This work is devoted to the study of a quasilinear elliptic system of resonant type. We prove the existence of infinitely many solutions of a related nonlinear eigenvalue problem. Applying an abstract minimax theorem, we obtain a solution of the quasilinear system

\[-\Delta_p u = F_u(x, u, v), \quad -\Delta_q v = F_v(x, u, v),\]

under conditions involving the first and the second eigenvalues.

1. Introduction

1.1. The problem and some previous results. We consider a gradient elliptic system

\[-\Delta_p u = F_u(x, u, v), \quad -\Delta_q v = F_v(x, u, v).\]  \hspace{1cm} (1.1)

Elliptic problems involving the \( p \)-Laplacian have been studied by several authors (cf. [3, 7, 8, 10, 11]). We recall some results from the work of Boccardo and de Figueiredo [4].

It is well known that the solutions of (1.1) in \( W = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \) are the critical points of the functional

\[\Phi(u, v) = \frac{1}{p} \int_\Omega |\nabla u|^p + \frac{1}{q} \int_\Omega |\nabla v|^q - \int_\Omega F(x, u, v)\]  \hspace{1cm} (1.2)

under the following three assumptions:

(1) \( \Omega \subset \mathbb{R}^N \) is a bounded domain, \( 1 < p, q < N \), so that the following continuous embeddings hold:

\[W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega), \quad W_0^{1,q}(\Omega) \subset L^{q^*}(\Omega);\]  \hspace{1cm} (1.3)
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(2) $F : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is $C^1$ and verifies the following growth condition:

$$|F(x,s,t)| \leq c(1 + |s|^\alpha + |t|^\beta) \quad \forall x \in \bar{\Omega}; s, t \in \mathbb{R};$$  (1.4)

(3) in order to have $\Phi \in C^1(W, \mathbb{R})$, we assume

$$|F_s(x,s,t)| \leq C(1 + |s|^{p^*-1} + |t|^{q^*(q'-1)/q'}) \quad \forall x \in \bar{\Omega}; s, t \in \mathbb{R},$$

$$|F_t(x,s,t)| \leq C(1 + |t|^{q^*-1} + |s|^{p^*(p'-1)/p'}) \quad \forall x \in \hat{\Omega}; s, t \in \mathbb{R}.$$  (1.5)

The geometry of $\Phi$ depends strongly on the values of $\alpha$ and $\beta$ in the estimate

$$|F(x,s,t)| \leq c(1 + |s|^\alpha + |t|^\beta) \quad \forall x \in \bar{\Omega}; s, t \in \mathbb{R},$$  (1.6)

where $\alpha \leq p^*$, $\beta \leq q^*$. In this work we are interested in the case $\alpha = p$, $\beta = q$ (systems of resonant type).

In our case, it is quite adequate to assume the following condition on $F$: consider the function

$$L(x,s,t) = \frac{1}{p} F_s(x,s,t)s + \frac{1}{q} F_t(x,s,t)t - F(x,s,t).$$  (1.7)

Assume that

$$\lim_{\|(s,t)\| \to \infty} L(x,s,t) = \pm \infty \quad \text{uniformly for } x \in \Omega.$$  (1.8)

This assumption implies that $\Phi$ satisfies the following compactness Cerami condition.

**Definition 1.1.** Let $X$ be a Banach space and $\Phi \in C^1(X, \mathbb{R})$. Given $c \in \mathbb{R}$, we say that $\Phi$ satisfies condition $(C_c)$, if

1. any bounded sequence $(u_n) \subset X$ such that $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ has a convergent subsequence;
2. there exist constants $\delta, R, \alpha > 0$ such that

$$\|\Phi'(u)\| \|u\| \geq \alpha \quad \forall u \in \Phi^{-1}([c-\delta, c+\delta]) \text{ with } \|u\| \geq R.$$  (1.9)

If $\Phi \in C^1(X, \mathbb{R})$ satisfies condition $(C_c)$ for every $c \in \mathbb{R}$, we say that $\Phi$ satisfies condition $(C)$.

Condition $(C)$ was introduced by Cerami [5]. It was shown in [2] that from condition $(C)$ it is possible to obtain a deformation lemma, that is fundamental in order to get minimax theorems.

In order to avoid resonance, Boccardo and de Figueiredo [4] introduced an assumption on $F$ involving an eigenvalue problem

$$-\Delta_p u - a G_u(u,v) = \lambda |u|^{p-2} u,$$

$$-\Delta_q v - a G_v(u,v) = \lambda |v|^{q-2} v,$$  (1.10)
where \( a = a(x) \in L^\infty(\Omega) \) and \( G \) is a \( C^1 \) even function \( G : \mathbb{R} \to [0, \infty) \) such that
\[
G(c^{1/p}s, c^{1/q}t) = cG(s, t) \quad \forall c > 0,
\]
\[
G(s, t) \leq K \left( \frac{1}{p} |s|^p + \frac{1}{q} |t|^q \right).
\]

We call such a \( G \) a \((p, q)\) homogeneous function.

It is easy to see that (1.11) implies (1.12). A \((p, q)\)-homogeneous function satisfies
\[
\frac{1}{p} G_s(s, t)s + \frac{1}{q} G_t(s, t)t = G(s, t).
\]

Examples of \((p, q)\) homogeneous functions are
\[
\begin{align*}
(1) & \quad G(s, t) = c_1 |s|^p + c_2 |t|^q, \\
(2) & \quad G(s, t) = c |s|^\alpha |t|^\beta \text{ with } \frac{\alpha}{p} + \frac{\beta}{q} = 1 \text{ where } c, c_1, c_2 \text{ are constants.}
\end{align*}
\]

The following results are proved in [4].

**Theorem 1.2.** Problem (1.10), with \( G \) as above, has a first eigenvalue \( \lambda_1(a) \), characterized variationally by
\[
\lambda_1(a) = \inf_{(u, v) \neq (0, 0)} \frac{(1/p) \int_{\Omega} |\nabla u|^p + (1/q) \int_{\Omega} |\nabla v|^q - \int_{\Omega} aG(u, v)}{(1/p) \int_{\Omega} |u|^p + (1/q) \int_{\Omega} |v|^q}
\]
which depends continuously on \( a \) in the \( L^\infty\)-norm.

**Theorem 1.3.** Assume (1.5), (1.6) with \( \alpha = p, \beta = q \), and that the following conditions hold:
\[
\begin{align*}
(1) & \quad \text{there exist positive numbers } c, R, \mu, \text{ and } \nu \text{ such that } \\
& \quad \frac{1}{p} \varepsilon F_s(x, s, t) + \frac{1}{q} t F_t(x, s, t) - F(x, s, t) \geq c(|s|^{\mu} + |t|^\nu) \quad \text{for } |s|, |t| > R,
\end{align*}
\]
\[
(2) \quad \text{there exists } G \text{ as above, such that}
\]
\[
\limsup_{|s|, |t| \to \infty} \frac{F(x, s, t)}{G(s, t)} \leq \lambda_1(a) \in L^\infty(\Omega),
\]
where \( \lambda_1(a) > 0 \).

Then the functional \( \Phi \) is bounded from below and the infimum is achieved.

**1.2. The existence of infinitely many eigenfunctions.** Let \( \mathcal{C} \) be the class of compact symmetric \((C = -C)\) subsets of the space \( W \). We recall that for \( C \in \mathcal{C} \) the Krasnoselskii genus \( \text{gen}(C) \) is defined as the minimum integer \( n \) such that there exists an odd continuous mapping \( \varphi : C \to (\mathbb{R}^n - \{0\}) \) (cf. [1]). We note
\[
\mathcal{C}_k = \{ C \in \mathcal{C} : \text{gen}(C) \geq k \}.
\]
For an arbitrary symmetric subset $S$ of $W - \{0\}$ the genus over compact sets $\gamma(S)$ is defined by
\[
\gamma(S) = \sup \{ \text{gen}(C) : C \subset S, C \in \mathcal{C}, C \text{ compact} \}. \tag{1.18}
\]

Now we may state our main result on the eigenvalue problem.

**Theorem 1.4.** The eigenvalue problem (1.10), with $G$ as above, has infinitely many eigenfunctions given by
\[
\lambda_k(a, G) = \inf_{C \in \mathcal{C}} \sup_{(u,v) \in C} \frac{(1/p) \int_\Omega |\nabla u|^p + (1/q) \int_\Omega |\nabla v|^q - \int_\Omega aG(u,v)}{(1/p) \int_\Omega |u|^p + (1/q) \int_\Omega |v|^q} \tag{1.19}
\]
and $\lambda_k(a, G) \to \infty$ as $k \to \infty$. Moreover, $\lambda_k$ depends continuously on $a$ in the $L^\infty$-norm.

**Remark 1.5.** Equivalently if we define
\[
S = \left\{ (u,v) \in W : \frac{1}{p} \int_\Omega |u|^p + \frac{1}{q} \int_\Omega |v|^q = 1 \right\}, \tag{1.20}
\]
we have
\[
\lambda_k(a, G) = \inf_{C \in \mathcal{C}} \sup_{(u,v) \in C} \frac{1}{p} \int_\Omega |\nabla u|^p + \frac{1}{q} \int_\Omega |\nabla v|^q - \int_\Omega aG(u,v). \tag{1.21}
\]
We will write $\lambda_k(a)$ instead of $\lambda_k(a, G)$, when the dependence on the $(p,q)$-homogeneous function $G$ is clear from the context.

1.3. **The existence result for resonant systems.** Applying Theorem 1.4 and an abstract minimax principle from [9], we prove the following theorem.

**Theorem 1.6.** Assume that $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ verifies (1.5), (1.6) with $\alpha = p$, $\beta = q$, (1.8), and that $a_1, a_2 \in L^\infty(\Omega)$ satisfy
\[
a_1(x) \leq \lim_{|s|,|t| \to \infty} \inf \frac{F(x,s,t)}{G_1(s,t)} \leq \lim_{|s|,|t| \to \infty} \sup \frac{F(x,s,t)}{G_2(s,t)} \leq a_2(x) \tag{1.22}
\]
with $G_1$ and $G_2$ two $(p,q)$-homogeneous functions and $\lambda_1(a_1, G_1) < 0 < \lambda_2(a_2, G_2)$, where $\lambda_1(a_1, G_1), \lambda_2(a_2, G_2)$ are given by (1.19). Then problem (1.1) has at least one solution.

**Remark 1.7.** The conditions above could be reformulated in terms of a different eigenvalue problem, for $a \in L^\infty(\Omega)$, $a(x) > 0$
\[
-\Delta_p u = \mu a G_u(u,v), \quad -\Delta_q v = \mu a G_v(u,v). \tag{1.23}
\]
This problem also has infinitely many eigenvalues given by
\[
\mu_k(a) = \inf_{C \in \mathcal{C}} \sup_{(u,v) \in C} \frac{(1/p) \int_\Omega |\nabla u|^p + (1/q) \int_\Omega |\nabla v|^q}{\int_\Omega aG(u,v)}. \tag{1.24}
\]
The condition $\lambda_1(a) < 0$ is equivalent to $\mu_1(a) < 1$, and the condition $\lambda_2(a) > 0$ is equivalent to $\mu_2(a) > 1$.

**Remark 1.8.** As an example for Theorem 1.6, we may take

$$G_1(s, t) = G_2(s, t) = |s|^{\alpha}|t|^{\beta}$$

with $\alpha/p + \beta/q = 1$;

$$F(x, s, t) = \lambda|s|^{\alpha}|t|^{\beta} + c|s|^{\mu}|t|^{\delta},$$

where $c \neq 0$ is a constant, and we assume that

$$\mu_1(1) < \lambda < \mu_2(1)$$

and $\mu < \alpha, \delta < \beta$, where $\mu_1(1), \mu_2(1)$ are defined as above (with $a \equiv 1$).

2. The eigenvalue problem

2.1. The functional framework. We apply the following abstract theorem due to Amann [1].

**Theorem 2.1.** Suppose that the following hypotheses are satisfied:

- $X$ is a real Banach space of infinite dimension, that is uniformly convex;
- $A : X \to X^*$ is an odd potential operator (i.e., $A$ is the Gateaux derivative of $\mathcal{A} : X \to \mathbb{R}$) which is uniformly continuous on bounded sets, and satisfies condition $(S)_1$: if $u_j \rightharpoonup u$ (weakly in $X$) and $A(u_j) \to v$, then $u_j \to u$ (strongly in $X$).
- For a given constant $\alpha > 0$, the level set

$$M_\alpha = \{ u \in X : \mathcal{A}(u) = \alpha \}$$

is bounded and each ray through the origin intersects $M_\alpha$. Moreover, for every $u \neq 0$, $\langle A(u), u \rangle > 0$ and there exists a constant $\rho_\alpha > 0$ such that $\langle A(u), u \rangle \geq \rho_\alpha$ on $M_\alpha$.
- The mapping $B : X \to X^*$ is a strongly sequentially continuous odd potential operator (with potential $\mathcal{B}$), such that $\mathcal{B}(u) \neq 0$ implies that $B(u) \neq 0$.

Let

$$\beta_k = \sup_{C \subset \mathcal{C}, C \subset M_\alpha} \inf_{u \in C} \mathcal{B}(u).$$

Then if $\beta_k > 0$, there exists an eigenfunction $u_k \in M_\alpha$ with $\mathcal{B}(u) = \beta_k$. If

$$\gamma(\{ u \in M_\alpha : \mathcal{B}(u) \neq 0 \}) = \infty,$$

then there exist infinitely many eigenfunctions.
We will work in the Banach space
\[ W = W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega) \]
equipped with the norm
\[ \| (u, v) \|_W = \sqrt{\| u \|_p^2 + \| v \|_q^2}. \]
As each factor is uniformly convex, we can conclude that \( W \) is uniformly convex (see [6]).

Given \((u^*, v^*) \in W^{-1,p'}(\Omega) \oplus W^{-1,q'}(\Omega)\) we may think of it as an element of \( W^* \):
\[ \langle (u^*, v^*), (u, v) \rangle = \langle u^*, u \rangle + \langle v^*, v \rangle. \]
Then we have \( W^* \cong W^{-1,p'}(\Omega) \oplus W^{-1,q'}(\Omega) \) (isometric isomorphism), where the norm in \( W^* \) is given by
\[ \| (u^*, v^*) \|_{W^*} = \sqrt{\| u^* \|_p^2 + \| v^* \|_q^2}. \]
With the notations of Theorem 2.1, we define
\[ A_0(u, v) = \frac{1}{p} \int_{\Omega} |\nabla u|^p + \frac{1}{q} \int_{\Omega} |\nabla v|^q, \]
\[ A(u, v) = A_0(u, v) - \int_{\Omega} aG(u, v) + M \left( \frac{1}{p} \int_{\Omega} |u|^p + \frac{1}{q} \int_{\Omega} |v|^q \right), \]
with \( a \) and \( G \) as in the statement of Theorem 1.4, and \( M \) a fixed constant such that \( M > K \|a\|_{L^{\infty}} \), where \( K \) is the constant in (1.12).

We write \( A_a \) instead of \( A \) when we want to remark the dependence on the weight \( a \)
\[ A(a, u, v) = (-\Delta_p u - aG_a(u, v) + M|u|^{p-2}u, -\Delta_q v - aG_a(u, v) + M|v|^{q-2}v), \]
\[ B(u, v) = \frac{1}{p} \int_{\Omega} |u|^p + \frac{1}{q} \int_{\Omega} |v|^q, \]
\[ B(u, v) = (|u|^{p-2}u, |v|^{q-2}v). \]

In order to apply Theorem 2.1, we prove the following two lemmas.

**Lemma 2.2.**
1. \( A \) is uniformly continuous on bounded sets.
2. \( A \) verifies the \((S)_1\) condition.

**Proof.** We write \( A = A_1 - A_2 \), where
\[ A_1(u, v) = (-\Delta_p u, -\Delta_q v), \]
\[ A_2(u, v) = (aG_a(u, v) - M|u|^{p-2}u, aG_a(u, v) - M|v|^{q-2}v). \]
We claim that $A_2 : W \to W^*$ verifies that: if $(u_j, v_j) \rightharpoonup (u, v)$ in $W$, then $A_2(u_j, v_j) \to A_2(u, v)$ in $W^*$.

Indeed, if $(u_j, v_j) \rightharpoonup (u, v)$, then

$$(u_j, v_j) \rightharpoonup (u, v) \quad \text{in} \quad L^p(\Omega) \times L^q(\Omega) \quad (2.12)$$

and we obtain that

$$G_u(u_j, v_j) \rightharpoonup G_u(u, v) \quad \text{in} \quad L^p(\Omega),$$
$$G_v(u_j, v_j) \rightharpoonup G_v(u, v) \quad \text{in} \quad L^q(\Omega). \quad (2.13)$$

Hence, $A_2(u_j, v_j) \to A_2(u, v)$ in $W^*$.

Let $(u_j, v_j) \rightharpoonup (u, v)$ in $W$ such that $A(u_j, v_j) \rightharpoonup (z, w)$. Therefore $A_2(u_j, v_j) \to A_2(u, v)$ and then $A_1(u_j, v_j) \to (z, w) + A_2(u, v)$. Since $A_1$ verifies condition (S)$_1$, it follows that $(u_j, v_j) \to (u, v)$.

\[\square\]

**Lemma 2.3.**

\begin{enumerate}
\item The set $M_\alpha = \{(u, v) \in W : \mathcal{A} = \alpha\}$ is bounded.
\item Every ray $t \cdot (u, v)$ with $(u, v) \neq 0$ intersects $M_\alpha$.
\item There exists a constant $\rho_\alpha > 0$ such that
\[\langle A(u, v), (u, v) \rangle \leq \rho_\alpha. \quad (2.15)\]
\item Condition (2.3) is satisfied.
\end{enumerate}

**Proof.**

\begin{enumerate}
\item As we have fixed $M > K\|a\|_{L^\infty}$ on $M_\alpha$, then
\[\alpha = \mathcal{A}(u, v) \geq \frac{1}{p} |\nabla u|^p + \frac{1}{q} |\nabla v|^q \quad (2.16)\]

and the proof is complete.

\item Let $f(c) = \mathcal{A}(c(u, v))$, $f(0) = 0$,
\[f(c) = \frac{c^p}{p} \int_\Omega |\nabla u|^p + \frac{c^q}{q} \int_\Omega |\nabla v|^q - \int_\Omega aG(cu, cv) + M \left( \frac{c^p}{p} \int_\Omega |u|^p + \frac{c^q}{q} \int_\Omega |v|^q \right) \quad (2.17)\]

From (1.12) and the choice of $M$, we have
\[f(c) \geq \frac{c^p}{p} \int_\Omega |\nabla u|^p + \frac{c^q}{q} \int_\Omega |\nabla v|^q \to +\infty \quad (2.18)\]

as $c \to \infty$. Since $f$ is continuous, there exists $c \in \mathbb{R}$ such that $f(c) = \alpha$.\]
We have
\[
\left(A(u,v), (u,v)\right) = \int_{\Omega} |\nabla u|^p + \int_{\Omega} |\nabla v|^q - \int_{\Omega} a\left[G_u(u,v)u + G_v(u,v)v\right] + M\left(\int_{\Omega} |u|^p + \int_{\Omega} |v|^q\right).
\]  
(2.19)

Then, using (1.13)
\[
\left(A(u,v), (u,v)\right) \geq \min\{p, q\} / H_{5101}(u,v) = \min\{p, q\} \alpha.
\]  
(2.20)

(4) In order to see that \(\gamma(M_\alpha) \geq k\), it is enough to show that \(M_\alpha\) contains subsets homeomorphic to the unit sphere in \(\mathbb{R}^k\) by an odd homeomorphism. Hence, the proof is completed. \(\square\)

2.2. The continuous dependence of \(\lambda_k(a)\) on \(a\). In this section we prove that the eigenvalue \(\lambda_k(a)\) depends continuously on the weight \(a\) in the \(L^\infty\)-norm. This result will be used for proving Lemma 3.3.

Proposition 2.4. The eigenvalue \(\lambda_k(a)\) depends continuously on \(a\) in the \(L^\infty\)-norm.

Proof. We have
\[
|\mathcal{A}_a(u,v) - \mathcal{A}_b(u,v)| \leq K \|a - b\|_{L^\infty} \left(\frac{1}{p} \int_{\Omega} |u|^p + \frac{1}{q} \int_{\Omega} |v|^q\right),
\]  
(2.21)

where \(K\) is given by condition (1.12), with \(\mathcal{A}_a, \mathcal{A}_b\) as above. Let \(\varepsilon > 0\). Then there exists \(C \in \mathcal{C}_k, C \subset S\) such that
\[
\sup_{(u,v) \in C} \mathcal{A}_a(u,v) \leq \lambda_k(a) + \frac{\varepsilon}{2}.
\]  
(2.22)

Then for any \((u,v) \in C\), if \(\|a - b\|_{L^\infty} \leq \delta = \varepsilon/2K\) we get
\[
\mathcal{A}_b(u,v) \leq \mathcal{A}_a(u,b) + \frac{\varepsilon}{2} \leq \lambda_k(a) + \varepsilon.
\]  
(2.23)

It follows that
\[
\sup_{(u,v) \in C} \mathcal{A}_b(u,v) \leq \lambda_2(a) + \varepsilon
\]  
(2.24)

and we obtain
\[
\lambda_k(b) \leq \lambda_k(a) + \varepsilon.
\]  
(2.25)

By reversing the roles of \(a\) and \(b\), we get \(|\lambda_k(a) - \lambda_k(b)| \leq \varepsilon\). \(\square\)
3. Proof of the existence theorem

3.1. A minimax principle. Our main tool for proving Theorem 1.6 will be an abstract minimax principle due to El Amrouss and Moussaoui [9].

Theorem 3.1. Let $\Phi$ be a $C^1$ functional on $X$ satisfying condition (C), let $Q$ be a closed connected subset of $X$ such that $\partial Q \cap \partial (-Q) \neq \emptyset$, and let $\beta \in \mathbb{R}$. Assume that

1. for every $K \in \mathfrak{K}_2$ there exists $v_K \in K$ such that $\Phi(v_K) \geq \beta$ and $\Phi(-v_K) \geq \beta$,
2. $a = \sup_{\partial Q} \Phi < \beta$,
3. $\sup_Q \Phi < \infty$.

Then $\Phi$ has a critical value $c \geq \beta$ given by

$$c = \inf_{h \in \Gamma} \sup_{x \in Q} \Phi(h(x)), \quad (3.1)$$

where $\Gamma = \{ h \in C(X,X) : h(x) = x \text{ for every } x \in \partial Q \}$.

3.2. Compactness conditions

Lemma 3.2. Suppose that $F$ satisfies (1.6), (1.8), and (1.22). Then the functional $\Phi$, given by (1.2), satisfies the Cerami condition.

Proof. In a similar way to [9, Lemma 3.1], we see that the first condition in Definition 1.1 holds.

We will prove that the second condition in Definition 1.1 holds, in the case $L(x,s,t) \to -\infty$ as $\|(s,t)\| \to \infty$ (the case $L(x,s,t) \to +\infty$ is similar). To do that, assume by contradiction that there exists a sequence $(u_n,v_n)_{n \in \mathbb{N}} \subset W$ such that

$$\Phi(u_n,v_n) \to c, \quad \varepsilon_n = \|\Phi'(u_n,v_n)\| \| (u_n,v_n) \| \to 0, \quad \| (u_n,v_n) \| \to \infty. \quad (3.2)$$

Therefore,

$$\left| \frac{1}{p} \langle \Phi_u(u_n,v_n), u_n \rangle + \frac{1}{q} \langle \Phi_v(u_n,v_n), v_n \rangle - \Phi(u_n,v_n) \right| \to c \quad (3.3)$$

or equivalently

$$\lim_{n \to \infty} \left| \int_{\Omega} \frac{1}{p} F_u(x,u_n,v_n) u_n + \frac{1}{q} F_v(x,u_n,v_n) v_n - F(x,u_n,v_n) \right| = c. \quad (3.4)$$

We define

$$z_n = \alpha_n^{1/p} u_n, \quad w_n = \alpha_n^{1/q} v_n, \quad (3.5)$$

where

$$\alpha_n = \frac{1}{\mathcal{A}_0(u_n,v_n)} \to 0 \quad (3.6)$$
with \( S_{\varepsilon} \) given by definition (2.8). We have that \( S_{\varepsilon}(z_n, w_n) = 1 \) so \( (z_n, w_n) \) is bounded in \( W \). After passing to a subsequence, we may assume that

\[
\begin{align*}
z_n &\rightarrow z \quad \text{in } W^{1,p}(\Omega), \\
w_n &\rightarrow w \quad \text{in } W^{1,q}(\Omega), \\
z_n &\rightarrow z \quad \text{in } L^p(\Omega), \text{ a.e. in } \Omega, \\
w_n &\rightarrow w \quad \text{in } L^q(\Omega), \text{ a.e. in } \Omega.
\end{align*}
\]

(3.7)

Now we show that \((z, w) \neq (0, 0)\)

\[
\frac{\Phi(u_n, v_n)}{S_{\varepsilon}(u_n, v_n)} = 1 - \int_\Omega F(x, u_n, v_n) \frac{1}{S_{\varepsilon}(u_n, v_n)}.
\]

(3.8)

From (1.22), we get that for any \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that

\[
F(x, s, t) \leq (a_2(x) + \varepsilon) G_2(s, t) + C_\varepsilon.
\]

(3.9)

As a consequence

\[
\int_\Omega F(x, u_n, v_n) \leq \int_\Omega (a_2(x) + \varepsilon) G_2(u_n, v_n) + C_\varepsilon |\Omega|,
\]

(3.10)

then

\[
\int_\Omega F(x, u_n, v_n) \frac{1}{S_{\varepsilon}(u_n, v_n)} \leq \alpha_n \int_\Omega (a_2(x) + \varepsilon) G_2(u_n, v_n) + C_\varepsilon |\Omega| \alpha_n.
\]

(3.11)

Since

\[
\alpha_n \int_\Omega (a_2(x) + \varepsilon) G_2(u_n, v_n) = \int_\Omega (a_2(x) + \varepsilon) G_2(z_n, w_n)
\]

(3.12)

in the limit we get

\[
0 \geq 1 - \int_\Omega (a_2(x) + \varepsilon) G_2(z, w)
\]

(3.13)

and we conclude that \( G_2(z, w) \neq 0 \).

Let

\[
L(x, s, t) = \frac{1}{p} F_s(x, s, t)s + \frac{1}{q} F_t(x, s, t)t - F(x, s, t).
\]

(3.14)

By (1.8) (and since \( L \) is continuous), \( L(x, s, t) \leq -M \). It follows that

\[
\int_\Omega L(x, u_n, v_n) \leq \int_{\{G_2(z, w) \neq 0\}} L(x, u_n, v_n) + M \left| \{x : G_2(z(x), w(x)) = 0\} \right|.
\]

(3.15)

Note that

\[
\alpha_n G_2(u_n, v_n) \rightarrow G_2(z, w).
\]

(3.16)
So in the set \( \{ x : G_2(z(x), w(x)) \neq 0 \} \), \( G_2(u_n, v_n) \to +\infty \), and then by (1.11), we have that \( u_n(x), v_n(x) \to \infty \). It follows that \( L(u_n, v_n) \to -\infty \) by condition (1.8). Hence the first integral tends to \(-\infty\) by Fatou lemma, and we get

\[
\lim_{n \to \infty} \int_{\Omega} L(x, u_n, v_n) = -\infty.
\]

(3.17)

This contradicts (3.4), and the proof is completed.

\[\square\]

3.3. Geometric conditions. In this section we show that the functional \( \Phi \) satisfies the geometric conditions of Theorem 3.1.

Lemma 3.3. Let \( F \) satisfy the assumptions of Theorem 1.6. Then the functional \( \Phi \), given by (1.2), satisfies

1. \( \Phi(c_1/p \phi, c_1/q \psi) \to -\infty \) as \( c \to +\infty \);
2. for every \( K \in \mathcal{C}_2 \) there exists \( (u_K, v_K) \in K \) and \( \beta \in \mathbb{R} \) such that \( \Phi(u_K, v_K) \geq \beta \) and \( \Phi(-u_K, -v_K) \geq \beta \).

Proof. (1) As \( \lambda_1(a, G_1) < 0 \), we may choose \( \varepsilon > 0 \) such that \( \lambda_1(a_1 - \varepsilon, G_1) < 0 \). Let \( (\phi, \psi) \) be the first eigenfunction for the problem

\[
\begin{align*}
-\Delta_p u - (a_1(x) - \varepsilon) G_1(u, v) &= \lambda |u|^{p-2} u \quad \text{in } \Omega, \\
-\Delta_q v - (a_1(x) - \varepsilon) G_1(v, u) &= \lambda |v|^{q-2} v \quad \text{in } \Omega, \\
u = v = 0 &\quad \text{in } \partial \Omega,
\end{align*}
\]

normalized by

\[
\frac{1}{p} \int_{\Omega} |\nabla \phi|^p + \frac{1}{q} \int_{\Omega} |\nabla \psi|^q = 1.
\]

(3.19)

Then, using (1.13), we get

\[
\frac{1}{p} \int_{\Omega} |\nabla \phi|^p + \frac{1}{q} \int_{\Omega} |\nabla \psi|^q - \int_{\Omega} (a_1(x) - \varepsilon) G_1(u, v) = \lambda_1(a_1 - \varepsilon, G_1).
\]

(3.20)

By (1.22), we have

\[
F(x, s, t) \geq (a_1(x) - \varepsilon) G_1(s, t) - C_\varepsilon.
\]

(3.21)

It follows that

\[
\Phi(c_1/p \phi, c_1/q \psi) \leq c \left( \frac{1}{p} \int_{\Omega} |\nabla \phi|^p + \frac{1}{q} \int_{\Omega} |\nabla \psi|^q \\
- \int_{\Omega} (a_1(x) - \varepsilon) G_1(\phi, \psi) \right) + C_\varepsilon |\Omega|
\]

\[
\leq c \lambda_1(a_1 - \varepsilon, G_1) + C_\varepsilon |\Omega|,
\]

(3.22)

and so \( \Phi(c_1/p \phi, c_1/q \psi) \to -\infty \) as \( c \to +\infty \).
Since \( \lambda_2(a_2, G_2) > 0 \), we may choose \( \epsilon > 0 \) such that \( \lambda_2(a_2 + \epsilon, G_2) > 0 \). Given \( K \in \mathcal{C}_2 \) and this \( \epsilon > 0 \), we claim that there exists \((u_K, v_K) \in K \) verifying
\[
\lambda_2(a_2 + \epsilon, G_2) \left( \frac{1}{p} \int_{\Omega} |u_K|^p + \frac{1}{q} \int_{\Omega} |v_K|^q \right) \\
\leq \frac{1}{p} \int_{\Omega} |\nabla u_K|^p + \frac{1}{q} \int_{\Omega} |\nabla v_K|^q - \int_{\Omega} (a_2(x) + \epsilon) G_2(u_K, v_K).
\]
By (1.22), we have
\[
F(x, s, t) \leq (a_2(x) + \epsilon) G_2(s, t) + C_\epsilon.
\]
It follows that
\[
\Phi(u_K, v_K) \geq \frac{1}{p} \int_{\Omega} |\nabla u_K|^p + \frac{1}{q} \int_{\Omega} |\nabla v_K|^q \\
- \int_{\Omega} (a_2(x) + \epsilon) G_2(u_K, v_K) - C_\epsilon |\Omega| \\
\geq \lambda_2(a_2 + \epsilon, G_2) \left( \frac{1}{p} \int_{\Omega} |u_K|^p + \frac{1}{q} \int_{\Omega} |v_K|^q \right) - C_\epsilon |\Omega| \\
\geq -C_\epsilon |\Omega| = \beta.
\]
Similarly,
\[
\Phi(-u_K, -v_K) \geq -C_\epsilon |\Omega| = \beta.
\]

3.4. Proof of Theorem 1.6. We apply Theorem 3.1. We take
\[
Q = \{(|c|^{1/p-1} c \varphi, |c|^{1/q-1} c \psi), \ -R \leq c \leq R\},
\]
where \((\varphi, \psi)\) is given by Lemma 3.3. \( Q \) is closed and compact (it is the image of \([-R, R]\) under a continuous mapping). Also \( \partial Q = \partial(-Q) = \{ (\pm R^{1/p} \varphi, \pm R^{1/q} \psi) \} \neq \emptyset \). By Lemma 3.3 if we choose \( R \) big enough, we have
\[
\sup_{\partial Q} \Phi < \beta.
\]
Also \( \sup_Q \Phi < +\infty \) since \( Q \) is compact and \( \Phi \) is continuous. The functional \( \Phi \) verifies condition \((C)\) by Lemma 3.2. Then all the conditions of Theorem 3.1 are fulfilled and the proof is completed.

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References


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