Research Article

Stability of Solutions for a Family of Nonlinear Difference Equations

Taixiang Sun,1 Hongjian Xi,1, 2 and Caihong Han1

1 College of Mathematics and Information Science, Guangxi University, Nanning, Guangxi 530004, China
2 Department of Mathematics, Guangxi College of Finance and Economics, Nanning, Guangxi 530003, China

Correspondence should be addressed to Taixiang Sun, stx1963@163.com

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We consider the family of nonlinear difference equations:

\[ x_{n+1} = \left( \sum_{i=1}^{3} f_i(x_n, \ldots, x_{n-k}) + \sum_{i=3}^{5} f_i(x_n, \ldots, x_{n-k}) \right) / \left( \sum_{i=1}^{2} f_i(x_n, \ldots, x_{n-k}) + \sum_{i=3}^{5} f_i(x_n, \ldots, x_{n-k}) \right), \]

where \( f_i \in C([0, +\infty)^k, (0, +\infty)) \), for \( i \in \{1, 2, 4, 5\} \), and the initial values \( x_{-k}, x_{-k+1}, \ldots, x_0 \in (0, +\infty) \). We give sufficient conditions under which the unique equilibrium \( \bar{x} = 1 \) of these equations is globally asymptotically stable, which extends and includes corresponding results obtained in the cited references.

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1. Introduction

In [1], Papaschinopoulos and Schinas investigated the global asymptotic stability of the following nonlinear difference equation:

\[ x_{n+1} = \frac{\sum_{i \in \mathbb{Z}_k - \{j, j-1\}} x_{n-i} + x_{n-j}x_{n-j+1} + 1}{\sum_{i \in \mathbb{Z}_k} x_{n-i}}, \quad n = 0, 1, \ldots, \] (1.1)

where \( k \in \{1, 2, 3, \ldots\} \), \( \{j, j-1\} \subset \mathbb{Z}_k \equiv \{0, 1, \ldots, k\} \), and the initial values \( x_{-k}, x_{-k+1}, \ldots, x_0 \in \mathbb{R}_+ \equiv (0, +\infty) \).

Moreover, Kruse and Nesemann [2] studied the global asymptotic stability of the unique equilibrium of a discrete dynamical system, and as a special result they proved that the unique equilibrium \( \bar{x} = 1 \) of the Putnam difference equation
\begin{equation}
\label{eq:1.2}
x_{n+1} = \frac{x_n x_{n-1} + x_{n-2} x_{n-3}}{n = 0, 1, \ldots,}
\end{equation}

is globally asymptotically stable, where the initial values \(x_0, x_{-1}, x_{-2}, x_{-3} \in \mathbb{R}_+\).

In [3], Çinar et al. investigated the global asymptotic stability of the following nonlinear difference equation:

\begin{equation}
\label{eq:2.1}
x_{n+1} = \frac{x_n \sum_{i=1}^k x_{n-i} + 1}{x_n + x_{n-1} + x_n \sum_{i=2}^k x_{n-i}}, \quad n = 0, 1, \ldots,
\end{equation}

where \(k \in \{1, 2, 3, \ldots\}\) and the initial values \(x_{-k}, x_{-k+1}, \ldots, x_n \in \mathbb{R}_+\). For closely related results, see [4–10].

In this paper, we consider the family of nonlinear difference equations:

\begin{equation}
\label{eq:2.2}
x_{n+1} = \frac{\sum_{i=1}^k f_i(x_{n-i}, \ldots, x_{n-k}) + f_4(x_{n-i}, \ldots, x_{n-k}) f_5(x_{n-i}, \ldots, x_{n-k})}{f_1(x_{n-i}, \ldots, x_{n-k}) f_2(x_{n-i}, \ldots, x_{n-k}) + \sum_{i=3}^k f_i(x_{n-i}, \ldots, x_{n-k})}, \quad n = 0, 1, \ldots
\end{equation}

where \(f_i \in C((0, +\infty)^k, (0, +\infty))\), for \(i \in \{1, 2, 4, 5\}\), \(f_3 \in C((0, +\infty)^k, (0, +\infty))\), \(k \in \{1, 2, \ldots\}\), and the initial values \(x_{-k}, x_{-k+1}, \ldots, x_n \in (0, +\infty)\). Our main result is the following theorem.

**Theorem 1.1.** Let \(u^* = \max\{u_0, 1/u\}\), for any \(u \in \mathbb{R}_+\). If \(f(u_0, u_1, \ldots, u_k) \leq \max\{u_0^*, u_1^*, \ldots, u_k^*\}\), for \(i = 1, 2, 4, 5\), then \(\alpha = 1\) is the unique positive equilibrium of (1.4) which is globally asymptotically stable.

**2. The proof of Theorem 1.1**

In this section, we will prove Theorem 1.1. To do this, we need the following lemma.

**Lemma 2.1.** Let \((a, b, c, d) \in \mathbb{R}_+^4 - \{(1, 1, 1, 1)\}, e \in [0, \infty),\) and \(\alpha = \max\{a^*, b^*, c^*, d^*\}\). Then,

\begin{equation}
\label{eq:2.1}
\frac{1}{\alpha} < \frac{c + d + e + ab}{cd + e + a + b} < \alpha.
\end{equation}

**Proof.** Since \((a, b, c, d) \in \mathbb{R}_+^4 - \{(1, 1, 1, 1)\}, e \in [0, \infty),\) and \(\alpha = \max\{a^*, b^*, c^*, d^*\}\), we have \(\alpha > 1\) and either \(\alpha \geq \beta > 1/\alpha\) or \(\alpha > \beta \geq 1/\alpha\), for every \(\beta \in \{a, b, c, d\}\). If \(c < 1\) or \(d < 1\), then

\begin{equation}
\label{eq:2.2}
acd + aa + ab + ae > ab + c + d + e.
\end{equation}

It follows that

\begin{equation}
\label{eq:2.3}
\frac{c + d + e + ab}{cd + e + a + b} < \alpha.
\end{equation}

If \(c \geq 1\) and \(d \geq 1\), then \(\alpha \geq c > 1\) or \(\alpha > c \geq 1\) and \(\alpha \geq d > 1\) or \(\alpha > d \geq 1\). Thus, we have the following inequalities:

\begin{equation}
\label{eq:2.4}
\begin{align*}
a(a + b) & \geq 2ab, \\
acd + aa & \geq ac + 1 > 2c, \\
acd + ab & \geq ad + 1 > 2d.
\end{align*}
\end{equation}
It follows from (2.4) that

\[ acd + aa + ab + ae > ab + c + d + e, \]  

which implies

\[ \frac{c + d + e + ab}{cd + e + a + b} < a. \]  

By the symmetry, we have also that

\[ \frac{1}{a} < \frac{c + d + e + ab}{cd + e + a + b}. \]  

This completes the proof.

Proof of Theorem 1.1. Let \( \{x_n\}_{n=k}^{\infty} \) be a positive solution of (1.4) with the initial values \( x_k, x_{k+1}, \ldots, x_0 \in \mathbb{R}^+ \). For any \( n > 0 \), write

\[ p_n = \max\{x^*_n, x^*_{n-1}, \ldots, x^*_k\}. \]  

From Lemma 2.1, it follows that for any \( n \geq 0 \),

\[
x_{n+1} = \frac{\sum_{i=1}^5 f_i(x_{n+1}, \ldots, x_{n+k}) + f_4(x_{n+1}, \ldots, x_{n+k}) f_5(x_{n+1}, \ldots, x_{n+k})}{f_1(x_{n+1}, \ldots, x_{n+k}) f_2(x_{n+1}, \ldots, x_{n+k}) + \sum_{i=3}^5 f_i(x_{n+1}, \ldots, x_{n+k})}
\]

\[ \leq \max \left\{ \left[ f_i(x_{n+1}, \ldots, x_{n+k}) \right]^* : i = 1, 2, 4, 5 \right\} \]

\[ \leq \max \{x^*_{n-i} : 0 \leq i \leq k\} = p_n, \]

\[
x_{n+1} + 1 = \frac{\sum_{i=1}^5 f_i(x_{n+1}, \ldots, x_{n+k}) + f_4(x_{n+1}, \ldots, x_{n+k}) f_5(x_{n+1}, \ldots, x_{n+k})}{f_1(x_{n+1}, \ldots, x_{n+k}) f_2(x_{n+1}, \ldots, x_{n+k}) + \sum_{i=3}^5 f_i(x_{n+1}, \ldots, x_{n+k})}
\]

\[ \geq \frac{1}{\max \{f_i(x_{n+1}, \ldots, x_{n+k})^* : i = 1, 2, 4, 5\}} \]

\[ \geq \frac{1}{\max \{x^*_{n-i} : 0 \leq i \leq k\}} = \frac{1}{p_n}. \]

By (2.9), we have that for any \( n \geq 0 \),

\[ 1 \leq x^*_{n+1} \leq p_n, \quad p_{n+1} \leq p_n. \]  

From (2.10), we may assume that

\[ \lim_{n \to \infty} p_n = M \geq 1. \]  

Then,

\[ \frac{1}{M} \leq \liminf_{n \to \infty} x_n \leq \limsup_{n \to \infty} x_n \leq M. \]  

Since $p_n = \max\{x^*_n, x^*_{n-1}, \ldots, x^*_{n-4}\}$, there exists a sequence $l_n \to \infty$ such that

$$\lim_{s \to \infty} x_s = M$$

(2.13)

or

$$\lim_{s \to \infty} x_{l_n} = \frac{1}{M}.$$  

(2.14)

We may suppose (by taking a subsequence) that for $1 \leq i \leq k + 1$,

$$\lim_{s \to \infty} x_{l_n - i} = M_i.$$  

(2.15)

From (2.12), it follows that $1/M \leq M_i \leq M$.

We claim that $M = 1$. Indeed, if $M > 1$, then $f_i(M_1, \ldots, M_k) \neq 1$, for some $i \in \{1, 2, 4, 5\}$. If $\lim_{s \to \infty} x_{l_n} = M$, then it follows from Lemma 2.1 and (1.4) that

$$M = \frac{\sum_{i=1}^3 f_i(M_1, \ldots, M_{k+1}) + f_4(M_1, \ldots, M_{k+1}) f_5(M_1, \ldots, M_{k+1})}{f_1(M_1, \ldots, M_{k+1}) f_2(M_1, \ldots, M_{k+1}) + \sum_{i=3}^{5} f_i(M_1, \ldots, M_{k+1})}$$

$$< \max \{ [f_i(M_1, \ldots, M_{k+1})]^* : i = 1, 2, 4, 5 \}$$

$$\leq \max \{ M_i : 1 \leq i \leq k + 1 \} \leq M,$$

which is a contradiction.

If $\lim_{s \to \infty} x_{l_n} = 1/M$, then it follows from Lemma 2.1 and (1.4) that

$$\frac{1}{M} = \frac{\sum_{i=1}^3 f_i(M_1, \ldots, M_{k+1}) + f_4(M_1, \ldots, M_{k+1}) f_5(M_1, \ldots, M_{k+1})}{f_1(M_1, \ldots, M_{k+1}) f_2(M_1, \ldots, M_{k+1}) + \sum_{i=3}^{5} f_i(M_1, \ldots, M_{k+1})}$$

$$> \frac{1}{\max \{ [f_i(M_1, \ldots, M_{k+1})]^* : i = 1, 2, 4, 5 \}}$$

$$\geq \frac{1}{\max \{ M_i : 1 \leq i \leq k + 1 \}} \geq \frac{1}{M},$$

which is a contradiction. This completes the proof of the claim.

By (1.4) and (2.12), it follows that $\lim_{n \to \infty} x_n = 1$ and

$$1 = \frac{\sum_{i=1}^3 f_i(1, \ldots, 1) + f_4(1, \ldots, 1) f_5(1, \ldots, 1)}{f_1(1, \ldots, 1) f_2(1, \ldots, 1) + \sum_{i=3}^{5} f_i(1, \ldots, 1)}.$$  

(2.18)

Thus, $\bar{x} = 1$ is the unique positive equilibrium of (1.4).

For any $0 < \varepsilon < 1$, choose $\delta = \varepsilon/(\varepsilon + 1)$ and let $\{x_n\}_{n=k}^\infty$ be a solution of (1.4) with the initial values $x_{-k}, x_{-k+1}, \ldots, x_0 \in (1 - \delta, 1 + \delta)$. Then, for any $-k \leq i \leq 0$, we have that $x_i < 1 + \varepsilon$ and $1/x_i < 1/(1 - \delta) = 1 + \varepsilon$. By (2.9) it follows that for any $n \geq 0$,

$$1 - \varepsilon < \frac{1}{p_0} \leq \frac{1}{p_n} \leq x_{n+1} \leq p_n \leq p_0 < 1 + \varepsilon,$$

(2.19)

which implies that $\bar{x} = 1$ is globally asymptotically stable. This completes the proof.
3. Example

In this section, we will give an application of Theorem 1.1.

Example 3.1. Consider the following equation:

\[
x_{n+1} = x_{n-1} + x_{n-j} + g(x_n, \ldots, x_{n-k}) + x_{n-s}x_{n-t} \quad n = 0, 1, \ldots,
\]

where \( k \in \{1, 2, \ldots\} \), \( i, j, s, t \in \{0, 1, \ldots, k\} \), the initial conditions \( x_{-k}, x_{-k+1}, \ldots, x_0 \in \mathbb{R}_+ \), and \( g \in C([0, +\infty)^{k+1}, [0, +\infty]) \). Then, \( \bar{x} = 1 \) is the unique positive equilibrium of (3.1) which is globally asymptotically stable.

Proof. Let \( f_1(u_0, u_1, \ldots, u_k) = u_i, f_2(u_0, u_1, \ldots, u_k) = u_j, f_3(u_0, u_1, \ldots, u_k) = g(u_0, u_1, \ldots, u_k) \), \( f_4(u_0, u_1, \ldots, u_k) = u_s \), and \( f_5(u_0, u_1, \ldots, u_k) = u_t \). It is easy to verify that \( [f_i(u_0, u_1, \ldots, u_k)]^* \leq \max\{u_0, u_1^*, \ldots, u_k^*\} \), for \( i = 1, 2, 4, 5 \). By Theorem 1.1, we know that \( \bar{x} = 1 \) is the unique positive equilibrium of (3.1) which is globally asymptotically stable.

Remark 3.2. Let \( k \geq 3 \), \( f_1(u_0, u_1, \ldots, u_k) = 1, f_2(u_0, u_1, \ldots, u_k) = u_j, \) for some \( t \in \mathbb{Z}_k - \{j - 1, j\} \). \( f_3(u_0, u_1, \ldots, u_k) = \sum_{i=1}^{k} u_{j-i+1}, f_4(u_0, u_1, \ldots, u_k) = u_{j-1}, \) and \( f_5(u_0, u_1, \ldots, u_k) = u_i \). Then, (1.4) is (1.1), since \( [f_i(u_0, u_1, \ldots, u_k)]^* \leq \max\{u_0, u_1^*, \ldots, u_k^*\} \), for \( i = 1, 2, 4, 5 \). By Theorem 1.1, we know that the unique positive equilibrium \( \bar{x} = 1 \) of (1.1) is globally asymptotically stable.

Remark 3.3. Let \( k = 3 \), \( f_1(u_0, u_1, u_2) = u_0, f_2(u_0, u_1, u_2, u_3) = u_1, f_3(u_0, u_1, u_2, u_3) = 0, f_4(u_0, u_1, u_2, u_3) = u_2, \) and \( f_5(u_0, u_1, u_2, u_3) = u_3 \). Then, (1.4) is (1.2), since \( [f_i(u_0, u_1, \ldots, u_k)]^* \leq \max\{u_0, u_1^*, \ldots, u_k^*\} \), for \( i = 1, 2, 4, 5 \). By Theorem 1.1, we know that the unique positive equilibrium \( \bar{x} = 1 \) of (1.2) is globally asymptotically stable.

Remark 3.4. Let \( f_1(u_0, u_1, \ldots, u_k) = 1/u_0, f_2(u_0, u_1, \ldots, u_k) = u_1, f_3(u_0, u_1, \ldots, u_k) = u_2 + \cdots + u_{k-1}, f_4(u_0, u_1, \ldots, u_k) = u_k, \) and \( f_5(u_0, u_1, \ldots, u_k) = 1 \). Then, (1.4) is (1.3), since \( [f_i(u_0, u_1, \ldots, u_k)]^* \leq \max\{u_0, u_1^*, \ldots, u_k^*\} \), for \( i = 1, 2, 4, 5 \). By Theorem 1.1, we know that the unique positive equilibrium \( \bar{x} = 1 \) of (1.3) is globally asymptotically stable.

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References


