1. Introduction

We consider the quasilinear elliptic boundary value problem,
\[-\Delta_p u = \alpha_+(x)(u^+)^{p-1} - \alpha_-(x)(u^-)^{p-1} + f(x, u), \quad u \in W^{1,p}_0(\Omega), \tag{1.1}\]
where \(\Omega\) is a bounded domain in \(\mathbb{R}^n, n \geq 1\), \(\Delta_p u = \text{div}(\nabla |\nabla u|^{p-2} \nabla u)\) is the \(p\)-Laplacian, \(1 < p < \infty\), \(u^\pm = \max\{\pm u, 0\}\), \(\alpha_\pm \in L^\infty(\Omega)\), and \(f\) is a Carathéodory function on \(\Omega \times \mathbb{R}\) satisfying a growth condition,
\[|f(x, t)| \leq q V(x)^{p-q}|t|^{q-1} + W(x)^{p-1}, \tag{1.2}\]
with \(1 \leq q < p\) and \(V, W \in L^p(\Omega)\). We assume that (1.1) is resonant from one side in the sense that either
\[\lambda_l \leq \alpha_\pm(x) \leq \lambda_{l+1} - \varepsilon \tag{1.3}\]
or
\[\lambda_l + \varepsilon \leq \alpha_\pm(x) \leq \lambda_{l+1}, \tag{1.4}\]
for two consecutive variational eigenvalues, \(\lambda_l < \lambda_{l+1}\) of \(-\Delta_p\) on \(W^{1,p}_0(\Omega)\), and some \(\varepsilon > 0\) (see Section 2 for the definition of the variational spectrum).

The special case where \(\alpha_+(x) = \alpha_-(x) \equiv \lambda_l\) and \(q = 1\) was recently studied by Arcoya and Orsina [1], Bouchala and Drábek [3], and Drábek and Robinson [8] (see also Cuesta et al. [6] and Dancer and Perera [7]). In the present paper, we prove a single existence theorem for the general case that includes all their results and much more.
Denote by $N$ the set of nontrivial solutions of the asymptotic problem

$$-\Delta_p u = \alpha_+(x)(u^+)^{p-1} - \alpha_-(x)(u^-)^{p-1}, \quad u \in W^{1,p}_0(\Omega),$$  \hspace{1cm} (1.5)

and set

$$F(x,t) := \int_0^t f(x,s) \, ds, \quad H(x,t) := pF(x,t) - tf(x,t).$$ \hspace{1cm} (1.6)

Our main result is the following theorem.

**Theorem 1.1.** Problem (1.1) has a solution in the following cases:

(i) equation (1.3) holds and $\int_\Omega H(x,u_j) \to +\infty$,

(ii) equation (1.4) holds and $\int_\Omega H(x,u_j) \to -\infty$ for every sequence $(u_j)$ in $W^{1,p}_0(\Omega)$ such that $\|u_j\| \to \infty$ and $u_j/\|u_j\|$ converges to some element of $N$. In particular, (1.1) is solvable when (1.3) or (1.4) holds and $N$ is empty.

As is usually the case in resonance problems, the main difficulty here is the lack of compactness of the associated variational functional, which we will overcome by constructing a sequence of approximating nonresonance problems, finding approximate solutions for them using linking and min-max type arguments, and passing to the limit (see Rabinowitz [10] for standard details of the variational theory). But first we give some corollaries and deduce the results of [1, 3, 8]. In what follows, $(u_j)$ is as in the theorem, that is, $\rho_j := \|u_j\| \to \infty$ and $v_j := u_j/\rho_j \to v \in N$.

First, we give simple pointwise assumptions on $H$ that imply the limits in the theorem.

**Corollary 1.2.** Problem (1.1) has a solution in the following cases:

(i) equation (1.3) holds, $H(x,t) \to +\infty$ a.e. as $|t| \to \infty$, and $H(x,t) \geq -C(x)$,

(ii) equation (1.4) holds, $H(x,t) \to -\infty$ a.e. as $|t| \to \infty$, and $H(x,t) \leq C(x)$ for some $C \in L^1(\Omega)$.

Note that this corollary makes no reference to $N$.

**Proof.** If (i) holds, then $H(x,u_j(x)) = H(x,\rho_j v_j(x)) \to +\infty$ for a.e. $x$ such that $v(x) \neq 0$ and $H(x,u_j(x)) \geq -C(x)$, so

$$\int_\Omega H(x,u_j) \geq \int_{v \neq 0} H(x,u_j) - \int_{v = 0} C(x) \to +\infty$$ \hspace{1cm} (1.7)

by Fatou’s lemma. Similarly, $\int_\Omega H(x,u_j) \to -\infty$ if (ii) holds. \hfill $\square$

Note that the above argument goes through as long as the limits in (i) and (ii) hold on subsets of $\{x \in \Omega : v(x) \neq 0\}$ with positive measure. Now, taking $w = v^\pm$ in

$$\int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla w = \int_\Omega \left[ \alpha_+(x)(v^+)^{p-1} - \alpha_-(x)(v^-)^{p-1} \right] w$$ \hspace{1cm} (1.8)
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gives
\[
\| v^+ \|^p = \int_{\Omega^+} \alpha_+(x)(v^+)^p \leq \| \alpha_+ \|_\infty \| v^+ \|^p \mu(\Omega^+) \frac{p}{p/n}
\leq \| \alpha_+ \|_\infty S^{-1} \| v^+ \|^p \mu(\Omega^+) \frac{p}{p/n},
\]

(1.9)

where \( \Omega^+ = \{ x \in \Omega : v(x) \geq 0 \} \), \( p^* = np/(n-p) \) is the critical Sobolev exponent, \( S \) is the best constant for the embedding \( W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) \), and \( \mu \) is the Lebesgue measure in \( \mathbb{R}^n \). So

\[
\mu(\Omega^+) \geq \left( S \| \alpha_+ \|_\infty^{-1} \right)^{n/p},
\]

(1.10)

and hence

\[
\mu(\{ x \in \Omega : v(x) = 0 \}) \leq \mu(\Omega) - S^n p \left( \| \alpha_+ \|_\infty^{-n/p} + \| \alpha_- \|_\infty^{-n/p} \right).
\]

(1.11)

Thus, we have the following corollary.

**Corollary 1.3.** Problem (1.1) has a solution in the following cases:

(i) equation (1.3) holds, \( H(x,t) \to +\infty \) in \( \Omega' \) as \( |t| \to \infty \), and \( H(x,t) \geq -C(x) \),

(ii) equation (1.4) holds, \( H(x,t) \to -\infty \) in \( \Omega' \) as \( |t| \to \infty \), and \( H(x,t) \leq C(x) \) for some \( \Omega' \subset \Omega \) with \( \mu(\Omega') > \mu(\Omega) - S^n p \left( \| \alpha_+ \|_\infty^{-n/p} + \| \alpha_- \|_\infty^{-n/p} \right) \) and \( C \in L^1(\Omega) \).

Similar conditions on \( H \) were recently used by Furtado and Silva [9] in the semilinear case \( p = 2 \).

Next, note that

\[
H_+(x)(v^+(x))^q + H_-(x)(v^-(x))^q
\leq \liminf \frac{H(x,u_j(x))}{\rho_j^q} \leq \limsup \frac{H(x,u_j(x))}{\rho_j^q}
\]

\[
\leq \overline{H}_+(x)(v^+(x))^q + \overline{H}_-(x)(v^-(x))^q,
\]

(1.12)

where

\[
\underline{H}_+(x) = \liminf_{t \to +\infty} \frac{H(x,t)}{|t|^q}, \quad \overline{H}_+(x) = \limsup_{t \to +\infty} \frac{H(x,t)}{|t|^q}.
\]

(1.13)

Moreover,

\[
\frac{|H(x,u_j(x))|}{\rho_j^q} \leq (p+q)V(x)^{p-q}|v_j(x)|^q + \frac{(p+1)W(x)^{p-1}}{\rho_j^{q-1}}|v_j(x)|
\]

(1.14)
by (1.2), so it follows that

\[
\int_{\Omega} H^+ (v^+) q + H^- (v^-) q \leq \liminf \frac{\int_{\Omega} H(x, u_j)}{\rho_j^q} \leq \limsup \frac{\int_{\Omega} H(x, u_j)}{\rho_j^q} \leq \int_{\Omega} H^+ (v^+) q + H^- (v^-) q.
\]

Thus we have the following corollary.

**Corollary 1.4.** Problem (1.1) has a solution in the following cases:

(i) equation (1.3) holds and \(\int_{\Omega} H^+ (v^+) q + H^- (v^-) q > 0\) for all \(v \in N\),

(ii) equation (1.4) holds and \(\int_{\Omega} H^+ (v^+) q + H^- (v^-) q < 0\) for all \(v \in N\).

When \(\alpha_+(x) = \alpha_-(x) \equiv \lambda_1\) and \(q = 1\) this reduces to the result of Bouchala and Drábel [3].

Finally, we note that if

\[
\frac{tf(x, t)}{|t|^q} \to f_\pm(x) \quad \text{a.e. as } t \to \pm \infty,
\]

then

\[
\frac{F(x, t)}{|t|^q} = \frac{1}{|t|^q} \int_0^t \left[ f'(x, s) \right] s^{q-2} s ds + \frac{f_\pm(x)}{q} \to \frac{f_\pm(x)}{q}
\]

and hence

\[
\frac{H(x, t)}{|t|^q} \to \left( \frac{p}{q} - 1 \right) f_\pm(x),
\]

so Corollary 1.4 implies the following corollary.

**Corollary 1.5.** Problem (1.1) has a solution in the following cases:

(i) equation (1.3) holds and \(\int_{\Omega} f_+(v^+) q + f_- (v^-) q > 0\) for all \(v \in N\),

(ii) equation (1.4) holds and \(\int_{\Omega} f_+(v^+) q + f_- (v^-) q < 0\) for all \(v \in N\).

This was proved in Arcoya and Orsina [1] and Drábel and Robinson [8] for the special case \(\alpha_+(x) = \alpha_-(x) \equiv \lambda_1\) and \(q = 1\).

2. Proof of Theorem 1.1

First we recall some facts about the variational spectrum of the \(p\)-Laplacian. It is easily seen from the Lagrange multiplier rule that the eigenvalues of \(-\Delta_p\) on \(W^{1,p}_0(\Omega)\) correspond to the critical values of

\[
J(u) = \int_{\Omega} |\nabla u|^p, \quad u \in S := \{ u \in W^{1,p}_0(\Omega) : \|u\|_p = 1 \}.
\]
Moreover, $J$ satisfies the Palais-Smale condition (cf. Drábek and Robinson [8]). Thus, we can define an unbounded sequence of min-max eigenvalues by

$$
\lambda_l := \inf_{A \in \mathcal{F}_l} \max_{u \in A} J(u), \quad l \in \mathbb{N},
$$

(2.2)

where

$$
\mathcal{F}_l = \{ A \subset S : \exists \text{ a continuous odd surjection } h : S^{l-1} \to A \} \quad (2.3)
$$

and $S^{l-1}$ is the unit sphere in $\mathbb{R}^l$.

**Lemma 2.1.** $\lambda_l$ is an eigenvalue of $-\Delta_p$ and $\lambda_l \to \infty$.

*Proof.* If $\lambda_l$ is a regular value of $J$, then there is an $\varepsilon > 0$ and an odd homeomorphism $\eta : S \to S$ such that $\eta(J^{\lambda_l+\varepsilon}) \subset J^{\lambda_l-\varepsilon}$ by [2, Theorem 2.5] (the standard first deformation lemma is not sufficient because the manifold $S$ is not of class $C^{1,1}$ when $p < 2$). But then taking $A \in \mathcal{F}_l$ with $\max J(A) \leq \lambda_l + \varepsilon$ and setting $\tilde{A} = \eta(A)$, we get a set in $\mathcal{F}_l$ for which $\max J(\tilde{A}) \leq \lambda_l - \varepsilon$, contradicting the definition of $\lambda_l$. Finally, denoting by $\mu_l \to \infty$ the usual Lusternik-Schnirelmann eigenvalues, we have $\lambda_l \geq \mu_l$ since the genus of each $A$ in $\mathcal{F}_l$ is $l$, so $\lambda_l \to \infty$. □

It is not known whether this is a complete list of eigenvalues when $p \neq 2$ and $n \geq 2$. However, the variational structure provided by this portion of the spectrum is sufficient to show that the associated functional admits a linking geometry in the nonresonant case. We only consider (i) as the proof for (ii) is similar. Let

$$
\alpha_j^\pm(x) = \begin{cases} 
\alpha_{\pm}(x), & \text{if } \alpha_{\pm}(x) \geq \lambda_l + \frac{1}{j}, \\
\lambda_l + \frac{1}{j}, & \text{if } \alpha_{\pm}(x) < \lambda_l + \frac{1}{j},
\end{cases} \quad (2.4)
$$

so that

$$
\lambda_l + \frac{1}{j} \leq \alpha_j^\pm(x) \leq \lambda_{l+1} - \varepsilon, \quad |\alpha_j^\pm(x) - \alpha_{\pm}(x)| \leq \frac{1}{j},
$$

(2.5)

and let

$$
\Phi_j(u) = \int_{\Omega} |\nabla u|^p - \alpha_j^+(x)(u^+)^p - \alpha_j^-(x)(u^-)^p - pF(x,u), \quad u \in W^{1,p}_0(\Omega). \quad (2.6)
$$

First, we show that there is a $u_j \in W^{1,p}_0(\Omega)$ such that

$$
\|u_j\|_{\Phi_j'(u_j)} \to 0, \quad \inf \Phi_j(u_j) > -\infty. \quad (2.7)
$$

By (2.2), there is an $A \in \mathcal{F}_l$ such that

$$
J(u) \leq \lambda_l + \frac{1}{2j}, \quad u \in A. \quad (2.8)
$$
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For $u \in A$ and $R > 0$,

$$
\Phi_j(Ru) = R^p \left[ J(u) - \int_\Omega \alpha_+^j(x)(u^+)^p + \alpha_-^j(x)(u^-)^p \right] - \int_\Omega p F(x, Ru)
\leq -\frac{R^p}{2^j} + p \left( \|V\|_p^{p-q} R^q + \|W\|_p^{p-1} R \right)
$$

by (1.2), (2.5), and (2.8), so

$$
\max_{u \in A} \Phi_j(Ru) \longrightarrow -\infty \quad \text{as } R \longrightarrow \infty.
$$

Next, let

$$
\mathcal{F} = \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega |\nabla u|^p \geq \lambda_{l+1} \int_\Omega |u|^p \right\}.
$$

For $u \in \mathcal{F}$,

$$
\Phi_j(u) \geq \varepsilon \|u\|_p^p - p \left( \|V\|_p^{p-q} \|u\|_p^q + \|W\|_p^{p-1} \|u\|_p \right),
$$

so

$$
\inf_{u \in \mathcal{F}} \Phi_j(u) \geq C := \min_{r \geq 0} \left[ \varepsilon r^p - p \left( \|V\|_p^{p-q} r^q + \|W\|_p^{p-1} r \right) \right] > -\infty.
$$

Now use (2.10) to fix $R > 0$ so large that

$$
\max_{u \in A} \Phi_j(Ru) < C,
$$

where $RA = \{ Ru : u \in A \}$.

Since $A \in \mathcal{F}_l$, there is a continuous odd surjection $h : S^{l-1} \rightarrow A$. Let

$$
\Gamma = \left\{ \varphi \in C(D^l, W_0^{1,p}(\Omega)) : \varphi|_{S^{l-1}} = Rh \right\},
$$

where $D^l$ is the unit disk in $\mathbb{R}^l$ with boundary $S^{l-1}$. We claim that $RA$ links $\mathcal{F}$ with respect to $\Gamma$, that is,

$$
\varphi(D^l) \cap \mathcal{F} \neq \emptyset \quad \forall \varphi \in \Gamma.
$$

To see this, first note that the proof is done if $0 \in \varphi(D^l)$. Otherwise, denoting by $\pi$ the radial projection onto $S$, $\tilde{A} := \pi(\varphi(D^l)) \cup -\pi(\varphi(D^l)) \in \mathcal{F}_{l+1}$, and hence

$$
\max_{u \in \pi(\varphi(D^l))} J(u) = \max_{u \in \tilde{A}} J(u) \geq \lambda_{l+1},
$$

so $\pi(\varphi(D^l)) \cap \mathcal{F} \neq \emptyset$, which implies that $\varphi(D^l) \cap \mathcal{F} \neq \emptyset$.

Now it follows from a deformation argument of Cerami [5] that there is a $u_j$ such that

$$
\|u_j\| \rightarrow 0, \quad \|\Phi_j'(u_j)\| \rightarrow 0,
$$

and

$$
\Phi_j(u_j) - c_j \rightarrow 0,
$$

where

$$
c_j := \inf_{u \in \mathcal{F}} \Phi_j(u) = \frac{-\varepsilon r^p}{p} - \lambda_{l+1} r, \quad r = \max_{u \in \mathcal{F}} \max_{u \in \mathcal{F}} |u|.
$$
where
\[ c_j := \inf_{\varphi \in \Gamma} \max_{u \in \varphi(D^p)} \Phi_j(u) \geq C, \]  
(2.19)

from which (2.7) follows.

We complete the proof by showing that a subsequence of \((u_j)\) converges to a solution of (1.1). It is easy to see that this is the case if \((u_j)\) is bounded, so suppose that \(\rho_j := \|u_j\| \to \infty\). Setting \(v_j := u_j/\rho_j\) and passing to a subsequence, we may assume that \(v_j \to v\) weakly in \(W^{1,p}_0(\Omega)\), strongly in \(L^p(\Omega)\), and a.e. in \(\Omega\). Then

\[
\int_\Omega |\nabla v_j|^{p-2} \nabla v_j \cdot \nabla (v_j - v) = \frac{\langle \Phi_j'(u_j), v_j - v \rangle}{p \rho_j^{p-1}} + \int_\Omega \left[ \alpha^+_j(x) (v_j^+)^{p-1} - \alpha^-_j(x) (v_j^-)^{p-1} + \frac{f(x,u_j)}{\rho_j^{p-1}} \right] (v_j - v) \to 0, \tag{2.20}
\]

and we deduce that \(v_j \to v\) strongly in \(W^{1,p}_0(\Omega)\) (cf. Browder [4]). In particular, \(\|v\| = 1\), so \(v \neq 0\). Moreover, for each \(w \in W^{1,p}_0(\Omega)\), passing to the limit in

\[
\frac{\langle \Phi_j'(u_j), w \rangle}{p \rho_j^{p-1}} = \int_\Omega |\nabla v_j|^{p-2} \nabla v_j \cdot \nabla w - \left[ \alpha^+_j(x) (v_j^+)^{p-1} - \alpha^-_j(x) (v_j^-)^{p-1} + \frac{f(x,u_j)}{\rho_j^{p-1}} \right] w \tag{2.21}
\]

gives that

\[
\int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla w - \left[ \alpha^+(x) (v^+)^{p-1} - \alpha^-(x) (v^-)^{p-1} \right] w = 0, \tag{2.22}
\]

so \(v \in N\). Thus,

\[
\frac{\langle \Phi_j'(u_j), u_j \rangle}{p} - \Phi_j(u_j) = \int_\Omega H(x,u_j) \to +\infty, \tag{2.23}
\]

contradicting (2.7).

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