An inverse problem, the determination of the shape and a convective coefficient on a part of the boundary from partial measurements of the solution, is studied using 2-person optimal control techniques.

1. Introduction

Let $H, \mathcal{H}_j, \mathcal{U}_j; j = 1, \ldots, N$ be Hilbert spaces and let $\phi$ be a lower semi-continuous (l.s.c.) function from $H \times \prod_{j=1}^{N} \mathcal{U}_j$ into $\mathbb{R}^+$ with $\phi(\cdot; u)$ convex on $H$.

Consider the initial-value problem

$$y' + \partial \phi(y; u) + f(t, y; u) \ni 0 \quad \text{on } (0, T), \quad y(0) = y_0.$$  \hspace{1cm} (1.1)

With some conditions on $\phi$ and on $f$, the set $\mathcal{R}(u)$ of all “strong” solutions of (1.1) is nonempty. Let $f_j$ be mappings of $L^2(0, T; \mathcal{H}_j) \times \mathcal{U}$ into $\mathbb{R}^+$ and associate with (1.1) the cost functionals

$$J_j(y; u) = \int_0^T f_j(y(s); u) \, ds, \quad j = 1, \ldots, N,$$  \hspace{1cm} (1.2)

with $D(\phi(\cdot, u)) \subset \mathcal{H}_j$ for all $u \in \mathcal{U} = \prod_{j=1}^{N} \mathcal{U}_j$.

The existence of an open loop of (1.1), (1.2) with $\phi$ independent of the control $u$, has been established in Ton [7]. With optimal shape design and with inverse problems in mind, we will consider the case when $\phi$ depends on the control $u$ as it appears in the top order term of the partial differential operators involved in the problems.

Optimal design of domains has been investigated by Barbu and Friedman [1], Canadas et al. [2], Gunzburger and Kim [3], Pironneau [6], and others.
problems have been studied by Canadas et al. [2], Lenhart et al. [4], Lenhart and Wilson [5], and others.

In contrast with all the cited works where a single cost functional is involved, we will consider the \( N \)-person optimal control approach. It is well known that for \( N \)-control, open and closed loops are two different notions. In this paper, the existence of an open loop of (1.1), (1.2) is established in Section 3, that is, there exists \( \tilde{u} \in U \) such that

\[
J_j(\tilde{y}; \pi_j \tilde{u}, \tilde{u}) \leq J_j(y; \pi_j \tilde{u}, v_j), \quad \forall y \in \mathcal{R}(\pi_j \tilde{u}, v_j), \quad \forall v_j \in U_j; \quad j = 1, \ldots, N, \tag{1.3}
\]

where \( U_j \) are given compact convex subsets of the control spaces \( \mathcal{U}_j \) and \( \pi_j \) is the projection of \( \mathcal{U} \) onto \( \prod_{k \neq j} \mathcal{U}_k \).

With a cost functional \( f_j \) defined by

\[
f_j(y) = \| y(\cdot, t) - h(\cdot, t) \|_{L^2(\mathcal{H}; L^2(\Omega))}^2, \tag{1.4}
\]

where \( \Omega \) is a proper subset of the domain and where \( h \) is a measurement of the solution \( y \) of (1.1) in the subdomain \( \Omega \), then (1.1), (1.2) becomes an inverse problem.

Applications to parabolic inequalities are carried out in Section 4 and the notation and the main assumptions of the paper are given in Section 2.

2. Notation and assumptions

Let \( H, \mathcal{H}_j, \mathcal{U}_j; \ j = 1, \ldots, N \) be Hilbert spaces. The norm in \( H \) is denoted by \( \| \cdot \| \) and \((\cdot, \cdot)\) is the inner product in the space. Throughout, \( U \) is a given compact convex subset of the control space \( \mathcal{U} = \prod_{j=1}^N \mathcal{U}_j \).

**Assumption 2.1.** Let \( \varphi \) be a mapping of \( H \times \mathcal{U} \) into \( \mathbb{R}^+ \). We assume that

1. for each \( u \in \mathcal{U} \), \( D(\varphi(\cdot; u)) \) is dense in \( H \);
2. \( \varphi(y; u) \) is an l.s.c. function from \( H \times \mathcal{U} \) into \( \mathbb{R}^+ \) and is convex on \( H \) for each given \( u \in \mathcal{U} \);
3. there exists a positive constant \( c \) such that
   \[
c \| y \|^2 \leq \varphi(y; u), \quad \forall y \in D(\varphi(y; u)), \quad \forall u \in \mathcal{U}; \tag{2.1}
\]
4. for each positive constant \( C \),
   \[
   \{ y : \varphi(y; u) \leq C \} \tag{2.2}
   \]
   is a compact convex subset of \( H \) for each given \( u \in \mathcal{U} \);
5. if \( u_n \to u \) in \( \mathcal{U} \), then
   \[
   \int_0^T \varphi(y(s); u) \, ds = \lim_{n \to \infty} \int_0^T \varphi(y(s); u_n) \, ds, \quad \forall y \in \bigcap_{u_n \in \mathcal{U}} D(\varphi(\cdot; u_n)) \cap L^2(0, T; H). \tag{2.3}
   \]
The subdifferential of \( \phi(y;u) \) at \( y \) is the set
\[
\partial \phi(y;u) = \{ g : g \in H, \phi(x;u) - \phi(y;u) \geq (g, x - y), \forall x \in D(\phi(;u)) \}. \tag{2.4}
\]

It is known that \( A(y;u) = \partial \phi(y;u) \) is maximal monotone in \( H \). The images of \( A(y;u) \) are closed, convex subsets of \( H \).

Let \( f(y;u) \) be a mapping of \( L^2(0,T;H) \times U \) into \( L^2(0,T;H) \) satisfying the following assumption.

**Assumption 2.2.** We assume that there exists a constant \( C \) such that
\[
\| f(y;u) \|_H^2 \leq C \{ 1 + \| u \|_{H^1}^2 + \phi(y;u) \} \tag{2.5}
\]
for all \( y \in D(\phi(;u)) \), all \( u \in U \).

Throughout, the set of solutions of (1.1) is denoted by \( \mathcal{R}(u) \).

**Assumption 2.3.** Let \( f_j \) be mappings of \( L^2(0,T;H^j) \times U \) into \( \mathbb{R}^+ \). We assume that
1. \( D(\phi(;u)) \subset H^j \) for all \( u \in U \);
2. suppose that
\[
\phi(y^n;u^n) + \| (y^n)' \|_{L^2(0,T;H)} \leq C, \quad u^n \in U, \quad \{ y^n, u^n \} \rightarrow \{ y, u \} \quad \text{in} \quad L^2(0,T;H) \times U,
\]
then
\[
\int_0^T f_j(y;u) dt = \lim_{n \rightarrow \infty} \int_0^T f_j(y^n;u^n) dt. \tag{2.7}
\]

### 3. Open loop control

The main result of this section is the following theorem.

**Theorem 3.1.** Let \( \phi, f \) be as in Assumptions 2.1 and 2.2, and let \( f_j \) be continuous mappings of \( L^2(0,T;H^j) \times U \) into \( \mathbb{R}^+ \). Suppose that \( y_0 \in D(\phi(;u)) \) for all \( u \in U \). Then there exists \( \{ \tilde{y}, \tilde{u} \} \in \{ L^2(0,T;H) \cap \mathcal{R}(\tilde{u}) \} \times U \) such that
\[
J_j(\tilde{y};\pi_j \tilde{u}, \tilde{u}_j) \leq J_j(y;\pi_j \tilde{u}, \nu_j), \quad \forall y \in \mathcal{R}(\pi_j \tilde{u}, \nu_j), \quad \forall \nu_j \in U_j, \quad j = 1, \ldots, N. \tag{3.1}
\]

Moreover, there exists a positive constant \( C \), independent of \( u \) such that
\[
\text{ess sup} \phi(\tilde{y}(t); \tilde{u}) + \| \tilde{y}' \|_{L^2(0,T;H)}^2 + \| A(\tilde{y}; \tilde{u}) \|_{L^1(0,T;H)}^2 \leq C \left\{ 1 + \sup_{u \in U} \phi(y_0;u) \right\}, \tag{3.2}
\]
where \( A(\tilde{y}; \tilde{u}) \) is an element of the set \( \partial \phi(\tilde{y}; \tilde{u}) \).

First, we will show that the set \( \mathcal{R}(u) \) is nonempty.
Theorem 3.2. Suppose all the hypotheses of Theorem 3.1 are satisfied. Then for each given \( u \in U \), there exists a solution \( y \) of (1.1) with
\[
\|y\|_{L^2(0,T;H)}^2 + \|A(y;u)\|_{L^2(0,T;H)}^2 + \text{ess sup} \varphi(y(t);u) \leq C \{ 1 + \|u\|_U^2 \}.
\] (3.3)
The constant \( C \) is independent of \( u \) and \( A(y;u) \) is an element of \( \partial \varphi(y;u) \).

Proof. For a given \( u \in U \), the existence of a solution \( y \) of (1.1) with \( \{ y, y', A(y;u) \} \in L^\infty(0,T;H) \times (L^2(0,T;H))^2 \) (3.4) is known (cf. Yamada [8]).

We will now establish the estimate of Theorem 3.2. We have
\[
(y', \partial \varphi(y;u)) + \|\partial \varphi(y;u)\|^2 + (f(y;u), \partial \varphi(y;u)) = 0.
\] (3.5)

With our hypotheses on \( f \), we get
\[
\frac{d}{dt} \varphi(y;u) + \|\partial \varphi(y;u)\|^2 \leq C \{ 1 + \|u\|_U^2 + \varphi(y(t);u) \}.
\] (3.6)

It follows from the Gronwall lemma that
\[
\text{ess sup}_{t \in [0,T]} \varphi(y(t);u) + \|\partial \varphi(y;u)\|_{L^2(0,T;H)}^2 \leq C \{ 1 + \|u\|_U^2 \}.
\] (3.7)

The different constants \( C \) are all independent of \( u \).

With the estimate (2.1), we deduce from (1.1) and from Assumption 2.2 that
\[
\|y\|_{L^2(0,T;H)}^2 \leq C \{ 1 + \|u\|_U^2 \}.
\] (3.8)

The theorem is thus proved. \(\square\)

Set
\[
B_C = \left\{ y : \|y\|_{L^2(0,T;H)} + \text{ess sup}_{u \in U} \sup_{t \in [0,T]} \varphi(y;u) \leq C \left( 1 + \sup_{u \in U} \|u\|_U \right) \right\},
\] (3.9)

Consider the evolution inclusion
\[
y' + \partial \varphi(y;u) + f(x;u) \ni 0 \quad \text{on} \ (0,T), \ y(0) = y_0
\] (3.10)
with \( x \in B_C \).

In view of Theorem 3.2, inclusion (3.10) has a unique solution which we will write as \( y = R(x;u) \).

Denote by
\[
J_j(x; y; u) = \int_0^T f_j(y(s); u) \, ds, \quad j = 1, \ldots, N,
\] (3.11)
the cost functionals associated with (3.10) and where $y = R(x; u)$ is the unique solution of (3.10).

Let

$$
\Psi(x; u, v) = \sum_{j=1}^{N} J_j(x; y_j; \pi_j u, v_j),
$$

where $y_j = R(x; \pi_j u, v_j)$.

**Lemma 3.3.** Suppose all the hypotheses of Theorem 3.1 are satisfied. Then for each given $\{x, u\} \in B_C \times U$, there exists $v^* \in U$ such that

$$
\Psi(x; u, v^*) = d(x; u) = \inf \{\Psi(x; u, v) : v \in U\}.
$$

**Proof.** Let $\{v^n\}$ be a minimizing sequence of (3.13) with

$$
d(x; u) \leq \Psi(x; u, v^n) \leq d(x; u) + n^{-1}.
$$

Since $v^n \in U$ and $U$ is a compact subset of $\mathcal{U}$, we obtain by taking subsequences that $v^{n_k} \rightarrow v^*$ in $\mathcal{U}$. Let $y_j^n = R(x; \pi_j u, v_j^n)$, then from the estimates of Theorem 3.2 we obtain, by taking subsequences, that

$$
\{y_j^n, (y_j^n)' A(y_j^n; \pi_j u, v_j^n)\} \rightarrow \{y_j^*, (y_j^*)', \chi_j\} \text{ in } L^2(0, T; H) \times (L^2(0, T; H))^2 \text{ weak}.
$$

(3.15)

From the definition of subdifferential, we have

$$
\int_0^T \varphi(z(t); \pi_j u, v_j^n) dt - \int_0^T \varphi(y_j^n(t); \pi_j u, v_j^n) dt \\
\geq \int_0^T (A(y_j^n(t); \pi_j u, v_j^n), z - y_j^n) dt,
$$

for all $z \in L^2(0, T; H)$.

It follows from Assumption 2.1 that

$$
\int_0^T \varphi(z(t); \pi_j u, v_j^*) dt - \int_0^T \varphi(y_j^*(t); \pi_j u, v_j^*) dt \\
\geq \int_0^T (\chi_j, z - y_j^*(t)) dt.
$$

(3.17)

Hence

$$
\chi_j = A(y_j^*; \pi_j u, v_j^*).
$$

(3.18)

It is clear that $y_j^* = R(x; \pi_j u, v_j^*)$ and thus,

$$
d(x; u) = \Psi(x; u, v^*) = \sum_{j=1}^{N} J_j(x; y_j, \pi_j u, v_j^*),
$$

(3.19)

where $y_j = R(x; \pi_j u, v_j^*)$.

The lemma is proved.
An inverse problem for evolution inclusions

Let

\[ X(x; u) = \{ v^* : \Psi(x; u, v^*) \leq \Psi(x; u, v), \ \forall v \in U \}. \]  

(3.20)

**Lemma 3.4.** Let \( g_j \) be a continuous mapping of \( U_j \) into \( \mathbb{R}^+ \) and suppose that \( g_j \) is 1-1. Then there exists a unique \( \hat{v} \in X(x; u) \) such that

\[ g_j(\hat{v}) = \inf \{ g_j(v^*) : v^* \in X(x; u) \}. \]  

(3.21)

**Proof.** The set \( X(x; u) \) is nonempty and with our hypothesis on \( g_j \), it is clear that

\[ d_j(x; u) = \inf \{ g_j(v^*) : v^* \in X(x; u) \} \]  

exists.

Let \( v^n_j \) be a minimizing sequence of the optimization problem (3.22) with

\[ d_j(x; u) \leq g_j(v^n_j) \leq d_j(x; u) + n^{-1}, \quad j = 1, \ldots, N, \]  

(3.23)

and \( v^n \in X(x, u) \).

Let \( y^n_j = R(x; \pi_j u, v^n_j) \) be the unique solution of (3.10) with controls \( \{ \pi_j u, v^n_j \} \) and \( f(x; \pi_j u, v^n_j) \). Then from the estimates of Theorem 3.2, we obtain, by taking subsequences, that

\[ \{ y^n_j, (y^n_j)', A(y^n_j; \pi_j u, v^n_j) \} \rightarrow \{ \hat{y}^j, \hat{y}'^j, \chi_j \} \quad \text{in} \quad L^2(0, T; H) \times (L^2(0, T; H))^2 \]  

(3.24)

Since \( v^n \in U \), we get by taking subsequences that \( v^n \rightarrow \hat{v} \) in \( \mathfrak{U} \).

A proof, as in that of Lemma 3.3, shows that

\[ \chi_j = A(\hat{y}^j; \pi_j u, \hat{v}^j), \quad \hat{y}^j = R(x; \pi_j u, \hat{v}^j). \]  

(3.25)

Hence \( \hat{v} \in X(x; u) \). We now have

\[ g_j(\hat{v}) = d_j(x; u) = \inf \{ g_j(v^*) : v^* \in X(x; u) \}. \]  

(3.26)

Since \( g_j \) is 1-1, \( \hat{v} \) is unique. The lemma is proved. \( \square \)

Let \( \mathcal{L} \) be the nonlinear mapping of \( \mathcal{B}_C \times U \) into \( \mathcal{B}_C \times U \), defined by

\[ \mathcal{L}(x, u) = \{ \hat{y}, \hat{v} \}, \]  

(3.27)

where \( \hat{v} \) is the element of \( U \) given by Lemma 3.4 and \( \hat{y} = R(x; \pi_j u, \hat{v}^j) \) is the unique solution of (3.10) with control \( \{ \pi_j u, \hat{v}^j \} \) and \( f(x; \pi_j u, \hat{v}^j) \).

**Lemma 3.5.** Suppose all the hypotheses of Theorem 3.1 are satisfied. Then \( \mathcal{L} \), defined by (3.27), has a fixed point, that is, there exists \( \{ \tilde{y}, \tilde{u} \} \in \mathcal{B}_C \times U \) such that \( \mathcal{L}(\tilde{y}, \tilde{u}) = \{ \tilde{y}, \tilde{u} \} \).
Proof. (1) We now show that $L$ has a fixed point by applying Schauder’s theorem. Since $B_C \times U$ is a compact convex subset of $L^2(0, T; H) \times \mathcal{U}$ and since $L$ takes $B_C \times U$ into itself, it suffices to show that $L$ is continuous.

(2) Let $\{x^n, u^n\}$ be in $B_C \times U$ and let

$$y_j^n = R(x^n; \pi_j u^n, \hat{v}_j^n), \quad \hat{v}^n \text{ as in Lemma 3.4.} \quad (3.28)$$

Since $\{x^n u^n\} \in B_C \times U$ and $B_C \times U$ is a compact subset of $L^2(0, T; H) \times \mathcal{U}$, there exists a subsequence such that

$$\{x^n, u^n, \hat{v}^n\} \rightarrow \{x^*, u^*, \hat{v}\} \text{ in } L^2(0, T; H) \times \mathcal{U} \times \mathcal{U}. \quad (3.29)$$

From the estimates of Theorem 3.2, we get

$$\{y_j^n, (y_j^n)' , A(y_j^n; u^n)\} \rightarrow \{y_j^*, (y_j^*)' , \chi_j\} \text{ in } L^2(0, T; H) \times (L^2(0, T; H))^2 \text{ weak.} \quad (3.30)$$

A proof, as in that of Lemma 3.3, shows that

$$\chi_j = A(y_j^*; u^*), \quad y_j^* = R(x^*; \pi_j u^*, \hat{v}_j). \quad (3.31)$$

(3) We now show that $u^* \in X(x^*, \hat{v})$. Since

$$L\{u^n, x^n\} = \{v^n, \hat{v}^n\}, \quad (3.32)$$

it follows from the definition of $L$ that

$$\Psi(x^n; u^n, v^n) \leq \Psi(x^n; u^n, v), \quad \forall v \in U, \quad (3.33)$$

where $z^n_j = R(x^n; \pi_j u^n, v_j)$ is the unique solution of (3.10) with controls $\{\pi_j u^n, v_j\}$ and $f(x^n; \pi_j u^n, v_j)$.

Again from the estimates of Theorem 3.2, we deduce as above that

$$\{z^n_j, (z^n_j)' , A(z^n_j; u^n)\} \rightarrow \{z_j, z_j', A(z_j; u^*)\} \text{ in } L^2(0, T; H) \times (L^2(0, T; H))^2 \text{ weak.} \quad (3.34)$$

It then follows from (3.33) that

$$\sum_{j=1}^N J_j(x^n; y^n_j; \pi_j u^n, v^n_j) \leq \sum_{j=1}^N J_j(x^n; z^n_j; \pi_j u^n, v^n_j), \quad \forall v \in U, \quad (3.35)$$

that is,

$$\Psi(x^*; u^*, \hat{v}) \leq \Psi(x^*; u^*, v), \quad \forall v \in U. \quad (3.36)$$
Hence
\[ d(x^*, u^*) = \Psi(x^*; u^*, \hat{v}) = \inf \{ \Psi(x^*; u^*, v) : v \in U \}. \] (3.37)

Moreover, we have
\[ \lim_{n \to \infty} g_j(v^n_j) = g_j(\hat{v}_j), \quad j = 1, \ldots, N. \] (3.38)

By hypothesis, \( g_j \) is 1-1 and so \( \hat{v} \), the unique element of \( X(x^*; u^*) \), with
\[ g_j(\hat{v}_j) = \inf \{ g_j(v_j) : v \in X(x^*; u^*) \}, \] (3.39)
is in \( X(x^*; u^*) \). It follows that \( \mathcal{L}\{x^*, u^*\} = \{ y^*, \hat{v} \} \).

The operator \( \mathcal{L} \) is continuous and thus, it has a fixed point by Schauder’s theorem. The lemma is thus proved. \( \square \)

**Proof of Theorem 3.1.** Let \( \mathcal{L} \) be as in (3.33). Then it follows from Lemma 3.5 that \( \mathcal{L} \) has a fixed point, that is, there exists \( \{ \tilde{y}, \tilde{u} \} \) with
\[ \mathcal{L}\{ \tilde{y}, \tilde{u} \} = \{ \tilde{y}, \tilde{u} \}. \] (3.40)

Thus,
\[ \tilde{y}' + A(\tilde{y}; \tilde{u}) + f(\tilde{y}; \tilde{u}) = 0 \quad \text{on } (0, T); \quad y(0) = y_0. \] (3.41)

Moreover,
\[ \sum_{j=1}^{N} J_j(\tilde{y}; \pi_j\tilde{u}, \tilde{u}_j) \leq \sum_{j=1}^{N} J_j(y_j; \pi_j\tilde{u}, v_j), \quad \forall y_j \in \mathcal{R}(\pi_j\tilde{u}, v_j), \quad \forall v \in U. \] (3.42)

Take \( v = (\pi_j\tilde{u}, v_j) \) and we obtain from (3.42) that
\[ J_j(\tilde{y}; \pi_j\tilde{u}, \tilde{u}_j) \leq J_j(y_j; \pi_j\tilde{u}, v_j), \quad \forall y_j \in \mathcal{R}(\pi_j\tilde{u}, v_j). \] (3.43)

Repeating the process \( N \) times we get the theorem. \( \square \)

### 4. Applications

In this section, we give some applications of Theorem 3.1 to parabolic initial boundary value problems. For simplicity, we take \( N = 2 \).

Let \( G \) be a bounded open subset of \( \mathbb{R}^2 \) with a smooth boundary and let
\[ Q = G \times (0, 2), \quad \Gamma = G \times \{ 2 \}, \]
\[ Q(u_1) = \{ (\xi, \eta) : \xi \in G, \quad 0 < \eta < u_1(\xi) \}, \] (4.1)

where \( u_1 \) is a continuous function of \( G \) into \([1, 2]\). The top of the cylinders \( Q(u_1) \), \( Q \) are
\[ \Gamma(u_1) = \{ (\xi, u_1(\xi)) : \xi \in G \}, \quad \Gamma. \] (4.2)
Make the change of variable $\zeta = 2\eta/u_1$ and set

$$y(\xi, \eta) = y\left(\frac{\xi}{2}, \frac{u_1\zeta}{u_1}\right) = Y(\xi, \zeta).$$

(4.3)

As done in great details in [4, pages 946–948], we get

$$\nabla^2 y = \nabla_{\xi,\zeta} F(\xi, \zeta; u_1) \nabla_{\xi,\zeta} Y(\xi, \zeta) + u_1^{-1} F \nabla Y \cdot \nabla u_1,$$

(4.4)

where $F(\xi, \zeta; u_1)$ is the matrix

$$\begin{pmatrix}
1 & 0 & -\zeta(\partial_\xi u_1)u_1^{-1} \\
0 & 1 & -\zeta(\partial_\zeta u_1)u_1^{-1} \\
-\zeta(\partial_\xi u_1)u_1^{-1} & -\zeta(\partial_\zeta u_1)u_1^{-1} & \zeta^2|\nabla u_1|^2 u_1^{-2} + 4u_1^{-2}
\end{pmatrix}.$$  

(4.5)

Set

$$\mu(u_1) = 2u_1^{-1}\sqrt{1 + |\nabla u_1|^2}.$$  

(4.6)

4.1. An inverse problem for a nonlinear heat equation. Consider the initial boundary value problem

$$y' - \Delta y = \tilde{f}(y) \quad \text{on } Q(u_1) \times (0, T),$$

$$y = 0 \quad \text{on } \partial Q(u_1) / T \times (0, T),$$

$$\frac{\partial y}{\partial n} \in u_2 \beta(y) \quad \text{on } \Gamma(u_1) \times (0, T),$$

$$y(\cdot, 0) = y_0 \quad \text{on } Q(u_1),$$

(4.7)

where $\beta \in \partial j(r)$ and $j(r)$ is an l.s.c. convex function from $\mathbb{R}^+ \to [0, \infty]$. Let

$$J_1(y; u_1, u_2) = \int_0^T \int_G |y(\xi, u_1(\xi))|^2 d\xi dt,$$

$$J_2(y; u_1, u_2) = \int_0^T \int_\Omega |y - h(\xi, \eta)|^2 d\xi d\eta dt$$

be the cost functionals associated with (4.7) and let $h$ be the measurement of the solution $y$ of (4.7) in the sub-region $\Omega$.

We denote

$$U_j = \{ u_j : \|u_j\|_{H^1(G)} \leq C, \ 1 \leq u_1(\xi) \leq 2, \ 0 \leq u_2(\xi) \leq C \}$$

(4.9)

and let $\mathcal{U}_j = L^2(G)$. It is clear that the $U_j$ are compact convex subsets of the space of controls $\mathcal{U}_j$.

We will take

$$H = L^2(Q), \quad \mathcal{H}_1 = L^2(G), \quad \mathcal{H}_2 = L^2(\Omega), \quad \Omega \subset Q.$$  

(4.10)

The main result of this subsection is the following theorem.
An inverse problem for evolution inclusions

Theorem 4.1. Let \( y_0 \) be in \( H^1_0(Q) \) and let \( \tilde{f} \) be a continuous function of \( y, u \) with

\[
|\tilde{f}(y; u)| \leq C\{1 + |y| + |u|\}. \tag{4.11}
\]

Let \( h \) be a given function in \( L^2(0; T; L^2(\Omega)) \) where \( \Omega \) is a proper subset of \( Q \) and let \( j(r) \) be an l.s.c. convex function on \( \mathbb{R} \) with values in \( [0, +\infty] \). Then there exists

\[
\{\hat{y}, \hat{y}', \hat{u}\} \in L^2(0; T; H^1(Q(\hat{u}_1))) \cap L^\infty(0; T; L^2(Q(\hat{u}_1)))
\times L^2(0; T; L^2(Q(\hat{u}_1))) \times U \tag{4.12}
\]

such that \( \hat{y} \) is a solution of the initial boundary value problem (4.7) in \( Q(\hat{u}_1) \times (0, T) \); and

\[
J_1(\hat{y}; \hat{u}_1, \hat{u}_2) \leq J_1(y; \hat{u}_1, v_2), \quad \forall v_2 \in U_2,
\]

\[
J_2(\hat{y}; \hat{u}_1, \hat{u}_2) \leq J_2(x; v_1, \hat{u}_2), \quad \forall v_1 \in U_1,
\tag{4.13}
\]

where \( x, y \) are the solutions of (4.7) with controls \( \{v_1, \hat{u}_2\}, \{\hat{u}_1, v_2\} \in Q(v_1) \times (0, T) \) and in \( Q(\hat{u}_1) \times (0, T) \), respectively.

Problems of type (4.7) arise in the study of heat transfer between solids and gases under nonlinear boundary conditions.

As carried out in [4], we make the change of variable \( \zeta = 2u_1^{-1}\eta \) and set \( y(\xi, \eta) = Y(\xi, \zeta) \). Then (4.7) is transformed into the following problem:

\[
\begin{align*}
Y' - \nabla (F(u_1) \cdot \nabla Y) + u_1^{-1}F \nabla Y \cdot \nabla u_1 &= \tilde{f}(Y, u) & \text{on } Q \times (0, T), \\
Y &= 0 & \text{on } \partial Q \times (0, T), \\
-\frac{\partial Y}{\partial n} &\in \mu(u_1) u_2 \beta(Y) & \text{on } \Gamma \times (0, T), \\
Y(\cdot, 0) &= y_0 & \text{on } Q
\end{align*}
\tag{4.14}
\]

with cost functionals

\[
J_1(Y; u_1, u_2) = \int_0^T \int_G |Y(\xi, 2)|^2 d\xi dt,
\tag{4.15}
\]

\[
J_2(Y; u_1, u_2) = \int_0^T \int_{\Omega} \left| Y \left( \xi, \frac{2\eta}{u_1} \right) - h(\xi, \eta; t) \right|^2 d\xi d\eta dt,
\tag{4.16}
\]

where \( \mu \) is as in expression (4.6).

Our aim is to find the controls \( u_1, u_2 \) so that the solution \( y \) of (4.7), if it is unique, is as close to the measurement \( h \) in \( \Omega \) as possible.

Let \( \varphi \) be the mapping of \( H \times U_1 \times U_2 \) into \( \mathbb{R}^+ \) given by

\[
\varphi(Y; u_1, u_2) = \begin{cases} 
\frac{1}{2} \|F(u)\nabla Y\|_{L^2(Q)}^2 + \int_{\Gamma} \mu(u_1) u_2 j(Y) d\sigma, & j(Y) \in L^1(\Gamma), \\
+\infty, & \text{otherwise},
\end{cases}
\tag{4.17}
\]

where \( j(r) \) is an l.s.c. convex function from \( \mathbb{R} \) to \( [0, +\infty] \) with \( j(0) = 0 \).
By abuse of notation, we will write $y$ for $Y(\xi, \zeta, t)$ when there is no confusion possible.

**Lemma 4.2.** Let $\varphi$ be as in (4.17). Then $\varphi$ satisfies Assumption 2.1.

**Proof.**
(1) It is clear that $\varphi(y; u)$ is an l.s.c. function from $H \times U$ into $\mathbb{R}^+$ and that $C_0^\infty(Q) \subset D(\varphi(\cdot, u))$ for all $u \in U$.

(2) It was shown in [4, pages 949–952] that

$$\int_Q F(u) |\nabla y|^2 d\xi d\zeta \geq c \|y\|^2_{H^1(Q)}, \quad \text{for all } y \in D(\varphi). \tag{4.18}$$

Since $j(r)$ and $\mu$ are both positive functions, we get

$$c \|y\|^2_{H^1(Q)} \leq \varphi(y; u), \quad \forall y \in D(\varphi). \tag{4.19}$$

(3) By the Sobolev imbedding theorem, the set

$$\{ y : \varphi(y; u) \leq C \} \tag{4.20}$$

is a compact subset of $H = L^2(Q)$.

(4) Suppose that $u^n_i \to u$ in $H$ with $u^n_i \in U_1$. Since $u^n_i$ is in $U_1$, it follows from the definition of $U_1$ and from the Sobolev imbedding theorem that there exists a subsequence such that $u^n_i \to u^i \in H^2(G)$ and in $C^1(\overline{G})$.

With $F(u)$, $\mu(u)$ as above, it is trivial to check that we have

$$\lim_{n \to \infty} \int_0^T \varphi(y(s); u^n_i) ds = \int_0^T \lim_{n \to \infty} \varphi(y(s); u^n_i) ds. \tag{4.21}$$

**Lemma 4.3.** Let $\varphi$ be as in (4.16). Then $\partial \varphi(y; u) = -\nabla \cdot (F(u)\nabla y) = A(y; u)$ with

$$D(A(y; u)) = \left\{ y : \nabla \cdot (F(u)\nabla y) \in H, y = 0 \text{ on } \partial Q/\Gamma, \right.$$  \ 

$$\left. -\frac{\partial y}{\partial n} \in \mu(u_1)u_2 \beta(y) \text{ on } \Gamma \right\}. \tag{4.22}$$

**Proof.** For $y \in H^1(Q)$ with $\nabla \cdot F(u)\nabla y$ in $L^2(Q)$, we know that $F(u)\nabla y \cdot n \in H^{-1/2, 2}(\partial Q)$.

Let $A(y; u) = -\nabla \cdot F(u)\nabla y$ with

$$D(A(y; u)) = \left\{ y : y \in H, \nabla \cdot (F(u)\nabla y) \in H, y = 0 \text{ on } \partial Q/\Gamma, \right.$$  \ 

$$\left. -\frac{\partial y}{\partial n} y \in \mu(u_1)u_2 y \text{ on } \Gamma \right\}. \tag{4.23}$$

We now show that $A$ is maximal monotone on $H$ and that $A \subset \partial \varphi(y; u)$. 
Nonexpansive for all \( \lambda > 0 \).

Thus, we have used the nonexpansive property of \((\cdot,\cdot)\) is the pairing between \( H^{-1/2,2}(\Gamma) \) and its dual.

It follows that

\[-(\nabla \cdot F(u) \nabla y, x - y) \leq \varphi(x;u) - \varphi(y;u).\]  

Hence \( A(y;u) \subseteq \partial \varphi(y;u) \).

To show that \( A(y;u) \) is maximal monotone, it suffices to show that \( I + A(y;u) \) is onto.

Since \( \beta(y) \in \partial j(y) \) is maximal monotone, its resolvent operator \((I + \lambda \beta)^{-1} \) is nonexpansive for all \( \lambda > 0 \).

Consider the elliptic boundary value problem

\[-\nabla \cdot (F(u) \nabla y_\lambda) = f \quad \text{on } Q, \quad y_\lambda = 0 \quad \text{on } \partial Q/\Gamma, \quad \mu(u_1) u_2 \lambda y_\lambda + \lambda \frac{\partial}{\partial n} y_\lambda = \mu(u_1) u_2 (I + \lambda \beta)^{-1} x \quad \text{on } \Gamma.\]  

(4.26)

For \((f,x) \in L^2(Q) \times L^2(\Gamma)\), there exists a unique solution \( y_\lambda \) of \((4.17)\) with \( y_\lambda \in H^1(Q) \). Let \( L \) be the mapping of \( L^2(\Gamma) \) into itself given by

\[ L\left(\sqrt{\mu(u_1) u_2} x\right) = \sqrt{\mu(u_1) u_2} y_\lambda|_\Gamma.\]  

(4.27)

We now show that \( L \) is a contraction. Let \( L \) be as above, then

\[ \int_Q F(u) \left| \nabla \left( y_\lambda^1 - y_\lambda^2 \right) \right|^2 - \left< \frac{\partial}{\partial n} \left( y_\lambda^1 - y_\lambda^2 \right), y_\lambda^1 - y_\lambda^2 \right> = 0. \]  

(4.28)

As shown in \([4, \text{pages 949 and 952}]\) we have

\[ c \left\| y_\lambda^1 - y_\lambda^2 \right\|_{H^1(Q)}^2 - \left< \frac{\partial}{\partial n} \left( y_\lambda^1 - y_\lambda^2 \right), y_\lambda^1 - y_\lambda^2 \right> \leq 0. \]  

(4.29)

Thus,

\[ c \left\| y_\lambda^1 - y_\lambda^2 \right\|_{H^1(Q)}^2 + \lambda^{-1} \left\| \sqrt{\mu(u_1) u_2} (y_\lambda^1 - y_\lambda^2) \right\|_{L^2(\Gamma)}^2 \leq \lambda^{-1} (\mu(u_1) u_2 \{ (I + \lambda \beta)^{-1} x^1 - (I + \lambda \beta)^{-1} x^2 \}, y_\lambda^1 - y_\lambda^2) \]  

(4.30)

\[ \leq \left\| \sqrt{\mu(u_1) u_2} \lambda^{-1} (y_\lambda^1 - y_\lambda^2) \right\|_{L^2(\Gamma)} \left\| \sqrt{\mu(u_1) u_2} (x^1 - x^2) \right\|_{L^2(\Gamma)}. \]

We have used the nonexpansive property of \((I + \lambda \beta)^{-1} \) in the above estimate. We know that

\[ a \left\| y_\lambda^1 - y_\lambda^2 \right\|_{L^2(\Gamma)}^2 \leq \left\| y_\lambda^1 - y_\lambda^2 \right\|_{H^1(Q)}^2, \]  

(4.31)

where \( a \) is a positive constant.
Thus,  
\[
\lambda \text{ac} \left\| y_1^1 - y_2^2 \right\|_{L^2(\Gamma)}^2 + \left\| \sqrt{\mu(u_1)u_2(y_1^1 - y_2^2)} \right\|_{L^2(\Gamma)} \leq \left\| \sqrt{\mu(u_1)u_2(y_1^1 - y_2^2)} \right\|_{L^2(\Gamma)}.
\]
(4.32)

It follows that  
\[
\left\| \sqrt{\mu(u_1)u_2(y_1^1 - y_2^2)} \right\|_{L^2(\Gamma)} \leq y \left\| \sqrt{\mu(u_1)u_2(x_1^1 - x_2^2)} \right\|_{L^2(\Gamma)}
\]
(4.33)

with  
\[
y = \frac{\|\mu(u_1)u_2\|_{L^\infty(G)}}{\lambda \text{ac} + \|\mu(u_1)u_2\|_{L^\infty(G)}} < 1.
\]
(4.34)

Thus, \( L \) is a contraction mapping. There exists a unique \( y_\lambda \) such that  
\[
- \nabla \cdot (F(u_1) \nabla y_\lambda) = f \quad \text{on } Q, \\
y_\lambda = 0 \quad \text{on } \partial Q/\Gamma,
\]
(4.35)

(4) By a standard argument, we get from (4.35) the following estimate:  
\[
\| y_\lambda \|_{H^1(Q)}^2 \leq C \| f \|_{L^2(Q)}.
\]
(4.36)

Let \( \lambda \to 0^+ \), and we get by taking subsequences that \( y_\lambda \to y \) in \( (H^1(Q))_{\text{weak}} \cap L^2(Q) \). It is clear that \( y = 0 \) on \( \partial Q/\Gamma \). On the other hand,  
\[
- \frac{\partial y_\lambda}{\partial n} = \mu(u_1)u_2(\beta_\lambda(y_\lambda) - 1) y_\lambda = \mu(u_1)u_2(\beta_\lambda(y_\lambda)),
\]
(4.37)

where \( \beta_\lambda \) is the Yosida approximation of \( \beta \).

Since  
\[
\beta_\lambda(y_\lambda) \in \beta((I + \lambda \beta)^{-1} y_\lambda), \quad (I + \lambda \beta)^{-1} y_\lambda \to y \quad \text{in } L^2(\Gamma),
\]
(4.38)

it follows from the maximal monotonicity of \( \beta \) that  
\[
- \frac{\partial}{\partial n} y \in \mu(u_1)u_2 \beta(y).
\]
(4.39)

The lemma is proved. \( \square \)

Proof of Theorem 4.1. Consider the optimal control problem  
\[
Y' - \nabla \cdot (F(u) \nabla Y) + g(Y; u) = 0 \quad \text{on } Q \times (0, T), \\
Y = 0 \quad \text{on } (\partial Q/\Gamma) \times (0, T), \\
- \frac{\partial}{\partial n} Y \in \mu(u_1)u_2 \beta(Y) \quad \text{on } \Gamma \times (0, T), \\
Y(\cdot, 0) = y_0 \quad \text{on } Q
\]
(4.40)
48 An inverse problem for evolution inclusions

with
\[ g(Y;u) = -u_1^{-1}F(u_1) \nabla Y \cdot \nabla u_1 - \tilde{f}(Y,u) \]  \hspace{1cm} (4.41)

and cost functionals
\[
J_1(Y;u_1,u_2) = \int_0^T \int_G \left| Y(\xi,2; t) \right|^2 d\xi dt,
\]
\[
J_2(Y;u_1,u_2) = \int_0^T \int_\Omega \left| Y(\xi,2; t) - h(\xi,\eta,t) \right|^2 d\xi d\eta dt.
\]  \hspace{1cm} (4.42)

It is easy to check that \( g \) and \( J_1, J_2 \) satisfy Assumptions 2.2 and 2.3, respectively. It follows from Lemmas 4.2 and 4.3 and from Theorem 3.1 that there exists an open loop control \( \tilde{u} \) of (4.36) and (4.40), that is, we have
\[
\tilde{Y} \in L^2(0,T; H^1(Q)) \cap L^\infty(0,T; L^2(Q)),
\]
\[
\{ \tilde{Y}', A(\tilde{Y}; \tilde{u}) \} \in (L^2(0,T; L^2(Q)))^2,
\]  \hspace{1cm} (4.43)
solution of (4.36) with controls \( \tilde{u} \). Moreover,
\[
J_1(\tilde{Y}; \tilde{u}_1, \tilde{u}_2) \leq J_1(y; u_1, v_2),
\]
\[
J_2(\tilde{Y}; \tilde{u}_1, \tilde{u}_2) \leq J_2(x; u_1, \tilde{u}_2),
\]  \hspace{1cm} (4.44)
for all \( y \in \mathcal{R}(\tilde{u}_1, v_2) \), for all \( v_2 \in U_2 \), all \( x \in \mathcal{R}(u_1, \tilde{u}_2) \), and all \( u_1 \in U_1 \).

Now set
\[
\hat{\psi}(\xi,\eta) = \tilde{Y}(\xi,\zeta) = \tilde{Y}\left(\xi, \frac{2\eta}{u_1}\right)
\]  \hspace{1cm} (4.45)
and we get the stated result. \( \square \)

4.2. Parabolic variational inequalities. Consider the initial boundary value problem
\[
y' - \Delta y = \tilde{f}(y) \quad \text{on } Q(u_1) \times (0,T),
\]
\[
y = 0 \quad \text{on } (\partial Q/T) \times (0,T),
\]
\[
y(\cdot, t) \geq u_2(\xi) \quad \text{on } \Gamma \times (0,T),
\]
\[
y(\cdot, 0) = y_0 \quad \text{on } Q
\]  \hspace{1cm} (4.46)
with cost functionals
\[
J_1(y;u_1,u_2) = \int_0^T \int_G \left| y(\xi,u_1(\xi);t) \right|^2 d\xi dt,
\]
\[
J_2(y;u_1,u_2) = \int_0^T \int_\Omega \left| y(\xi,\eta; t) - h(\xi,\eta) \right|^2 d\xi d\eta dt,
\]  \hspace{1cm} (4.47)
where $h$ is the partial measurement of the solution $y$ of (4.46) in the subdomain $\Omega \times (0,T)$, $U_1$ is as before and

$$U_2 = \{ v : \|v\|_{H^1(G)} \leq C, \ 0 \leq v \text{ on } G \}. \quad (4.48)$$

The main result of this subsection is the following theorem.

**Theorem 4.4.** Let $y_0$ be an element of $H^1(Q)$ with

$$y_0 = 0 \quad \text{on } \partial Q, \quad y_0 \geq v \geq 0 \quad \text{on } \Gamma, \quad \forall v \in U_2. \quad (4.49)$$

Let $h \in L^2(0,T;L^2(\Omega))$ where $\Omega$ is a proper subset of $Q(u_1)$ for all $u_1 \in U_1$ and let $\tilde{f}$ be as in Assumption 2.2. Then there exists

$$\{ \hat{y}, \hat{y}', \hat{u} \} \in L^2(0,T;H^1(Q(\hat{u}_1))) \cap L^\infty(0,T;L^2(Q(\hat{u}_1))) \\
\times L^2(0,T;L^2(Q(\hat{u}_1))) \times U \quad (4.50)$$

with

$$J_1(\hat{y}; \hat{u}_1, \hat{u}_2) \leq J_1(y; \hat{u}_1; v_2),$$
$$J_2(\hat{y}; \hat{u}_1, \hat{u}_2) \leq J_2(x; u_1, \hat{u}_2), \quad (4.51)$$

for all solutions $y$ of (4.46) with controls $\hat{u}_1$, $v_2$ all solutions $x$ of (4.42) with controls $u_1$, $\hat{u}_2$ and all $\{ u_1, v_2 \} \in U_1 \times U_2$.

As before, we make the change of variables $\zeta = 2\eta/u_1$ and as in Section 4.1, we transform (4.42) into a problem in a fixed domain

$$Y'' - \nabla \cdot F\left(\left(\frac{u_1}{u_1} \right) \nabla Y\right) = \tilde{f}(Y,u) + u^{-1} F(u_1) \nabla Y \cdot \nabla u_1 \quad \text{on } Q \times (0,T),$$
$$Y = 0 \quad \text{on } \partial Q \cap \Omega \times (0,T),$$
$$Y \geq u_2 \quad \text{a.e. on } \Gamma \times (0,T),$$
$$Y(\cdot,0) = y_0 \quad \text{on } Q. \quad (4.52)$$

The cost functionals become

$$J_1(Y; u_1, u_2) = \int_0^T \int_Q \left| Y(\xi,2; t) \right|^2 \, d\xi \, dt,$$
$$J_2(Y; u_1, u_2) = \int_0^T \int_\Omega \left| Y\left(\xi, \frac{2\eta}{u_1}; t\right) - h(\xi, \eta; t) \right|^2 \, d\xi \, d\eta \, dt. \quad (4.53)$$

Set

$$K(u_2) = \{ y : y \in L^2(0,T;L^2(Q)), \ y \geq u_2 \ \text{a.e. on } \Gamma \times (0,T) \}. \quad (4.54)$$
Then \( K(u_2) \) is a closed convex subset of \( L^2(0, T; H) \). Let

\[
\varphi(y; u) = \frac{1}{2} \int_0^T \int_Q F(u) |\nabla y|^2 \, d\xi \, d\zeta \, dt + I_{K(u_2)}(y),
\]

where \( I_{K(u_2)} \) is the indicator function of the closed convex set \( K(u_2) \) of \( L^2(0, T; H) \) and

\[
D(\varphi(y; u)) = \left\{ y : y \in L^2(0, T; H^1(Q)), \; y = 0 \text{ on } (\partial Q / \Gamma) \times (0, T), \; y \geq u_2 \text{ on } \Gamma \times (0, T) \right\}.
\]

**Lemma 4.5.** Let \( \varphi \) be as in (4.53). Then \( \varphi \) satisfies Assumption 2.1.

**Proof.** As in the proof of Lemma 4.2, we have

\[
\varphi(y; u) \geq c \| y \|^2_{H^1(Q)}, \quad \forall y \in D(\varphi(\cdot, u)).
\]

It is clear that

\[
\partial \varphi(y; u) = \nabla (F(u) \cdot \nabla y) + \partial I_{K(u_2)}(y).
\]

All the other conditions of Assumption 2.1 can be verified without any difficulty. \( \square \)

**Lemma 4.6.** Suppose all the hypotheses of Theorem 4.4 are satisfied. Then there exists a solution \( \tilde{Y} \) of

\[
\tilde{Y}' + \partial \varphi(\tilde{Y}; \tilde{u}) \ni \tilde{f}(\tilde{Y}, \tilde{u}) + \tilde{u}_1^{-1} F(\tilde{u}_1) \nabla \tilde{Y} \cdot \nabla \tilde{u}_1, \quad \tilde{Y}(\cdot, 0) = y_0, \quad \tilde{Y}(\cdot, 0) = y_0, \quad \tilde{Y}(\cdot, 0) = y_0,
\]

\[
\{ \tilde{Y}, \tilde{Y}', \partial \varphi(\tilde{Y}; \tilde{u}), \tilde{u} \} \in (L^2(0, T; H^1(Q))) \cap L^\infty(0, T; L^2(Q)))
\]

\[
\times (L^2(0, T; L^2(Q)))^2 \times U.
\]

Moreover,

\[
J_1(\tilde{Y}; \tilde{u}_1, \tilde{u}_2) \leq J_1(y; \tilde{u}_1, v_2), \quad J_2(\tilde{Y}; \tilde{u}_1, \tilde{u}_2) \leq J_2(x; u_1, \tilde{u}_2),
\]

for all solutions \( y, x \) of (4.55) with controls \( \{ \tilde{u}_1, v_2 \}, \{ u_1, \tilde{u}_2 \} \), respectively, and for all \( \{ u_1, v_2 \} \) in \( U_1 \times U_2 \).

**Proof.** The proof is an immediate consequence of Theorem 3.1 and Lemma 4.5. \( \square \)

**Proof of Theorem 4.4.** Let \( \{ \tilde{Y}, \tilde{u} \} \) be as in Lemma 4.6 and set \( \dot{y}(\xi, \eta; t) = \tilde{Y}(\xi, 2\eta / \tilde{u}_1) \). Then \( \dot{y}, \tilde{u} \) is a solution of (4.52) and (4.53). The theorem is proved. \( \square \)
References


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