GENERIC UNIQUENESS OF A MINIMAL SOLUTION FOR VARIATIONAL PROBLEMS ON A TORUS

ALEXANDER J. ZASLAVSKI

Received 17 July 2001

We study minimal solutions for one-dimensional variational problems on a torus. We show that, for a generic integrand and any rational number \( \alpha \), there exists a unique (up to translations) periodic minimal solution with rotation number \( \alpha \).

1. Introduction

In this paper, we consider functionals of the form

\[
I^f(a,b,x) = \int_a^b f(t,x(t),x'(t)) \, dt,
\]

where \( a \) and \( b \) are arbitrary real numbers satisfying \( a < b \), \( x \in W^{1,1}(a,b) \) and \( f \) belongs to a space of functions described below. By an appropriate choice of representatives, \( W^{1,1}(a,b) \) can be identified with the set of absolutely continuous functions \( x : [a,b] \to \mathbb{R} \), and henceforth we will assume that this has been done.

Denote by \( \mathcal{M} \) the set of integrands \( f = f(t,x,p) : \mathbb{R}^3 \to \mathbb{R} \) which satisfy the following assumptions:

(A1) \( f \in C^3 \) and \( f(t,x,p) \) has period 1 in \( t,x \);
(A2) \( \delta_f \leq f_{pp}(t,x,p) \leq \delta_f^{-1} \) for every \( (t,x,p) \in \mathbb{R}^3 \);
(A3) \( |f_{xp}| + |f_{tp}| \leq c_f (1 + |p|), \quad |f_{xx}| + |f_{xt}| \leq c_f (1 + p^2) \),

with some constants \( \delta_f \in (0,1), \ c_f > 0 \).

Clearly, these assumptions imply that

\[
\delta_f p^2 - \tilde{c}_f \leq f(t,x,p) \leq \delta_f^{-1} p^2 + \tilde{c}_f
\]

for every \( (t,x,p) \in \mathbb{R}^3 \) for some constants \( \tilde{c}_f > 0 \) and \( 0 < \delta_f < \delta_f \).

In this paper, we analyse extremals of variational problems with integrands \( f \in \mathcal{M} \). The following optimality criterion was introduced by Aubry and Le

Copyright © 2002 Hindawi Publishing Corporation
2000 Mathematics Subject Classification: 49J99, 58F99
URL: http://dx.doi.org/10.1155/S1085337502000842
144 Uniqueness of a minimal solution


Let \( f \in \mathcal{M} \). A function \( x(\cdot) \in W_{\text{loc}}^{1,1}(\mathbb{R}) \) is called an \((f)\)-minimal solution if

\[
I^f(a, b, y) \geq I^f(a, b, x)
\]

for each pair of numbers \( a < b \) and each \( y \in W^{1,1}(a, b) \) which satisfies \( y(a) = x(a) \) and \( y(b) = x(b) \) (see \([2, 9, 10, 12]\)).

Our work follows Moser \([9, 10]\), who studied the existence and structure of minimal solutions in the spirit of Aubry-Mather theory \([2, 7]\).

Consider any \( f \in \mathcal{M} \). It was shown in \([9, 10]\) that \((f)\)-minimal solutions possess numerous remarkable properties. Thus, for every \((f)\)-minimal solution \( x(\cdot) \), there is a real number \( \alpha \) satisfying

\[
\sup \{ ||x(t) - \alpha t|| : t \in \mathbb{R} \} < \infty
\]

which is called the rotation number of \( x(\cdot) \), and given any real \( \alpha \) there exists an \((f)\)-minimal solution with rotation number \( \alpha \). Senn \([11]\) established the existence of a strictly convex function \( E_f : \mathbb{R} \rightarrow \mathbb{R} \), which is called the minimal average action of \( f \) such that, for each real \( \alpha \) and each \((f)\)-minimal solution \( x \) with rotation number \( \alpha \),

\[
(T_2 - T_1)^{-1} I^f(T_1, T_2, x) \rightarrow E_f(\alpha) \quad \text{as} \quad T_2 - T_1 \rightarrow \infty.
\]

This result is an analogue of Mather’s theorem about the average energy function for Aubry-Mather sets generated by a diffeomorphism of the infinite cylinder \([8]\).

In this paper, we show that for a generic integrand \( f \) and any rational \( \alpha \), there exists a unique (up to translations) \((f)\)-minimal periodic solution with rotation number \( \alpha \).

Let \( k \geq 3 \) be an integer. Set \( \mathcal{M}_k = \mathcal{M} \cap C_k(\mathbb{R}^3) \). For \( f \in \mathcal{M}_k \) and \( q = (q_1, q_2, q_3) \in \{0, \ldots, k\}^3 \) satisfying \( q_1 + q_2 + q_3 \leq k \), we set

\[
|q| = q_1 + q_2 + q_3,
\]

\[
D^q f = \frac{\partial^{|q|} f}{\partial t^{q_1} \partial x^{q_2} \partial p^{q_3}}.
\]

For \( N, \epsilon > 0 \) we set

\[
E_k(N, \epsilon) = \{(f, g) \in \mathcal{M}_k \times \mathcal{M}_k : |D^q f(t, x, p) - D^q g(t, x, p)| \leq \epsilon + \epsilon \max \{|D^q f(t, x, p)|, |D^q g(t, x, p)|\} \}
\]

\[
\forall q \in \{0, 1, 2\}^3 \text{ satisfying } |q| \in \{0, 2\}, \forall (t, x, p) \in \mathbb{R}^3 \}
\]

\[
\cap \{(f, g) \in \mathcal{M}_k \times \mathcal{M}_k : |D^q f(t, x, p) - D^q g(t, x, p)| \leq \epsilon \}
\]

\[
\forall q \in \{0, \ldots, k\}^3 \text{ satisfying } |q| \leq k, \forall (t, x, p) \in \mathbb{R}^3
\]

such that \(|p| \leq N\).
It is easy to verify that, for the set $M_k$ there exists a uniformity which is determined by the base $E_k(N, e)$, $N, e > 0$, and that the uniform space $M_k$ is metrizable and complete [3]. We establish the existence of a set $F_k \subset M_k$ which is a countable intersection of open everywhere dense subsets of $M_k$ such that, for each $f \in F_k$ and each rational $\alpha \in \mathbb{R}^1$, there exists a unique (up to translations) $(f)$-minimal periodic solution with rotation number $\alpha$.

2. Properties of minimal solutions

Consider any $f \in M$. We note that, for each pair of integers $j$ and $k$ the translations $(t, x) \rightarrow (t + j, x + k)$ leave the variational problem invariant. Therefore, if $x(\cdot)$ is an $(f)$-minimal solution, so is $x(\cdot) + j + k$. Of course, on the torus, this represents the same curve as does $x(\cdot)$. This motivates the following terminology [9, 10].

We say that a function $x(\cdot) \in W^{1,1}_{\text{loc}}(\mathbb{R}^1)$ has no self-intersections if for all pairs of integers $j, k$ the function $t \rightarrow x(t + j) + k - x(t)$ is either always positive, or always negative, or identically zero.

Denote by $Z$ the set of all integers. We have the following result (see [6, Proposition 3.2] and [9, 10]).

**Proposition 2.1.** (i) Let $f \in M$. Given any real $\alpha$ there exists a nonself-intersecting $(f)$-minimal solution with rotation number $\alpha$.

(ii) For any $f \in M$ and any $(f)$-minimal solution $x$, there is the rotation number of $x$.

For each $f \in M$, each rational number $\alpha$, and each natural number $q$ satisfying $qa \in Z$, we define

$$\mathcal{N}(\alpha, q) = \{ x(\cdot) \in W^{1,1}_{\text{loc}}(\mathbb{R}^1) : x(t + q) = x(t) + qa, \ t \in \mathbb{R}^1 \},$$

$$\mathcal{M}_f(\alpha, q) = \{ x(\cdot) \in \mathcal{N}(\alpha, q) : I^f(0, q, x) \leq I^f(0, q, y) \ \forall y \in \mathcal{N}(\alpha, q) \}. \quad (2.1)$$

We have the following result [9, Theorems 5.1, 5.2, 5.4, and Corollaries 5.3 and 5.5].

**Proposition 2.2.** Let $f \in M$, let $\alpha$ be a rational number, and let $p, q \geq 1$ be integers satisfying $pa, qa \in Z$. Then $\mathcal{M}_f(\alpha, q) = \mathcal{M}_f(\alpha, p) \neq \emptyset$, each $x \in \mathcal{M}_f(\alpha, q)$ is a nonself-intersecting $(f)$-minimal solution with rotation number $\alpha$ and the set $\mathcal{M}_f(\alpha, q)$ is totally ordered, that is, if $x, y \in \mathcal{M}_f(\alpha, q)$, then either $x(t) < y(t)$ for all $t$, or $x(t) > y(t)$ for all $t$, or $x(t) = y(t)$ identically.

For any $f \in M$ and any rational number $\alpha$ we set $\mathcal{M}^{\text{per}}_f(\alpha) = \mathcal{M}_f(\alpha, q)$, where $q$ is a natural number satisfying $qa \in Z$.

We have the following result (see [6, Theorem 1.1]).

**Proposition 2.3.** Let $f \in M$. Then there exist a strictly convex function $E_f : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ satisfying $E_f(\alpha) \rightarrow \infty$ as $|\alpha| \rightarrow \infty$ and a monotonically increasing function $\Gamma_f : (0, \infty) \rightarrow [0, \infty)$ such that for each real $\alpha$, each $(f)$-minimal solution $x$ with
Uniqueness of a minimal solution

rotation number $\alpha$ and each pair of real numbers $S$ and $T$,

$$|I^f(S, S + T, x) - E_f(\alpha)T| \leq \Gamma_f(|\alpha|). \quad (2.2)$$

By Proposition 2.3 for each $f \in M$ there exists a unique number $\alpha(f)$ such that

$$E_f(\alpha(f)) = \min \{ E_f(\beta) : \beta \in \mathbb{R} \}. \quad (2.3)$$

Note that assumptions (A1), (A2), and (A3) play an important role in the proofs of Propositions 2.1, 2.2, and 2.3 (see [9, 10]).

3. The main results

Theorem 3.1. Let $k \geq 3$ be an integer and $\alpha$ be a rational number. Then there exists a set $\mathcal{H}_{5106} k \alpha \subset M_k$ which is a countable intersection of open everywhere dense subsets of $M_k$ such that for each $f \in M_k$ the following assertions hold:

1. If $x, y \in M_f^{(\per)}(\alpha)$, then there are integers $p, q$ such that $y(t) = x(t + p) - q$ for all $t \in \mathbb{R}^1$.

2. Let $x \in M_f^{(\per)}(\alpha)$ and $\epsilon > 0$. Then there exists a neighborhood $\mathcal{U}$ of $f$ in $M_k$ such that for each $g \in \mathcal{U}$ and each $y \in M_g^{(\per)}(\alpha)$ there are integers $p, q$ such that $|y(t) - x(t + p) + q| \leq \epsilon$ for all $t \in \mathbb{R}^1$.

It is not difficult to see that Theorem 3.1 implies the following result.

Theorem 3.2. Let $k \geq 3$ be an integer. Then there exists a set $\mathcal{H}_k \subset M_k$ which is a countable intersection of open everywhere dense subsets of $M_k$ such that, for each $f \in M_k$ and each rational number $\alpha$ the assertions (1) and (2) of Theorem 3.1 hold.

Note that minimal solutions with irrational rotation numbers were studied in [2, 7, 9, 10, 12].

4. An auxiliary result

Let $k \geq 3$ be an integer and $\beta \in \mathbb{R}^1$. For each $f \in M_k$, define $\mathcal{A} f \in C^3(\mathbb{R}^3)$ by

$$(\mathcal{A} f)(t, x, u) = f(t, x, u) - \beta u, \quad (t, x, u) \in \mathbb{R}^3. \quad (4.1)$$

Clearly $\mathcal{A} f \in M_k$ for each $f \in M_k$.

Proposition 4.1. The mapping $\mathcal{A} : M_k \to M_k$ is continuous.

Proof. Let $f \in M_k$ and let $N, \epsilon > 0$. In order to prove the proposition, it is sufficient to show that there exists $\epsilon_0 \in (0, \epsilon)$ such that

$$\mathcal{A}(\{ g \in M_k : (f, g) \in E_k(N, \epsilon_0) \}) \subset \{ h \in M_k : (h, \mathcal{A} f) \in E_k(N, \epsilon) \}. \quad (4.2)$$

Set

$$\Delta_0 = 2(|\beta| + 1). \quad (4.3)$$
Equation (1.2) implies that there exists \( c_0 > 0 \) such that

\[
\Delta_0 |u| - c_0 \leq f(t, x, u) \quad \forall (t, x, u) \in \mathbb{R}^3. \tag{4.4}
\]

Choose a number \( \epsilon_0 \) such that

\[
0 < \epsilon_0 < \min \{1, \epsilon\}, \quad 4\epsilon_0 + 4\epsilon_0(1 - \epsilon_0)^{-1}(4 + c_0) < \epsilon. \tag{4.5}
\]

It follows from (4.3) and (4.4) that for each \((t, x, u) \in \mathbb{R}^3\),

\[
|f(t, x, u) - \beta u| \geq |f(t, x, u)| - |\beta u| \geq |f(t, x, u)| - |\beta|\Delta_0^{-1}(f(t, x, u) + c_0)
\geq |f(t, x, u)|(1 - |\beta|\Delta_0^{-1}) - |\beta|\Delta_0^{-1}c_0 
\geq 2^{-1}|f(t, x, u)| - 2^{-1}c_0. \tag{4.6}
\]

Assume that

\[
g \in \mathcal{M}_k, \quad (f, g) \in E_k(N, \epsilon_0). \tag{4.7}
\]

By (1.7) and (4.7) for each \((t, x, u) \in \mathbb{R}^3\),

\[
|f(t, x, u) - g(t, x, u)| \leq \epsilon_0 + \epsilon_0 \max \{|f(t, x, u)|, |g(t, x, u)|\},
\max \{|f(t, x, u)|, |g(t, x, u)|\} - \min \{|f(t, x, u)|, |g(t, x, u)|\}
\leq \epsilon_0 + \epsilon_0 \max \{|f(t, x, u)|, |g(t, x, u)|\},
\]

\[
(1 - \epsilon_0) \max \{|f(t, x, u)|, |g(t, x, u)|\} \leq \min \{|f(t, x, u)|, |g(t, x, u)|\} + \epsilon_0,
\]

\[
|g(t, x, u)| \leq (1 - \epsilon_0)^{-1}|f(t, x, u)| + (1 - \epsilon_0)^{-1}\epsilon_0.
\]

We show that \((\mathcal{A}f, \mathcal{A}g) \in E_k(N, \epsilon)\). It follows from (1.7), (4.1), (4.5), and (4.7) that, for each \( q = (q_1, q_2, q_3) \in \{0, \ldots, k\}^3 \) satisfying \(|q| \leq k\) and each \((t, x, p) \in \mathbb{R}^3\) satisfying \(|p| \leq N\),

\[
|D^q(\mathcal{A}f)(t, x, p) - D^q(\mathcal{A}g)(t, x, p)| = |D^q f(t, x, p) - D^q g(t, x, p)| \leq \epsilon_0 < \epsilon. \tag{4.9}
\]

Let \( q \in \{0, 1, 2\}^3, |q| \in \{0, 2\}, \) and \((t, x, p) \in \mathbb{R}^3\). Equation (4.1) implies that

\[
|D^q(\mathcal{A}f)(t, x, p) - D^q(\mathcal{A}g)(t, x, p)| = |D^q f(t, x, p) - D^q g(t, x, p)|. \tag{4.10}
\]

If \(|q| = 2\), then by (1.7), (4.1), (4.5), (4.7), and (4.10),

\[
|D^q(\mathcal{A}f)(t, x, p) - D^q(\mathcal{A}g)(t, x, p)|
\leq \epsilon_0 + \epsilon_0 \max \{|D^q f(t, x, p)|, |D^q g(t, x, p)|\}
\leq \epsilon + \epsilon \max \{|D^q(\mathcal{A}f)(t, x, p)|, |D^q(\mathcal{A}g)(t, x, p)|\}. \tag{4.11}
\]
Assume that \( q = 0 \). By (1.7), (4.1), (4.5), (4.6), (4.7), and (4.8),

\[
\left| D^q(\mathcal{A}f)(t,x,p) - D^q(\mathcal{A}g)(t,x,p) \right| \\
\leq |f(t,x,p) - g(t,x,p)| \leq \varepsilon_0 + \varepsilon_0 \max \left\{ |f(t,x,p)|, |g(t,x,p)| \right\} \\
\leq \varepsilon_0 + \varepsilon_0 \max \left\{ |f(t,x,p)|, (1-\varepsilon_0)^{-1} f(t,x,p) + (1-\varepsilon_0)^{-1} \varepsilon_0 \right\} \\
= \varepsilon_0 + \varepsilon_0 (1-\varepsilon_0)^{-1} |f(t,x,p)| + \varepsilon_0^2 (1-\varepsilon_0)^{-1} (4.12) \\
\leq \varepsilon_0 + \varepsilon_0^2 (1-\varepsilon_0)^{-1} + \varepsilon_0 (1-\varepsilon_0)^{-1} \left[ 2|f(t,x,p) - \beta p| + 2c_0 \right] \\
\leq \varepsilon_0 + \varepsilon_0^2 (1-\varepsilon_0)^{-1} + 2\varepsilon_0 (1-\varepsilon_0)^{-1} c_0 + 2\varepsilon_0 (1-\varepsilon_0)^{-1} |f(t,x,p) - \beta p| \\
\leq 2\varepsilon_0 (1-\varepsilon_0)^{-1} (f(t,x,p) - \beta p) + \varepsilon \leq \varepsilon + \varepsilon |(\mathcal{A}f)(t,x,p)|.
\]

Equations (4.9), (4.11), and (4.12) imply that \((\mathcal{A}f, \mathcal{A}g) \in E_k(N, \varepsilon)\). Proposition 4.1 is proved. \(\square\)

Let \(-\infty < T_1 < T_2 < \infty \) and \( x \in W^{1,1}(T_1, T_2) \). By (4.1) we have

\[
I^{\mathcal{A}f}(T_1, T_2, x) = \int_{T_1}^{T_2} \left( f(t, x(t), x'(t)) - \beta x'(t) \right) dt \\
= I^{f}(T_1, T_2, x) - \beta x(T_2) + \beta x(T_1).
\]

Therefore, each \( x \in W^{1,1}_{\text{loc}}(\mathbb{R}^1) \) is an \((\mathcal{A}f)\)-minimal solution if and only if \( x(\cdot) \) is an \((f)\)-minimal solution.

Let \( x \in W^{1,1}_{\text{loc}}(\mathbb{R}^1) \) be an \((f)\)-minimal solution with rotation number \(\beta\). By Proposition 2.1 there exists \( c_1 > 0 \) such that for all \( s, t \in \mathbb{R}^1 \),

\[
| x(t+s) - x(t) - rs | \leq c_1. \tag{4.14}
\]

Proposition 2.3 implies that there exists a constant \( c_2 > 0 \) such that for each \( s, t \in \mathbb{R}^1 \) and each \( t > 0 \),

\[
\left| I^{f}(s, s+t, x) - E_{\beta r}(r) t \right| \leq c_2, \tag{4.15}
\]

\[
\left| I^{\mathcal{A}f}(s, s+t, x) - E_{\mathcal{A}f}(r) t \right| \leq c_2. \tag{4.16}
\]

It follows from (4.13), (4.14), (4.15), and (4.16) that, for each \( s, t \in \mathbb{R}^1 \) and each \( t > 0 \),

\[
\left| E_{\mathcal{A}f}(r) t + \beta t r - E_{\beta r}(r) t \right| \\
\leq \left| E_{\mathcal{A}f}(r) t - I^{\mathcal{A}f}(s, s+t, x) \right| + \left| I^{\mathcal{A}f}(s, s+t, x) + \beta t r - I^{f}(s, s+t, x) \right| \\
+ \left| I^{f}(s, s+t, x) - E_{f}(r) t \right| \\
\leq c_2 + |\beta t r - \beta [x(t+s) - x(s)]| + c_2 \leq 2c_2 + |\beta| c_1.
\]

These inequalities imply that

\[
E_{\mathcal{A}f}(r) = E_{f}(r) - \beta r \quad \forall r \in \mathbb{R}^1. \tag{4.18}
\]
5. Proof of Theorem 3.1

Let \( g \in \mathcal{M} \). We define

\[ \mu(g) = \inf \left\{ \liminf_{T \to \infty} T^{-1} I^g(0, T, x) : x(\cdot) \in \mathcal{W}^{1,1}_{\text{loc}}([0, \infty)) \right\}. \tag{5.1} \]

In [13, Section 5] we showed that the number \( \mu(g) \) is well defined and proved the following result [13, Theorem 5.1].

**Proposition 5.1.** Let \( f \in \mathcal{M} \). Then there exists a constant \( M_0 > 0 \) such that:

1. \( I^f(0, T, x) - \mu(f) T \geq -M_0 \) for each \( x \in \mathcal{W}^{1,1}_{\text{loc}}([0, \infty)) \) and each \( T > 0 \).
2. For each \( a \in \mathbb{R}^1 \) there exists \( x \in \mathcal{W}^{1,1}_{\text{loc}}([0, \infty)) \) such that \( x(0) = a \) and

\[ |I^f(0, T, x) - \mu(f) T| \leq 4M_0 \quad \forall T > 0. \tag{5.2} \]

Note that assertion (ii) of Proposition 5.1 holds by the periodicity of \( f \) in \( x \).

Let \( f \in \mathcal{M} \). A function \( x \in \mathcal{W}^{1,1}_{\text{loc}}([0, \infty)) \) is called \((f)\)-good (see [5]) if

\[ \sup \{|I^f(0, T, x) - \mu(f) T| : T \in (0, \infty)\} < \infty. \tag{5.3} \]

By [6, Theorem 4.1],

\[ E_f(\alpha(f)) = \mu(f) \quad \forall f \in \mathcal{M}. \tag{5.4} \]

For \( f \in \mathcal{M} \), \( x, y, T_1 \in \mathbb{R}^1 \), and \( T_2 > T_1 \) we set

\[ U^f(T_1, T_2, x, y) = \inf \{ I^f(T_1, T_2, \nu) : \nu \in \mathcal{W}^{1,1}_{\text{loc}}(T_1, T_2), \nu(T_1) = x, \nu(T_2) = y \}. \tag{5.5} \]

It is not difficult to see that for each \( x, y, T_1 \in \mathbb{R}^1 \), \( T_2 > T_1 \),

\[ U^f(T_1, T_2, x+1, y+1) = U^f(T_1, T_2, x, y), \quad U^f(T_1+1, T_2+1, x, y) = U^f(T_1, T_2, x, y), \quad -\infty < U^f(T_1, T_2, x, y) < \infty, \]

\[ \inf \{ U^f(T_1, T_2, a, b) : a, b \in \mathbb{R}^1 \} > -\infty. \tag{5.6} \]

Denote by \( \mathcal{M}_{\text{per}} \) the set of all \( f \in \mathcal{M} \) such that \( \alpha(f) \) is rational and denote by \( \mathcal{M}_{\text{per}}^0 \) the set of all \( g \in \mathcal{M}_{\text{per}} \) for which there exist an \((g)\)-minimal solution \( w \in C^2(\mathbb{R}^1) \), a continuous function \( \pi : \mathbb{R}^1 \to \mathbb{R}^1 \), and integers \( m, n \) such that the following properties hold:

1. \( \pi(x+1) = \pi(x), \ x \in \mathbb{R}^1; \)
2. \( n \geq 1 \) and \( \alpha(g) = mn^{-1} \) is an irreducible fraction;
3. \( w(t+n) = w(t) + m \) for all \( t \in \mathbb{R}^1; \)
4. \( U^g(0, 1, x, y) - \mu(g) - \pi(x) + \pi(y) \geq 0 \) for each \( x, y \in \mathbb{R}^1; \)
5. for any \( u \in \mathcal{W}^{1,1}_{\text{loc}}(0, n) \), the equality

\[ I^g(0, n, u) = n\mu(g) + \pi(u(0)) - \pi(u(n)) \tag{5.7} \]

holds if and only if there are integers \( i, j \) such that \( u(t) = w(t+i) - j \) for all \( t \in [0, n] \).
150  Uniqueness of a minimal solution

Consider the manifold \((\mathbb{R}^1 / \mathbb{Z})^2\) and the canonical mapping \(P : \mathbb{R}^2 \to (\mathbb{R}^1 / \mathbb{Z})^2\). We have the following result [13, Proposition 6.2].

**Proposition 5.2.** Let \(\Omega\) be a closed subset of \((\mathbb{R}^1 / \mathbb{Z})^2\). Then there exists a bounded nonnegative function \(\phi \in C^\infty((\mathbb{R}^1 / \mathbb{Z})^2)\) such that

\[
\Omega = \{ x \in (\mathbb{R}^1 / \mathbb{Z})^2 : \phi(x) = 0 \}. \tag{5.8}
\]

Proposition 5.2 is proved by using [1, Chapter 2, Section 3, Theorem 1] and the partition of unity (see [4, Appendix 1]).

We also have the following result (see [13, Proposition 6.3]).

**Proposition 5.3.** Suppose that \(f \in M_{\text{per}}, \alpha(f) = mn^{-1}\) is an irreducible fraction \((m, n\) are integers, \(n \geq 1\) and \(w \in W^{1,1}_\text{loc}(\mathbb{R}^1)\) is an \((f)\)-minimal solution satisfying \(w(t + n) = w(t) + m\) for all \(t \in \mathbb{R}^1\). Let \(\phi \in C^\infty((\mathbb{R}^1 / \mathbb{Z})^2)\) be as guaranteed in Proposition 5.2 with

\[
\Omega = \{ P(t, w(t)) : t \in [0, n] \}, \tag{5.9}
\]

and let

\[
g(t, x, p) = f(t, x, p) + \phi(P(t, x)), \quad (t, x, p) \in \mathbb{R}^3. \tag{5.10}
\]

Then \(g \in M_{\text{per}}^0\) and there is a continuous function \(\pi : \mathbb{R}^1 \to \mathbb{R}^1\) such that the properties (P1), (P2), (P3), (P4), and (P5) hold with \(g, w, \pi, m, n\) and \(\alpha(g) = \alpha(f)\).

In the sequel we need the following two lemmas proved in [13].

**Lemma 5.4** [13, Lemma 6.6]. Assume that \(k \geq 3\) is an integer, \(g \in M_{\text{per}}^0 \cap M_k\), and properties (P1), (P2), (P3), (P4), and (P5) hold with a g-minimal solution \(w(\cdot) \in C^2(\mathbb{R}^1)\), a continuous function \(\pi : \mathbb{R}^1 \to \mathbb{R}^1\) and integers \(m, n\). Then for each \(e \in (0, 1)\), there exists a neighborhood \(U\) of \(g\) in \(M_k\) such that for each \(h \in U\) and each \((h)\)-good function \(v \in W^{1,1}_\text{loc}([0, \infty))\) there are integers \(p, q\) such that

\[
| v(t) - w(t + p) - q | \leq e \quad \text{for all large enough } t. \tag{5.11}
\]

**Lemma 5.5** [13, Corollary 6.1]. Assume that \(k \geq 3\) is an integer, \(g \in M_{\text{per}}^0 \cap M_k\), and properties (P1), (P2), (P3), (P4), and (P5) hold with a g-minimal solution \(w(\cdot) \in C^2(\mathbb{R}^1)\), a continuous function \(\pi : \mathbb{R}^1 \to \mathbb{R}^1\) and integers \(m, n\). Then there exist a neighborhood \(U\) of \(g\) in \(M_k\) and a number \(L > 0\) such that for each \(h \in U\) and each \((h)\)-good function \(v \in W^{1,1}_\text{loc}([0, \infty))\), the following property holds.

There is a number \(T_0 > 0\) such that

\[
| v(t_2) - v(t_1) - \alpha(g)(t_2 - t_1) | \leq L \tag{5.12}
\]

for each \(t_1 \geq T_0\) and each \(t_2 > t_1\).
Completion of the proof of Theorem 3.1. Let $k \geq 3$ be an integer and let $\alpha = mn^{-1}$ be an irreducible fraction $(n \geq 1$ and $m$ are integers). Let $f \in M_k$. By Proposition 2.2 there exists an $(f)$-minimal solution $w_f(\cdot) \in W^{1,1}_{\text{loc}}(\mathbb{R}^1)$ such that

$$w_f(t+n) = w_f(t) + m \quad \forall t \in \mathbb{R}^1. \quad (5.13)$$

Choose

$$\beta \in \partial E_f(\alpha). \quad (5.14)$$

Consider a mapping $\mathcal{A} : M_k \to M_k$ defined by (4.1). By Proposition 4.1 the mapping $\mathcal{A}$ is continuous. Clearly there exists a continuous $\mathcal{A}^{-1} : M_k \to M_k$. Equations (5.14) and (4.18) imply that

$$0 \in \partial E_{\mathcal{A}f}(\alpha), \quad E_{\mathcal{A}f}(\alpha) = \min \{E_{\mathcal{A}f}(r) : r \in \mathbb{R}^1\} = \mu(\mathcal{A}f) \quad (5.15)$$

and that $\mathcal{A}f \in M_{\text{per}}$. It follows from Proposition 5.2 that there exists a bounded nonnegative function $\phi \in C^\infty((\mathbb{R}^1/\mathbb{Z})^2)$ such that

$$\{x \in (\mathbb{R}^1/\mathbb{Z})^2 : \phi(x) = 0\} = \{P(t, w_f(t)) : t \in [0, n]\}. \quad (5.16)$$

Set $f(\beta) = \mathcal{A}f$ and for each $\gamma \in (0, 1)$ define

$$f_\gamma(t, x, u) = f(t, x, u) + \gamma \phi(P(t, x)), \quad (t, x, u) \in \mathbb{R}^3, \quad f_\gamma^{(\beta)} = \mathcal{A}(f_\gamma). \quad (5.17)$$

Proposition 5.3 implies that for each $\gamma \in (0, 1)$,

$$f_\gamma^{(\beta)} \in M_{\text{per}}^0 \cap M_k, \quad f_\gamma \to f \quad \text{as} \quad \gamma \to 0^+, \quad f_\gamma^{(\beta)} \to f^{(\beta)} \quad \text{as} \quad \gamma \to 0^+ \text{ in } M_k. \quad (5.18)$$

Fix $\gamma \in (0, 1)$ and an integer $n \geq 1$. By Proposition 5.3 the properties (P1), (P2), (P3), (P4), and (P5) hold with $g = f_\gamma^{(\beta)}$, $\alpha(g) = \alpha$ and $w(\cdot) = w_f$.

By Lemmas 5.4 and 5.5, there exists an open neighborhood $V(f, \gamma, n)$ of $f_\gamma^{(\beta)}$ in $M_\gamma$ and a number $L(f, \gamma, n) > 0$ such that the following properties hold:

(i) for each $h \in V(f, \gamma, n)$ and each $(h)$-good function $v \in W^{1,1}_{\text{loc}}([0, \infty))$, there are integers $p, q$ such that

$$|v(t) - w_f(t+p) - q| \leq \frac{1}{n} \quad (5.19)$$

for all large enough $t$;

(ii) for each $h \in V(f, \gamma, n)$ and each $(h)$-good function $v \in W^{1,1}_{\text{loc}}([0, \infty))$, there is a number $T_0$ such that

$$|v(t_2) - v(t_1) - \alpha(f_\gamma^{(\beta)})(t_2-t_1)| \leq L \quad (5.20)$$

for each $t_1 \geq T_0$ and each $t_2 > t_1$. 
Let $h \in V(f, \gamma, n)$ and let $v \in W^{1,1}_{\text{loc}}(\mathbb{R}^1)$ be an $(h)$-minimal solution with rotation number $\alpha(h)$. Then by Proposition 2.3, (2.3), (5.4), and property (ii), $v|_{[0,\infty)}$ is an $(h)$-good function and there is $T_0$ such that (5.20) holds for each $t_1 \geq T_0$ and each $t_2 > t_1$. Since $v \in W^{1,1}_{\text{loc}}(\mathbb{R}^1)$ has rotation number $\alpha(h)$ it follows from Proposition 2.1 that there exists $c_1 > 0$ such that
\[
|v(t+s) - v(t) - \alpha(h)s| \leq c_1 \quad \forall s, t \in \mathbb{R}. \tag{5.21}
\]
Equations (5.15), (5.17), (5.20), and (5.21) imply that
\[
\alpha(h) = \alpha(f_y^{(\beta)}) = \alpha(f_y^{(\beta)}) = \alpha. \tag{5.22}
\]
Thus we have shown that
\[
\alpha(h) = \alpha \quad \forall h \in V(f, \gamma, n). \tag{5.23}
\]
Let $h \in V(f, \gamma, n)$ and let $v \in W^{1,1}_{\text{loc}}(\mathbb{R}^1)$ be an $(h)$-minimal solution with rotation number $\alpha$. It follows from Proposition 2.3, (2.3), and (5.4) that $v|_{[0,\infty)}$ is an $(h)$-good function. By property (i) there exist integers $p, q$ such that
\[
|v(t) - w_f(t + p) - q| \leq \frac{1}{n} \quad \text{for all large enough } t. \tag{5.24}
\]
Therefore we proved the following property:
(iii) for each $h \in V(f, \gamma, n)$ and each $(h)$-minimal solution $v \in \mathcal{M}_{h}^{\text{per}}(\alpha)$, there exist integers $p, q$ such that
\[
|v(t) - w_f(t + p) - q| \leq \frac{1}{n} \quad \forall t \in \mathbb{R}. \tag{5.25}
\]
Define
\[
\mathcal{U}(f, \gamma, n) = \mathcal{A}^{-1}(V(f, \gamma, n)). \tag{5.26}
\]
Clearly $\mathcal{U}(f, \gamma, n)$ is an open neighborhood of $f_y$ in $M_k$. By property (iii) the following property holds:
(iv) for each $\xi \in \mathcal{U}(f, \gamma, n)$ and each $(\xi)$-minimal solution $v \in \mathcal{M}_{\xi}^{\text{per}}(\alpha)$, there exist integers $p, q$ such that (5.25) holds.
Define
\[
\mathcal{F}_{ka} = \cap_{n=1}^{\infty} \bigcup \{ \mathcal{U}(f, \gamma, i) : f \in M_k, \gamma \in (0, 1), i \geq n \}. \tag{5.27}
\]
It is not difficult to see that $\mathcal{F}_{ka}$ is a countable intersection of open everywhere dense subsets of $M_k$. 
Let $g \in \mathcal{F}_{k_\alpha}$, $\epsilon \in (0, 1)$ and $x, y \in \mathcal{M}_g^{(\text{per})}(\alpha)$. Choose a natural number $n > 8\epsilon^{-1}$. By (5.27) there exist $f \in \mathcal{M}_k$, $\gamma \in (0, 1)$ and an integer $i \geq n$ such that
\begin{equation}
\tag{5.28}
g \in \mathcal{U}(f, \gamma, i).
\end{equation}
It follows from (5.28) and property (iv) that there exist integers $p_1, q_1, p_2, q_2$ such that
\begin{align}
|x(t) - w_f(t + p_1) - q_1| &\leq \frac{1}{i} \quad \forall t \in \mathbb{R}^1, \\
|y(t) - w_f(t + p_2) - q_2| &\leq \frac{1}{i} \quad \forall t \in \mathbb{R}^1,
\end{align}
where $w_f \in \mathcal{M}_f^{(\text{per})}(\alpha)$.
It follows from (5.29) and (5.30) that for all $t \in \mathbb{R}^1$,
\begin{align}
|x(t - p_1) - w_f(t) - q_1| &\leq \frac{1}{i}, \\
|y(t - p_2) - w_f(t) - q_2| &\leq \frac{1}{i}, \\
|x(t - p_1 - q_1) - (y(t - p_2) - q_2)| &\leq \frac{2}{i}, \\
|x(t + p_2 - p_1) - y(t) - q_1 + q_2| &\leq \frac{2}{i} \leq \frac{2}{n} < \epsilon.
\end{align}
Since $\epsilon$ is any number in $(0, 1)$, we conclude that there exist integers $p, q$ such that
\begin{equation}
\tag{5.32}
x(t + p) - q = y(t) \quad \forall t \in \mathbb{R}^1.
\end{equation}
Assume that $h \in \mathcal{U}(f, \gamma, i)$ and $z \in \mathcal{M}_h^{(\text{per})}(\alpha)$. By the property (iv) there exist integers $p_3, q_3$ such that
\begin{equation}
\tag{5.33}
|z(t) - w_f(t + p_3) - q_3| \leq \frac{1}{i} \quad \forall t \in \mathbb{R}^1.
\end{equation}
Combined with (5.29) this inequality implies that
\begin{equation}
\tag{5.34}
|z(t - p_3) - q_3 - x(t - p_1) + q_1| \leq \frac{2}{i} \leq \frac{2}{n} < \epsilon
\end{equation}
for all $t \in \mathbb{R}^1$. This completes the proof of Theorem 3.1.

References


154  Uniqueness of a minimal solution


[6] A. Leizarowitz and A. J. Zaslavski, Infinite-horizon variational problems with noncon-


[7] J. N. Mather, Existence of quasiperiodic orbits for twist homeomorphisms of the annulus,


[11] W. Senn, Strikte Konvexität für Variationsprobleme auf dem n-dimensionalen torus,

Manuscripta Math. 71 (1991), no. 1, 45–65 (German).


323–354.


Alexander J. Zaslavski: Department of Mathematics, Technion-Israel Institute of

Technology, Haifa 32000, Israel

E-mail address: ajzasl@technunix.technion.ac.il