properties:
   a. The initial points \( M_0 \) for \( F_a^2 \) are dense in \( \mathbb{R}^3_+ \);
   b. The initial points \( M_0 \) whose orbit for \( F_a^2 \) is dense in \( \mathcal{S}^+(M_0) \) are dense in \( \mathbb{R}^3_+ \setminus \mathcal{K}_a \);
   c. For every \( a \neq 1 \) there exists an integer \( N(a) \) such that for every integer \( n \geq N(a) \) there is a starting point \( M_0 \) which is \( n \)-periodic for \( F_a^2 \);
   d. Each point \( M_0 \in \mathbb{R}^3_+ \setminus \mathcal{K}_a \) has sensitivity to initial conditions: it exists a number \( \delta_a(M_0) > 0 \) such that in every neighborhood of \( M_0 \) there exists \( M'_0 \), such that \( d(F_a^{2n}(M_0), F_a^{2n}(M'_0)) \geq \delta_a(M_0) \) for infinitely many \( n \).

These four properties are denoted in [17]: invariant pointwise chaotic behavior (IPCB).

**Proof.** Its principle is the same as in papers [1, 17]. From the analyticity, the two exclusive possibilities are obvious (recall that the values of \( \theta \) are the same in \( \mathcal{K}_a \) and in \( \mathcal{K}_a^+ \)). What does
Proposition 7.18. One has the following limit result:

\[
\lim_{M_0 \to z, M_0 \notin \mathcal{K}_a} \theta_a(M_0) = \frac{1}{\pi} \cos^{-1} \left( \frac{\sqrt{1 + a} + a - 1}{2a} \right). \tag{7.40}
\]

So the image of the function \( \theta_a \) contains the interval \( I_a^+ := \left[ 1/4, (1/\pi) \cos^{-1}((\sqrt{1 + a} + a - 1)/2a) \right] \) if \( a > 1 \), or the interval \( I_a^- := \left] (1/\pi)\cos^{-1}((\sqrt{1 + a} + a - 1)/2a), 1/4 \right[ \) if \( a < 1 \).
Then one has

\[
\bigcup_{a > 0} \text{Im(}\theta_a) \supset \left[ \frac{1}{\pi} \cos^{-1} \frac{3}{4}, \frac{1}{3} \right].
\]

(7.41)

and \((1/\pi)\cos^{-1}(3/4) = 0.230053456\ldots\). Moreover, \(1/3 \notin \bigcup_{a > 0} \text{Im}(\theta_a)\) and \(1/5 \notin \bigcup_{a > 0} \text{Im}(\theta_a)\).

Proof. The formula (7.40) comes from the extension of Zeeman’s differential method and from the eigenvalues of \(dF_a(L)\), the intervals \(I^*_a\) and \(I_a\) comes from Lemma 7.16 and the connectivity of \(\mathbb{R}^* \setminus I_a\), and the formula (7.41) comes from the variations of the function of \(a\) in formula (7.40).

Of course, formula (7.41) enables us to determine a set of even periods of solutions \((u_n)_n\) of order 3 Lyness’ difference equation, for some initial points \((u_2, u_1, u_0)\) and some \(a > 0\).

Corollary 7.19. Every even integer \(p \geq 42\) is the minimal period of some solution of an order 3 Lyness’ difference equation \(u_{n+3}u_n = a + u_{n+2} + u_{n+1}\), for some \(a > 0\).

Proof. It suffices to use exactly the same method as in [1, Theorem 4]. We do not make here the calculations, and we give only their principle: first, the use of the prime numbers theorem and of a computer gives \(N = 1541\); then the method of successives blocks gives \(N = 54\); at last, tests on the integers less than 54 give \(N = 21\).

Problem 1. What is exactly the interval \(\bigcup_{a > 0} \text{Im}(\theta_a)\)?

So we stop here the study of order 3 Lyness’ difference equation by other methods than in [4] and refer the lector to [4] for another study and more informations.

8. General Conjectures about Order \(q\) Lyness’ Difference Equation

We give here conjectures about the behavior of the map \(F_a\) in \(\mathbb{R}^*_a\), which extend known results of [1, 19] in case \(q = 2\), and of [4], or Section 7 of the present paper, in case \(q > 3\). These conjectures seem to be common with these ones of [6]. The first one extends a conjecture in [5].

Conjecture 9. Let \(q = 2m - 1\) or \(q = 2m + 1\) \((m \geq 2, q \geq 4)\). Then there are \(m\) differential invariants \(G_1, G_2, \ldots, G_m\) which tend to \(+\infty\) at the infinite point of the locally compact space \(\mathbb{R}^*_a\) and satisfy the Conjectures 1, 2, 3, and 4 of Section 3.2.

The following conjecture describes the behavior of the map \(F_a\) when restricted to the invariant set defined by the previous invariant.

Conjecture 10. Let \(q = 2m\) or \(q = 2m + 1\), \(m \geq 1\). There is an open dense subset \(U^*_a\) of \(\mathbb{R}^*_a\) such that for every \(M_0 \in U^*_a\) the following properties are satisfied by the invariant set \(C^*(M_0) = \bigcap_{i=1}^{(q+1)/2} \{M \in \mathbb{R}^*_a \mid G_i(M) = G_i(M_0)\}\) (which contains \(M_0\)).

(1) If \(q = 2m\) (there are \(m\) invariants), \(C^*(M_0)\) is a \(m\)-manifold which is homeomorphic to the \(m\)-dimensional torus \(T^m\) \((T = \mathbb{R}/\mathbb{Z})\), and the restriction of \(F_a\) to \(C^*(M_0)\) is conjugated to a well defined \(m\)-rotation \(\mathcal{R}_{(\theta_1, \ldots, \theta_m)}\) on \(T^m\): \((z_1, z_2, \ldots, z_m) \mapsto (z_1e^{2\pi i\theta_1}, z_2e^{2\pi i\theta_2}, \ldots, z_m e^{2\pi i\theta_m})\).
(2) If \( q = 2m + 1 \) (then there are \( m + 1 \) invariants), \( C^+(M_0) \) is a \( m \)-manifold which is homeomorphic to the union of two \( m \)-dimensional torus, that is, \( \{-1, 1\} \times \mathbb{T}^m \), and the restriction of \( F_a \) to \( C^+(M_0) \) is conjugated to the following map:

\[
(\varepsilon, z_1, z_2, \ldots, z_m) \mapsto (-\varepsilon, \phi_1^n(z_1), \phi_2^n(z_2), \ldots, \phi_m^n(z_m)),
\]

where \( \varepsilon = \pm 1 \), \( \phi_1^n = 1d, \phi_2^n(z) = z e^{2\pi i\theta_j} \).

(3) In the two cases, the map \( M_0 \mapsto (\theta_1, \ldots, \theta_m) \) is continuous, and it is constant on no open set in \( U_a^d \).

Moreover, the limit of \( (\theta_1, \ldots, \theta_m) \) when \( M_0 \mapsto L \) is, apart from a permutation, the angles of Theorem 5.4 on \( dF_a(L) \) if \( q \) is even, and twice these angles if \( q \) is odd.

We give some comments. Put \( p = \lfloor (q + 1)/2 \rfloor \). If we define \( G = (G_1, \ldots, G_p) : \mathbb{R}^{*q} \to \mathbb{R}^{*p} \), the set \( V \) of points where the rank of \( dG \) is maximum is open, and the set \( \mathbb{R}^{*q} \setminus V \), is defined by \( \binom{q}{p} \) algebraic equations (nullity of determinants), and so it is a closed set with empty interior: \( V \) is dense. The set \( V \) is invariant under the action of \( F_a \), and so if \( M_0 \in V \) its orbit is included in \( V \). It is possible that one can take \( V \) for the set \( U_a^d \) of Conjecture 9. Perhaps another approach when \( q \) is odd could be to find sufficiently invariants for the map \( F_a^2 \).

References