This article investigates the existence of positive periodic solutions for a first-order functional differential equations of the form

\[ y'(t) = -a(t)y(t) + \lambda h(t) f(y(t - \tau(t))), \]  

where \( a = a(t), \ h = h(t), \) and \( \tau = \tau(t) \) are continuous \( T \)-periodic functions. We will also assume that \( T > 0, \lambda > 0, \ f = f(t) \) as well as \( h = h(t) \) are positive, \( \int_0^T a(t) \, dt > 0. \)

Functional differential equations with periodic delays appear in a number of ecological models. In particular, our equation can be interpreted as the standard Malthus population model \( y' = -a(t)y \) subject to perturbation with periodical delay. One important question is whether these equations can support positive periodic solutions. Such questions have been studied extensively by a number of authors (cf. [1, 2, 3, 4, 6, 7] and the references therein). In this paper, we are concerned with the existence and nonexistence of periodic solutions when the parameter \( \lambda \) varies. For this purpose, we call a continuously differentiable and \( T \)-periodic function a periodic solution of (1) associated with \( \lambda^* \) if it satisfies (1) when \( \lambda = \lambda^* \). We show that there exists \( \lambda^* > 0 \) such that (1) has at least one positive \( T \)-periodic solution for \( \lambda \in (0, \lambda^*] \) and does not have any \( T \)-periodic positive solutions for \( \lambda > \lambda^* \). Our technique is based on the well-known upper and lower solutions method (cf. [5]).

We proceed from (1) and obtain

\[ \left[ y(t) \exp \left( \int_0^t a(s) \, ds \right) \right]' = \lambda \exp \left( \int_0^t a(s) \, ds \right) h(t) f(y(t - \tau(t))). \]  

Copyright © 2002 Hindawi Publishing Corporation
2000 Mathematics Subject Classification: 34B15, 34K13
URL: http://dx.doi.org/10.1155/S1085337502000878
After integration from $t$ to $t + T$, we obtain

$$y(t) = \lambda \int_t^{t+T} G(t,s) h(s) f(y(s - \tau(s))) \, ds,$$  \hspace{1cm} (3)

where

$$G(t,s) = \frac{\exp\left(\int_t^s a(u) \, du\right)}{\exp\left(\int_0^T a(u) \, du\right) - 1}. \hspace{1cm} (4)$$

Note that the denominator in $G(t,s)$ is not zero since we have assumed that $\int_0^T a(t) \, dt > 0$.

It is not difficult to check that any $T$-periodic function $y(t)$ that satisfies (3) is also a $T$-periodic solution of (1). Note further that

$$0 < N \equiv \min_{0 \leq s, t \leq T} G(t,s) \leq G(t,s) \leq \max_{0 \leq s, t \leq T} G(t,s) \equiv M, \quad t \leq s \leq t + T,$$

$$1 \geq \frac{G(t,s)}{\max_{0 \leq s, t \leq T} G(t,s)} \geq \frac{\min_{0 \leq s, t \leq T} G(t,s)}{\max_{0 \leq s, t \leq T} G(t,s)} = \frac{N}{M} > 0. \hspace{1cm} (5)$$

Now let $X$ be the set of all real $T$-periodic continuous functions, endowed with the usual linear structure as well as the norm

$$\|y\| = \sup_{0 \leq t \leq T} |y(t)|. \hspace{1cm} (6)$$

Then $X$ is a Banach space with cones

$$\Phi = \{ y(t) \in X : y(t) \geq 0 \},$$

$$\Omega = \{ y(t) : y(t) \geq \sigma \|y\|, \quad t \in \mathbb{R} \}, \hspace{1cm} (7)$$

where $\sigma = N/M$.

Define a mapping $F : X \to X$ by

$$(Fy)(t) = \lambda \int_t^{t+T} G(t,s) h(s) f(y(s - \tau(s))) \, ds. \hspace{1cm} (8)$$

Then it is easily seen that $F$ is completely continuous on bounded subsets of $\Omega$ and for $y \in \Phi$,

$$(Fy)(t) \leq \lambda M \int_0^T h(s) f(y(s - \tau(s))) \, ds \hspace{1cm} (9)$$

so that

$$(Fy)(t) \geq \lambda N \int_0^T h(s) f(y(s - \tau(s))) \, ds \geq \sigma \|Fy\|. \hspace{1cm} (10)$$
That is, $F \Phi$ is contained in $\Omega$.

**Lemma 1.** *The mapping $F$ maps $\Phi$ into $\Omega$.***

**Lemma 2.** *Suppose that*

\[
\lim_{u \to +\infty} \frac{f(u)}{u} = +\infty. \tag{11}
\]

*Let $I$ be a compact subset of $(0, +\infty)$. Then there exists a constant $b_I > 0$ such that $\|u\| < b_I$ for all $\lambda \in I$ and all possible $T$-periodic positive solutions $u$ of (1) associated with $\lambda$.***

**Proof.** Suppose to the contrary that there is a sequence $\{u_n\}$ of $T$-periodic positive solutions of (1) associated with $\{\lambda_n\}$ such that $\lambda_n \in I$ for all $n$ and $\|u_n\| \to +\infty$ as $n \to \infty$. Since $u_n \in \Omega$,

\[
\min_{0 \leq t \leq T} u_n(t) \geq \sigma \|u_n\|. \tag{12}
\]

By (11), we may choose $R_f > 0$ such that $f(u) \geq \eta u$ for all $u \geq R_f$, and there exists $n_0$ such that $\sigma \|u_{n_0}\| \geq R_f$, where $\eta$ satisfies

\[
\sigma \eta N \lambda_{n_0} \int_0^T h(s) \, ds > 1. \tag{13}
\]

Thus, we have

\[
\|u_{n_0}\| \geq u_{n_0}(t) = \lambda_{n_0} \int_t^{t+T} G(t,s) h(s) f(u_{n_0}(s - \tau(s))) \, ds \geq \sigma \eta N \lambda_{n_0} \int_0^T h(s) \|u_{n_0}\| \, ds > \|u_{n_0}\|. \tag{14}
\]

This is a contradiction. The proof is complete. \square

**Lemma 3.** *Suppose that

\[f \text{ is nondecreasing on } [0, +\infty) \text{ and } f(0) > 0.\]***

*Let (1) have a $T$-periodic positive solution $y(t)$ associated with $\bar{\lambda} > 0$. Then (1) also has a positive $T$-periodic solution associated with $\lambda \in (0, \bar{\lambda})$.***

**Proof.** In view of (3) and (15), we have

\[
y(t) = \bar{\lambda} \int_t^{t+T} G(t,s) h(s) f(y(s - \tau(s))) \, ds \geq \lambda \int_t^{t+T} G(t,s) h(s) f(y(s - \tau(s))) \, ds, \quad 0 < \lambda \int_t^{t+T} G(t,s) h(s) f(0) \, ds. \tag{16}
\]
Bifurcation in functional differential equations

Let $y_0(t) = y(t)$,

$$\bar{y}_{k+1}(t) = \lambda \int_t^{t+T} G(t, s) h(s) f\left(\bar{y}_k(s - \tau(s))\right) ds, \quad k = 0, 1, 2, \ldots, \quad (17)$$

$y_0(t) = 0$, and

$$\underline{y}_{k+1}(t) = \lambda \int_t^{t+T} G(t, s) h(s) f\left(\underline{y}_k(s - \tau(s))\right) ds, \quad k = 0, 1, 2, \ldots \quad (18)$$

Clearly, we have

$$y_0(t) \geq y_1(t) \geq \cdots \geq y_k(t) \geq \cdots \geq y_{\infty}(t) \geq y_0(t). \quad (19)$$

If we now let $y(t) = \lim_{k \to \infty} \bar{y}_k(t)$, then $y(t)$ satisfies (3). Clearly, we have

$$y(t) \geq y_{\infty}(t) = \lambda \int_t^{t+T} G(t, s) h(s) f(0) ds > 0. \quad (20)$$

This completes our proof. □

**Lemma 4.** Suppose that (11) and (15) hold. Then there exists $\lambda_* > 0$ such that (1) has a $T$-periodic positive solution.

**Proof.** Let

$$\beta(t) = \int_t^{t+T} G(t, s) h(s) ds, \quad M_f = \max_{0 \leq t \leq T} f\left(\beta(t - \tau(t))\right), \quad \lambda_* = \frac{1}{M_f}. \quad (21)$$

We have

$$\beta(t) = \int_t^{t+T} G(t, s) h(s) ds \geq \lambda_* \int_t^{t+T} G(t, s) h(s) f\left(\beta(s - \tau(s))\right) ds, \quad (22)$$

$$0 < \lambda_* \int_t^{t+T} G(t, s) h(s) f(0) ds.$$

Let $\bar{y}_0(t) = \beta(t)$,

$$\bar{y}_{k+1}(t) = \lambda_* \int_t^{t+T} G(t, s) h(s) f\left(\bar{y}_k(s - \tau(s))\right) ds, \quad k = 0, 1, 2, \ldots \quad (23)$$

$\underline{y}_0(t) = 0$, and

$$\underline{y}_{k+1}(t) = \lambda_* \int_t^{t+T} G(t, s) h(s) f\left(\underline{y}_k(s - \tau(s))\right) ds, \quad k = 0, 1, 2, \ldots \quad (24)$$
Clearly, we have

$$y_0(t) \geq y_1(t) \geq \cdots \geq y_k(t) \geq \cdots \geq y_1(t) \geq y_0(t).$$  \hfill (25)$$

If we now let \( y(t) = \lim_{k \to \infty} y_k(t) \), then \( y(t) \) satisfies (3). Clearly, we have

$$y(t) \geq y_1(t) = \lambda^*_1 \int_{t}^{t+T} G(t,s)h(s)f(0) \, ds > 0.$$  \hfill (26)$$

The proof is complete. \hfill \square

**Theorem 5.** Suppose that (11) and (15) hold. Then there exists \( \lambda^* > 0 \) such that (1) has at least one positive \( T \)-periodic solution for \( \lambda \in (0, \lambda^*] \) and does not have any \( T \)-periodic positive solutions for \( \lambda > \lambda^* \).

**Proof.** Suppose to the contrary that there is a sequence \( \{u_n\} \) of \( T \)-periodic positive solutions of (1) associated with \( \{\lambda_n\} \) such that \( \lim_{n \to \infty} \lambda_n = \infty \). Then either we have \( \|u_{n_j}\| \to +\infty \) as \( j \to \infty \) or there is \( \tilde{M} > 0 \) such that \( \|u_n\| \leq \tilde{M} \). Assume the former case holds. Note that \( u_n \in \Omega \) and thus

$$\min_{0 \leq t \leq T} u_n(t) \geq \sigma \|u_n\|.$$  \hfill (27)$$

By (11), we may choose \( R_f > 0 \) and \( \eta_1 > 0 \) such that \( f(u) \geq \eta_1 u \) when \( \sigma u \geq R_f \). On the other hand, there exist \( \{t_n\} \subset [0,T] \) such that \( u_{n_j}(t_n) = \|u_{n_j}\| \) and \( u'_{n_j}(t_n) = 0 \) by the periodicity of \( \{u_{n_j}(t)\} \). In view of (1), we have

$$a(t_n) \|u_{n_j}\| = a(t_n) u(t_n) = \lambda_n h(t_n) f(u_{n_j}(t_n) - \tau(t_n))) \geq \lambda_n \eta_1 \sigma h(t_n) \|u_{n_j}\|.$$  \hfill (28)$$

for all large \( j \). That is, we have \( \lambda_{n_j} \leq a(t_n)/(\eta_1 \sigma h(t_n)) \). Note that \( a(t)/h(t) \) is bounded. Thus, we obtain a contradiction.

Next, suppose that the latter case holds. In view of (15), there exists \( \eta_2 > 0 \) such that \( f(0) \geq \eta_2 \tilde{M} \). Then as above, we will obtain

$$a(t_n) \|u_n\| = a(t_n) u(t_n) = \lambda_n h(t_n) f(u_n(t_n - \tau(t_n))) \geq \lambda_n \eta_2 h(t_n) \tilde{M} \geq \lambda_n \eta_2 h(t_n) \|u_n\|.$$  \hfill (29)$$

for all \( n \). A contradiction will again be reached.
Thus, there exists $\lambda^* > 0$ such that (1) has at least one positive $T$-periodic solution for $\lambda \in (0, \lambda^*)$ and no $T$-periodic positive solutions for $\lambda > \lambda^*$.

Finally, we assert that (1) has at least one $T$-periodic positive solution for $\lambda = \lambda^*$. Indeed, let $\{\lambda_n\}$ satisfy $0 < \lambda_1 < \cdots < \lambda_k < \lambda^*$ and $\lim_{k \to \infty} \lambda_k = \lambda^*$. Since $u_n(t)$ is $T$-periodic positive solution of (1) associated with $\lambda_n$ and Lemma 2 implies that the set $\{u_n(t)\}$ of solutions is uniformly bounded in $\Omega$, the sequence $\{u_n(t)\}$ has a subsequence converging to $u(t) \in \Omega$. We can now apply the Lebesgue convergence theorem to show that $u(t)$ is a $T$-periodic positive solution of (1) associated with $\lambda = \lambda^*$. The proof is complete. □

**Example 6.** Consider the equation

$$x'(t) + a(t)x(t) = \lambda h(t)\{x^\gamma(t - \tau(t)) + 1\}, \quad \gamma > 1,$$

(30)

where $a$, $h$, and $\tau$ satisfy the same assumptions stated for (1). In view of Theorem 5, there exists a $\lambda^* > 0$ such that (30) has at least one $T$-periodic positive solution for $\lambda \in (0, \lambda^*)$ and no $T$-periodic positive solution for $\lambda > \lambda^*$.

**Example 7.** Consider the equation

$$y'(t) = -ay(t) + \lambda b(y^2(t) + \varepsilon),$$

(31)

where $a$, $b$, and $\varepsilon > 0$. Note that the function $f(x) = (x^2 + \varepsilon)$ satisfies (11) and (15) in Theorem 5. Therefore Theorem 5 may be applied. However, we may give a direct proof that, for $\lambda > a/(2b\sqrt{\varepsilon})$, this equation cannot have any positive $2\pi$-periodic solutions associated with $\lambda$. Indeed, assume to the contrary that $y(t)$ is such a solution. Then $y'(\xi) = 0$ for some $\xi \in [0, 2\pi]$. Hence

$$-ay(\xi) + \lambda b y^2(\xi) + \lambda b \varepsilon = 0.$$  

(32)

However, since the discriminant of the quadratic equation

$$\lambda bx^2 - ax + \lambda b \varepsilon = 0$$

(33)

satisfies

$$a^2 - 4\lambda^2 b^2 \varepsilon < 0,$$

(34)

a contradiction is obtained. We remark that when $\varepsilon = 0$, our equation reduces to the well-known logistic equation.

Similarly, we can consider the equation

$$x'(t) = a(t)x(t) - \lambda h(t)f(x(t - \tau(t))),$$

(35)
where \( a = a(t), h = h(t), \) and \( f = f(t) \) satisfy the same assumptions stated for (1). By (35), we have

\[
x(t) = \int_{t}^{t+T} H(t,s) h(s) f(x(s - \tau(s))) \, ds,
\]

where

\[
H(t,s) = \frac{\exp \left( - \int_{t}^{s} a(u) \, du \right)}{1 - \exp \left( - \int_{0}^{T} a(u) \, du \right)} = \frac{\exp \left( \int_{s}^{t+T} a(u) \, du \right)}{\exp \left( \int_{0}^{T} a(u) \, du - 1 \right)}
\]

which satisfies

\[
M \geq H(t,s) \geq N, \quad t \leq s \leq t + T,
\]

for some \( M \) and \( N > 0 \), and \( \sigma = N/M \leq 1 \).

**Theorem 8.** Suppose that (11) and (15) hold. Then there exists \( \lambda^* > 0 \) such that (35) has at least one positive \( T \)-periodic solution for \( \lambda \in (0, \lambda^*] \) and no \( T \)-periodic positive solution for \( \lambda > \lambda^* \).

**Acknowledgment**

This work was supported by the Natural Science Foundation of Shanxi Province and Yanbei Normal College. Part of this work was done during the first author’s visit to the Institute of Applied Mathematics, Academy of Mathematics and System Sciences, Chinese Academy of Sciences. The first author wishes to express his thanks to Professor Daomin Cao for his kind invitation and nice hospitality. We also thank the referee for his helpful criticisms.

**References**

Bifurcation in functional differential equations


Guang Zhang: Department of Mathematics, Yanbei Normal College and Datong College, Datong, Shanxi 037000, China
E-mail address: dtgzhang@yahoo.com.cn

Sui Sun Cheng: Department of Mathematics, Tsing Hua University, Hsinchu, Taiwan 30043, Taiwan
E-mail address: sscheng@math.nthu.edu.tw