Solodki˘ı (1998) applied the modified projection scheme of Pereverzev (1995) for obtaining error estimates for a class of regularization methods for solving ill-posed operator equations. But, no a posteriori procedure for choosing the regularization parameter is discussed. In this paper, we consider Arcangeli’s type discrepancy principles for such a general class of regularization methods with modified projection scheme.

1. Introduction

Regularization methods are often employed for obtaining stable approximate solutions for ill-posed operator equations of the form

\[ Tx = y, \tag{1.1} \]

where \( T : X \rightarrow X \) is a compact linear operator on a Hilbert space \( X \). It is well known that if \( R(T) \) is infinite dimensional, then the problem of solving the above equation is ill-posed, in the sense that the generalized solution \( \hat{x} = T^* y \) does not depend continuously on the data \( y \). Here, \( T^* \) is the generalized Moore-Penrose inverse of \( T \) defined on the dense subspace \( D(T^*) = R(T) + R(T)^\perp \) of \( X \), and \( R(T) \) denotes the range of the operator \( T \). A typical example of such an ill-posed equation is the Fredholm integral equation of the first kind

\[ \int_a^b k(s,t)x(t) dt = y(s), \quad a \leq s \leq b, \tag{1.2} \]

with \( X = L^2[a, b] \), and \( k(\cdot, \cdot) \) a nondegenerate kernel belonging to \( L^2([a, b] \times [a, b]) \).
In a regularization method, corresponding to an inexact data \( \tilde{y} \), one looks for a stable approximation \( \tilde{x} \) of \( \hat{x} \) such that \( \|\hat{x} - \tilde{x}\| \) is "small" whenever the data error \( \|y - \tilde{y}\| \) is "small." A well-studied class of regularization methods for such a purpose is characterized by a class of Borel functions \( g_\alpha, \alpha > 0 \), defined on an interval \((0, b]\) where \( b \geq \|T\|_2^2 \). Corresponding to such functions \( g_\alpha \), the regularized solutions are defined by

\[
\begin{align*}
  x_\alpha &:= g_\alpha(T^*T)T^*y, \\
  \tilde{x}_\alpha &:= g_\alpha(T^*T)T^*\tilde{y}.
\end{align*}
\] (1.3)

(Cf. [1].) In order to perform error analysis, we impose certain conditions on the functions \( g_\alpha, \alpha > 0 \). Two primary assumptions are the following.

**Assumption 1.** There exists \( \nu_0 > 0 \) such that for every \( \nu \in (0, \nu_0] \), there exists \( c_\nu > 0 \) such that

\[
\sup_{0 \leq \lambda \leq b} \lambda^{1/2} |1 - \lambda g_\nu(\lambda)| \leq c_\nu \alpha^{\nu} \quad \forall \alpha > 0.
\] (1.4)

**Assumption 2.** There exists \( d > 0 \) such that

\[
\sup_{0 \leq \lambda \leq b} \lambda^{1/2} |g_\nu(\lambda)| \leq d \alpha^{-1/2} \quad \forall \alpha > 0.
\] (1.5)

These assumptions are general enough to include many regularization methods such as the ones given below.

For applying our discrepancy principle, we would like to impose two additional conditions.

**Assumption 3.** There exist \( \alpha_0 > 0 \) and \( \kappa_0 > 0 \) such that

\[
|1 - \lambda g_\nu(\lambda)| \geq \kappa_0 \alpha^{\nu_0} \quad \forall \lambda \in [0, b], \forall \alpha \leq \alpha_0.
\] (1.6)

**Assumption 4.** The function \( f(\alpha) = \alpha q(1 - \lambda g_\nu(\lambda)), q > 0 \), as a function of \( \alpha \), is continuous and differentiable and \( f(\alpha) \) is an increasing function.

Now we list a few regularization methods which are special cases of the above procedure.

**Tikhonov regularization**

\[
(T^*T + \alpha I)x_\alpha = T^*y.
\] (1.7)

Here

\[
g_\nu(\lambda) = \frac{1}{\lambda + \alpha}.
\] (1.8)

Assumptions 1, 2, 3, and 4 hold with \( \nu_0 = 1 \), and \( \kappa_0 \) in Assumption 3 can be taken as greater than or equal to \( 1/(\alpha_0 + \|T\|_2^2) \).
Generalized Tikhonov regularization

\[(T^*T)^{q} + \alpha^q I\]x\[\alpha\] = \(T^*y\).

Here

\[g_\alpha(\lambda) = \lambda^q + \alpha^q + 1\].

Assumptions 1, 2, 3, and 4 hold with \(\nu_0 = q + 1\), \(q \geq -\frac{1}{2}\), and \(\xi_0\) in Assumption 3 can be taken greater than or equal to \(1/(\alpha^q + 1)\).

Iterated Tikhonov regularization. In this method, the \(k\)th iterated approximation \(x^{(k)}\) is calculated from

\[(T^*T + \alpha I)x^{(k)} = \alpha x^{(i-1)} + T^*y, \quad i = 1, \ldots, k\].

with \(x^{(0)} = 0\). Here, with

\[g_\alpha(\lambda) = \frac{1}{1 - \left(\frac{\alpha}{\alpha + \lambda}\right)^q}\].

Assumptions 1, 2, 3, and 4 hold with \(\nu_0 = k\) and the constant \(\xi_0\) in Assumption 3 can be taken as any number greater than or equal to \(1/(\alpha^q + 1)\).

In order to obtain numerical approximations of \(\tilde{x}_\alpha = g_\alpha(T^*T)^{q} \tilde{y}\), one may have to replace \(T\) by an approximation of it, say by \(T_n\), where \((T_n)\) is a sequence of finite rank bounded operators which converges to \(T\) in some sense, and consider

\[\tilde{x}_{\alpha,n} = g_\alpha(T_n^*T_n)^{q} \tilde{y}\].

In [4], Periverzev considered Tikhonov regularization with

\[T_n = P_1T P_2n + \sum_{i=1}^{k}(P_2n - P_{2n-1})T P_{2n-i}\].

with \(R(P_{2n+1}) \subseteq R(P_{2n+1})\) and showed that the computational complexity for obtaining the solution

\[\tilde{x}_{\alpha,n} = (T_n^*T_n + \alpha I)^{-1}T_n^* \tilde{y}\]

is far less than that for ordinary projection method when \(T\) and \(T^*\) are having certain smoothness properties and \((P_1)\) is having certain approximation properties.
Arcangeli's type discrepancy principles

Recently, Solodki˘ı [6] applied the above modified projection approximation to the general regularization method, and obtained error estimate for the approximation

\[ \tilde{x}_{\alpha,n} = g_\alpha (T_n^* T_n) T_n^* \tilde{y} \]  \hspace{1cm} (1.16)

under an a priori choice of the regularization parameter \( \alpha \).

In this paper we not only consider the above class of regularization methods defined by \( \tilde{x}_{\alpha,n} = g_\alpha (T_n^* T_n) T_n^* \tilde{y} \) with \( T_n \) as in (1.14), but also consider a modified form of the generalized Arcangeli's discrepancy principle

\[ \| T_n \tilde{x}_{\alpha,n} - \tilde{f} \| = \left( \delta + a_n \right)^p, \quad p > 0, q > 0, \]  \hspace{1cm} (1.17)

for choosing the regularization parameter \( \alpha \). Here \( (a_n) \) is a sequence of positive real numbers such that \( a_n \to 0 \) as \( n \to \infty \). It is to be mentioned that, in [3], the authors considered the above discrepancy principle for Tikhonov regularization with \( T_n \) as in (1.14). The advantage of having a general sequence \( (a_n) \) instead of the traditional \( (\epsilon_n) \), where \( \| T_n - T \| = O(\epsilon_n) \), is that the order of convergence of the approximation is in terms of powers of \( \delta + a_n \), in place of powers of \( \delta + \epsilon_n \) with \( \epsilon_n \) smaller than \( \delta \). By properly choosing \( (a_n) \), it can happen that, for a small \( \delta \), the values of \( a_n \) for which \( a_n = O(\delta) \), can be much smaller than that required for \( \epsilon_n = O(\delta) \). In this paper we are going to use the estimate \( \| T_n - T \| = O(\epsilon_n) \), \( \epsilon_n = 2^{-nr} \), proved in [3], where \( r > 0 \) is a quantity depending on the smoothness property of \( T \), and take \( (a_n) \) such that \( 2^{-nr} = O(a_\lambda^n) \) for some \( \lambda > 0 \). For instance one may take \( a_n = 2^{-nr/\lambda} \) for any \( \lambda \in (0, 1] \).

In order to specify the smoothness properties of the operator \( T \) and approximate property of \( (P_n) \), we adopt the following setting as in [3, 4].

For \( r > 0 \), let \( X_r \) be a dense subspace of the Hilbert space \( X \) and \( L_r : X_r \to X \) a closed linear operator. On \( X_r \) consider the inner product

\[ \langle f, g \rangle_r := \langle f, g \rangle + \langle L_r f, L_r g \rangle, \quad f, g \in X_r, \]  \hspace{1cm} (1.18)

and the corresponding norm

\[ \| f \|_r := \| f \| + \| L_r f \|, \quad f \in X_r. \]  \hspace{1cm} (1.19)

It can be seen that, with respect to the above inner product \( \langle \cdot, \cdot \rangle_r \), \( X_r \) is a Hilbert space.

If \( A : X \to X, B : X_\sigma \to X, C : X \to X_r \) are bounded operators, then we will denote their norms by

\[ \| A \|, \quad \| B \|_{\sigma}, \quad \| C \|_{\lambda}. \]  \hspace{1cm} (1.20)

respectively.
We assume that $T : X \to X$ is a compact operator having the smoothness properties

$$
R(T) \subseteq X_r, \quad R(T^*) \subseteq X_r, \quad R((L_r T)^*) \subseteq X_r.
$$

(1.21)

with

$$
T : X \to X_r, \quad T^* : X \to X_r, \quad (L_r T)^* : X \to X_r
$$

(1.22)

being bounded operators, so that there exist positive real numbers $\gamma_1, \gamma_2, \gamma_3$ such that

$$
\|T\|_{0,r} \leq \gamma_1, \quad \|T^*\|_{0,r} \leq \gamma_2, \quad \|(L_r T)^*\|_{0,r} \leq \gamma_3.
$$

(1.23)

Further, we assume that $(P_n)$ is a sequence of orthogonal projections having the approximation property

$$
\|I - P_n\|_{0,0} \leq \epsilon_n R^{-*}.
$$

(1.24)

where $\epsilon_n > 0$ is independent of $n$.

2. Error estimate and discrepancy principle

2.1. Error estimate. Let $T : X \to X$ be a compact operator having the smoothness properties specified by (1.21) and (1.23) and $(P_n)$ a sequence of orthogonal projections having the approximation property (1.24). For each $n \in \mathbb{N}$, let $\hat{T}_n$ be defined by (1.14).

Let $y \in R(T)$ and $\tilde{y} \in X$ be such that

$$
\|y - \tilde{y}\| \leq \delta.
$$

(2.1)

Let $\{g_\alpha : \alpha > 0\}$ be a set of Borel measurable functions defined on $(0, b)$, where

$$
b \geq \max \left\{ \|T^{1/2}\|, \|T_0^{1/2}\| \right\} \forall n \in \mathbb{N},
$$

(2.2)

and satisfying Assumptions 1, 2, 3, and 4. Let

$$
\tilde{x} := T^{1/2}y, \quad x_\alpha := g_\alpha \left( (T^* T)^{1/2} \right) \tilde{x}, \quad x_{n, \alpha} := g_\alpha \left( (T_n^* T_n)^{1/2} \right) \tilde{x}.
$$

(2.3)

Further we assume that $\tilde{x} \in R((T^* T)^{1/2})$ for some $\nu \in (0, b)$ and

$$
\tilde{x} = (T^* T)^{1/2} u, \quad u \in X.
$$

(2.4)

In order to find an estimate for the error $\|\tilde{x} - x_{n, \alpha}\|$, we first observe that

$$
\|\tilde{x} - x_{n, \alpha}\| \leq \|\tilde{x} - x_{n, \alpha}\| + \|x_{n, \alpha} - x_{n, \alpha}\|.
$$

(2.5)
By the definition of \( x_{\alpha,n} \), \( \tilde{x}_{\alpha,n} \), and using spectral results, we have

\[
x_{\alpha,n} - \tilde{x}_{\alpha,n} = \xi_{\alpha}(T_{n}^{*}T_{n})(\bar{y} - \check{y}) = T_{n}^{*}\xi_{\alpha}(T_{n}^{*})(\bar{y} - \check{y}).
\]  

(2.6)

Therefore, using Assumption 2 on \( \xi_{\alpha} \), we get

\[
\|x_{\alpha,n} - \tilde{x}_{\alpha,n}\| = \|T_{n}^{*}\xi_{\alpha}(T_{n}^{*})(\bar{y} - \check{y})\| \\
\leq \sup_{0 \leq \lambda \leq b} \lambda^{1/2} \|\xi_{\alpha}(\lambda)\| \|y - \check{y}\| \leq d \delta \sqrt{\alpha}.
\]  

(2.7)

Thus, we have

\[
\|\hat{x} - \tilde{x}_{\alpha,n}\| \leq \|\hat{x} - x_{\alpha,n}\| + d \delta \sqrt{\alpha}.
\]  

(2.8)

The following theorem supplies an estimate for \( \|\hat{x} - x_{\alpha,n}\| \). For its proof we will make use of the result

\[
\|A_{\ell} - A_{\ell,n}\| \leq a_{\ell} \|A - A_{n}\| \min\{1, \ell\} \quad \ell > 0,
\]  

(2.9)

proven in [7] for positive, selfadjoint, bounded operators \( A \) and \( A_{n} \) on \( X \), with \((A_{n})\) uniformly bounded, where \( a_{\ell} > 0 \) is independent of \( n \).

**Proposition 2.1.** Let \( \hat{x} \) and \( x_{\alpha,n} \) be as in (2.3). Then

\[
\|\hat{x} - x_{\alpha,n}\| \leq c_{\alpha} \|T^{*}T - T_{n}^{*}T_{n}\| + a_{\alpha} \|T_{n} - P_{2}T(T^{*}T)^{1/2}d\| \sqrt{\alpha}.
\]  

(2.10)

**Proof.** We observe that

\[
\check{x} - x_{\alpha,n} = \check{x} - \xi_{\alpha}(T_{n}^{*}T_{n})\check{x} = [I - \xi_{\alpha}(T_{n}^{*}T_{n})]T_{n}^{*}\check{x} + \xi_{\alpha}(T_{n}^{*}T_{n})T_{n}^{*}(I - T_{n})\check{x}.
\]  

(2.11)

so that

\[
\|\hat{x} - x_{\alpha,n}\| \leq \|I - \xi_{\alpha}(T_{n}^{*}T_{n})\| \|T_{n}^{*}\check{x}\| + \|\xi_{\alpha}(T_{n}^{*}T_{n})T_{n}^{*}(I - T_{n})\| \|\check{x}\|.
\]  

(2.12)

Since \( \check{x} = (T^{*}T)\check{x} \),

\[
\|I - \xi_{\alpha}(T_{n}^{*}T_{n})\| \|T_{n}^{*}\check{x}\| = \|I - T_{n}^{*}\xi_{\alpha}(T_{n}^{*}T_{n})\| \|T^{*}T\| \|\check{x}\| \\
\leq \|I - T_{n}^{*}\xi_{\alpha}(T_{n}^{*}T_{n})\| \|T^{*}T\| \|\check{x}\| \\
\leq \|I - T_{n}^{*}\xi_{\alpha}(T_{n}^{*}T_{n})\| \|T^{*}T\| \|\check{x}\|.
\]  

(2.13)
Now, using Assumption 1 on $g_\alpha$, 
\[
\|I - T^*_n P_m (T^*_n T_n)\| \leq \sup_{0 < c < 1} c |1 - \lambda g_\alpha(\lambda)| \leq c_1 |\lambda|^{\alpha}, \tag{2.14}
\]
and by Assumption 1 on $g_\alpha$ and the result (2.9) with $A = T^*_n T_n$ and $\ell = v$, 
\[
\|\alpha (T^*_n T_n)\| (T^*_n T_n)^{\frac{1}{2}} \leq c_2 \|\alpha (T^*_n T_n)\| (T^*_n T_n)^{\frac{1}{2}} \|v\|
\leq c_2 \|\alpha (T^*_n T_n)\| (T^*_n T_n)^{\frac{1}{2}} \lambda \nu, \tag{2.15}
\]
Since $T^*_n P_m = T^*_n$, $\alpha = (T^*_n T_n)^{\frac{1}{2}}$ and using Assumption 2 on $g_\alpha$, we have 
\[
\|\alpha (T^*_n T_n)\| (T^*_n T_n)^{\frac{1}{2}} \leq c_3 \|\alpha (T^*_n T_n)\| (T^*_n T_n)^{\frac{1}{2}} \nu, \tag{2.16}
\]
Using the above estimates for $\|I - g_\alpha (T^*_n T_n)\| (I - T^*_n T_n)^{\frac{1}{2}}$ and $\|g_\alpha (T^*_n T_n)\| (T^*_n T_n)^{\frac{1}{2}}$ in relation (2.12) we get the required result. $\square$

In view of relation (2.8) and Proposition 2.1, we have to find estimates for the quantities
\[
\|T^* T - T^*_n T_n\|, \quad \|(T^*_n T_n) (T^*_n T_n)\|^{\frac{1}{2}}. \tag{2.17}
\]
It is proved in [4] (also see [6]) that 
\[
\|T^* T - T^*_n T_n\| = O(\lambda^{2\nu}) \tag{2.18}
\]
so that 
\[
\|T^* T - T^*_n T_n\|^{\frac{1}{2}} = O(\lambda^{2\nu \nu_1}), \quad \nu_1 = \min\{\nu, 1\}. \tag{2.19}
\]
Also, the estimate for $\|(T^*_n T_n) (T^*_n T_n)\|^\frac{1}{2}$ given in the following lemma can be deduced from a result of Solodkii [6]. Here we will give an independent and detailed proof for the same. We will use the estimates
\[
\|I - P_m\| = O(a^{-\nu}), \quad \|I - P_m\|_{0, \Omega} = O(a^{-\nu}) \tag{2.20}
\]
obtained by Pereverzev [4] (cf. also [3]) and the estimate
\[
\|I - P_m\| (T^*_n T_n) = O\left(\|I - P_m\|^{\frac{1}{2}} (T^*_n T_n)^{\frac{1}{2}} \right), \quad \ell > 0, \tag{2.21}
\]
given in [5].
Lemma 2.2. For \( \nu > 0 \),
\[
\| (T_n - P_2 T)(I - P_2) T (T^* T)^\nu \| = O(2^{-\nu(2+\nu)}) . \tag{2.22}
\]

Proof. It can be seen that
\[
P_2 T = P_1 T (I - P_2) + \sum_{k=1}^n (P_2 - P_2 k) T (I - P_2 k - 1) . \tag{2.23}
\]
Therefore,
\[
\| (P_2 T - T_n)(I - P_2) T (T^* T)^\nu \| \leq \| T (I - P_2) \| \| (I - P_2) (I - P_2 - 1) (T^* T)^\nu \| + \sum_{k=1}^n \| (I - P_2 k - 1) (I - P_2 k) (T^* T)^\nu \| . \tag{2.24}
\]
Now using (1.24), (2.20), and (2.21), it follows that
\[
\| (P_2 T - T_n)(I - P_2) T (T^* T)^\nu \| \leq \| T (I - P_2) \| \| (I - P_2) (I - P_2 - 1) (T^* T)^\nu \| + \sum_{k=1}^n \| (I - P_2 k - 1) (I - P_2 k) (T^* T)^\nu \| \leq \| T (I - P_2) \| \| (I - P_2) (I - P_2 - 1) (T^* T)^\nu \| + \sum_{k=1}^n \| (I - P_2 k - 1) (I - P_2 k) (T^* T)^\nu \|. \tag{2.25}
\]
Thus the lemma is proved.
\[\square\]

Theorem 2.3. Suppose that \( \hat{\epsilon} \in R(T^* T)^\nu \) and \( y \in R(T) \). Then
\[
\| \hat{\epsilon} - \xi_{\alpha, n} \| \leq \left( \epsilon^2 + 2^{-\nu(2+\nu)} \right)^{1/4} + \frac{\nu^2}{\sqrt{\nu^2}} + \frac{4}{\sqrt{\nu^2}} . \tag{2.26}
\]
where
\[ \nu_1 = \min\{\nu, 1\}, \quad \nu_2 = \min\{2\nu, 1\}. \] (2.27)

### 2.2. Discrepancy principle
We consider the discrepancy principle
\[ \| T_n \tilde{x}_{\alpha,n} - \tilde{y} \| = (\delta + a_n)^p, \quad p > 0, \quad q > 0, \] (2.28)
where \( (a_n) \) is a sequence of positive reals such that \( a_n \to 0 \) as \( n \to 0 \).

Let
\[ f_\alpha(u, \tilde{y}) = u^q \| T_n \tilde{x}_{\alpha,n} - \tilde{y} \|. \] (2.29)
We observe that
\[ T_n \tilde{x}_{\alpha,n} - \tilde{y} = \left[ T_n T^* \tilde{g}_\alpha \left( T_n T^* \right) - I \right] \tilde{y}. \] (2.30)
Hence, by Assumptions 1 and 3 on \( \tilde{g}_\alpha, \alpha > 0 \), and using spectral theory, we have
\[
\begin{align*}
\| T_n \tilde{x}_{\alpha,n} - \tilde{y} \| &= \left\| \left[ T_n T^* \tilde{g}_\alpha \left( T_n T^* \right) - I \right] \tilde{y} \right\| \\
&\leq \sup_{0 < \lambda < \lambda_0} \| 1 - \lambda \tilde{g}_\alpha(\lambda) \| \| \tilde{y} \| \\
&\leq c_0.
\end{align*}
\]
Therefore, it follows that
\[ \lim_{\alpha \to 0} f_\alpha(u, \tilde{y}) = 0, \quad \lim_{\alpha \to 0} f_\alpha(u, \tilde{y}) = \infty. \] (2.32)
Hence by the intermediate value theorem and Assumption 4 on \( \{g_n\} \), there exists a unique \( \alpha \) satisfying the discrepancy principle (2.28). It also follows that
\[
\frac{(\delta + a_n)^p}{\alpha^q} = \| T_n \tilde{x}_{\alpha,n} - \tilde{y} \| \geq c_0 \alpha^{\nu_0} \| \tilde{y} \|. \] (2.33)
so that
\[ \alpha = O\left( \frac{(\delta + a_n)^p}{\nu_0^{(q+\nu_0)}} \right). \] (2.34)
For the next result we make use of the estimate
\[ \| T - T_n \| = O(2^{-\nu}). \] (2.35)
proved in [3].
From the discrepancy principle (2.28) we have

\[ \frac{(\hat{\lambda} + \alpha)^q}{\alpha^q} = O((\hat{\lambda} + \alpha)^q), \] (2.36)

where

\[ \hat{\lambda} = \min \left\{ 1, \lambda, \frac{p}{q + \nu_0} \frac{p}{2(q + \nu_0) + 2\lambda \nu_2} \right\}, \]

\[ \nu_2 = \min \{\nu, 1\}, \quad \omega = \min \left\{ \frac{p}{q + \nu_0} \nu_0 \right\}. \] (2.37)

**Proof.** From the discrepancy principle (2.28) we have

\[ \frac{(\hat{\lambda} + \alpha)^q}{\alpha^q} = \left\| T_n g_\alpha - \hat{\gamma} \right\| \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \]

\[ = \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \right\| \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \right\| \left\| \left( (I - \sigma_\alpha(T_n T_n^*)T_n^*) \right) \hat{\gamma} \right\| \]

\[ \leq \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \right\| \left\| \left( (I - \sigma_\alpha(T_n T_n^*)T_n^*) \right) \hat{\gamma} \right\| \left\| \left( (I - \sigma_\alpha(T_n T_n^*)T_n^*) \right) \right\| \left\| \left( (I - \sigma_\alpha(T_n T_n^*)T_n^*) \right) \hat{\gamma} \right\| \].

(2.38)

We observe that

\[ \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \right\| \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \right\| \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \]

\[ = \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \]

\[ \leq \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \].

(2.39)

Now, using the fact that \( \hat{\gamma} = (T^* T)^\dagger \hat{\gamma} \), **Assumption 1** on \( \sigma_\alpha, \alpha > 0 \), and spectral results, we have

\[ \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \]

\[ \leq \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \].

(2.40)

where \( \hat{\lambda}_n = c_{\nu + 1/2} \) if \( \nu + 1/2 \leq \nu_0 \) and \( \hat{\lambda}_n = c_{\nu_0} \) if \( \nu + 1/2 \geq \nu_0 \), and \( \omega = \min \{\nu + 1/2, \nu_0\} \). Hence

\[ \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \]

\[ \leq \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \left\| (I - \sigma_\alpha(T_n T_n^*)T_n^*) \hat{\gamma} \right\| \].

(2.41)
Also, we have
\[
\left\| (I - g\alpha (T_nT^*)^n)T_nT^* (\tilde{y} - y) \right\| \leq c_0\delta. \tag{2.42}
\]
Thus
\[
\frac{(\delta + an)^p}{\alpha^q} \leq c_0 \left( (T - T_n)\delta + c_1\alpha^{1/2} \left\| \tilde{u} \right\| \right) \times \left\| (T^*T)^n - (T_nT^*)^n \right\| + c_0\delta. \tag{2.43}
\]
Now by the results (2.9), (2.34), (2.35), and the assumption that \(2^{-nr} = O(a\lambda n)\), we have
\[
\frac{(\delta + an)^p}{\alpha^q} \leq c_0 \left( (a\lambda n + \alpha \omega + \alpha^{1/2} a\lambda n + \delta) \right) \leq c_0 \left( (a\lambda n + \alpha \omega + \alpha^{1/2} a\lambda n + \delta) \right) \leq c_0 \left( (a\lambda n + \alpha \omega + \alpha^{1/2} a\lambda n + \delta) \right) \leq c_0 \left( (a\lambda n + \alpha \omega + \alpha^{1/2} a\lambda n + \delta) \right) \leq c_0 \left( (a\lambda n + \alpha \omega + \alpha^{1/2} a\lambda n + \delta) \right) \leq c_0 \left( (a\lambda n + \alpha \omega + \alpha^{1/2} a\lambda n + \delta) \right) \leq c_0 \left( (a\lambda n + \alpha \omega + \alpha^{1/2} a\lambda n + \delta) \right) \leq c_0 \left( (a\lambda n + \alpha \omega + \alpha^{1/2} a\lambda n + \delta) \right) \leq c_0 \left( (a\lambda n + \alpha \omega + \alpha^{1/2} a\lambda n + \delta) \right), \tag{2.44}
\]
where \(v_2 = \min\{v, 1\}, \omega = \min\{v + 1/2, v_0\}\). Thus
\[
\frac{(\delta + an)^p}{\alpha^q} = O((\delta + an)^p). \tag{2.45}
\]

**Theorem 2.5.** In addition to the assumptions in Proposition 2.4, suppose that
\[
p < s + 2q \min\{1, \lambda (2 + v_2)\}, \tag{2.46}
\]
where
\[
s = \min\left\{ 1, \lambda, \frac{p}{2(q + v_0)} + 2v_0, \frac{p}{q + v_0} + 2v_0 \right\}. \tag{2.47}
\]
Then
\[
\mu := \min\left\{ \frac{pv}{q + v_0}, 1 - \frac{p}{2q}, s, \lambda (2 + v_2) - \frac{p}{2q}, \frac{r}{2q} \right\} = 0, \tag{2.48}
\]
\[
\left\| \tilde{\tilde{z}} - \tilde{z}_{\omega,a} \right\| = O((\delta + an)^p). \tag{2.48}
\]
Arcangeli’s type discrepancy principles

Proof. Clearly, \( p \leq s + 2q \min \{ 1, \lambda(2 + \nu) \} \) implies \( \mu > 0 \). Now to obtain the estimate for \( \| \hat{x} - \hat{x}_{2n} \| \), first we recall from Theorem 2.3 that
\[
\| \hat{x} - \hat{x}_{2n} \| \leq c \left( \alpha \nu + \frac{\delta + \omega}{\sqrt{a}} \right) \left( 1 + \frac{\delta + \omega}{\sqrt{a}} \right) \left( 1 + \frac{\delta + \omega}{\sqrt{a}} \right)^{\frac{1}{2}} \left( 1 + \frac{\delta + \omega}{\sqrt{a}} \right) \left( 1 + \frac{\delta + \omega}{\sqrt{a}} \right) \left( 1 + \frac{\delta + \omega}{\sqrt{a}} \right) \left( 1 + \frac{\delta + \omega}{\sqrt{a}} \right). \tag{2.49}
\]

Now, using the assumption that \( 2^{-nr} = O(\lambda n) \) for some \( \lambda > 0 \), and relation (2.34), we have
\[
\| \hat{x} - \hat{x}_{2n} \| \leq c \left( \alpha \nu + \frac{\delta + \omega}{\sqrt{a}} \right) \left( 1 + \frac{\delta + \omega}{\sqrt{a}} \right) \left( 1 + \frac{\delta + \omega}{\sqrt{a}} \right) \left( 1 + \frac{\delta + \omega}{\sqrt{a}} \right) \left( 1 + \frac{\delta + \omega}{\sqrt{a}} \right) \left( 1 + \frac{\delta + \omega}{\sqrt{a}} \right) \left( 1 + \frac{\delta + \omega}{\sqrt{a}} \right) \left( 1 + \frac{\delta + \omega}{\sqrt{a}} \right). \tag{2.50}
\]
Since
\[
\frac{\delta + \omega}{\sqrt{a}} = O(\lambda(n + \nu)) \tag{2.51}
\]
for any \( \ell > 0 \), by Proposition 2.4,
\[
\frac{\delta + \omega}{\sqrt{a}} = O(\lambda(n + \nu)), \tag{2.52}
\]
Thus
\[
\| \hat{x} - \hat{x}_{2n} \| = O(\lambda n). \tag{2.53}
\]

The following corollary whose proof is immediate from the above theorem, specifies a condition required to be satisfied by \( \lambda \), and there by the sequence \( \{a_n\} \), so as to yield a somewhat realistic error estimate.

Corollary 2.6. In addition to the assumption in Theorem 2.5, suppose \( \lambda, p, q \) are such that
\[
\frac{p}{q} + \nu = \text{max} \left\{ \frac{1}{\lambda^2}, \frac{1}{\lambda} \right\} \leq 1. \tag{2.54}
\]
Then \( s \) and \( \mu \) in Theorem 2.5 are given by
\[
s = \frac{p
u}{q + \nu}, \quad \mu = \min \left\{ \frac{p\nu}{q + \nu}, 1 - \frac{p}{2(q + h)} \left( 1 + \frac{\nu - \omega}{\omega} \right) \right\}. \tag{2.55}
\]
In particular, with $\lambda$ as above, we have the following:

- \[
  \mu = \frac{p\nu}{q + \nu} \quad \text{whenever} \quad \frac{p}{q + \nu} \leq \frac{2}{2v + 1 + (\nu_0 - \omega)/q}, \tag{2.56}
\]
- \[
  \mu = \frac{2v}{2v + 1} \quad \text{whenever} \quad \frac{p}{q + \nu} = \frac{2}{2v + 1}, \quad \nu_0 - \frac{1}{2} \leq \nu \leq \nu_0, \tag{2.57}
\]
- \[
  \mu = \frac{2v}{2v + 1} \quad \text{whenever} \quad \frac{p}{q + \nu} = \frac{2}{2v + 1}, \quad q \geq \frac{1}{2}, \tag{2.58}
\]

We may observe that the result in (2.58) of Corollary 2.6 shows that the choice of $p, q$ in the discrepancy principle (2.28) does not depend on the smoothness of the unknown solution $\hat{x}$. Also, from the above corollary we can infer that for the Arcangeli’s discrepancy principle

\[
  \|T_n\hat{x}_{a,n} - \hat{x}\| = \delta + a_n \sqrt{\alpha}, \tag{2.59}
\]

one obtains the error estimate

\[
  \|\hat{x} - \hat{x}_{a,n}\| = O((\delta + a_n)\mu), \quad \mu = \frac{2v}{2v + 1}, \tag{2.60}
\]

provided $(a_n)$ satisfies

\[
  2^{1-\omega} = O(a_n^\nu), \quad \max\left\{\frac{2v}{2v + 1}, \frac{1}{2}\right\} \leq \lambda \leq 1. \tag{2.61}
\]

In particular, for Tikhonov regularization, where $\nu_0 = 1$, we have the order

\[
  O((\delta + a_n)^{\nu/3}) \quad \text{whenever} \quad \frac{2}{3} \leq \lambda \leq 1.
\]

### 3. Numerical example

In this section, we carry out some numerical experiments using JAVA programming for Tikhonov regularization, and implement our discrepancy principle. We also implement the a priori parameter choice strategy numerically.

Consider the Hilbert space $X = Y = L^2[0, 1]$ with the Haar orthonormal basis $\{e_1, e_2, \ldots\}$, of piecewise constant functions, where $e_1(t) = 1$ for all $t \in [0, 1)$, and for $m = 2^{k-1} + j$, $k = 1, 2, \ldots, j = 1, 2, \ldots, 2^{k-1} - 1$,

\[
  e_{\nu,n}(t) = \begin{cases} 
    2^{(k-1)/2} & \text{if } t \in \left[j - \frac{1}{2}, j - \frac{1}{2} + \frac{1}{2^{k-1}}\right] \\
    -2^{(k-1)/2} & \text{if } t \in \left[j - \frac{1}{2}, j + \frac{1}{2} - \frac{1}{2^{k-1}}\right] \\
    0 & \text{if } t \notin \left[j - \frac{1}{2}, j + \frac{1}{2}\right].
  \end{cases} \tag{3.1}
\]
Let \( T : X \to X \) be the integral operator,
\[
(Tx)(s) = \int_0^1 k(s, t)x(t)\,dt, \quad s \in [0, 1],
\]
with the kernel
\[
k(s, t) = \begin{cases} 
(1-s), & t \leq s, \\
(1-t), & t > s.
\end{cases}
\]
We take \( X_r \) with \( r = 1 \) as the Sobolev space of functions \( f \) with derivative \( f' \in L^2[0, 1] \). In all the following examples, we have \( \hat{x} \in R(T^*T)^\nu \) with \( 2\nu \leq 1 \). In this case the error estimate in Theorem 2.3 takes the form
\[
\|\hat{x} - \tilde{x}_{\alpha,n}\| \leq c(\alpha\nu + 2^n - 2^n\nu + 2^n - 2^n(1+\nu)\sqrt{\alpha} + \delta \sqrt{\alpha}).
\]
(3.4)

Taking the a priori choice of the parameter \( \alpha \) as
\[
\alpha \sim 2^{-2n}, \quad \alpha \sim \delta^2/(2\nu + 1),
\]
we get the optimal order
\[
\|\hat{x} - \tilde{x}_{\alpha,n}\| = O(\delta^2/(2\nu + 1)).
\]
(3.6)

In an a posteriori case, we find \( \alpha \) using Newton-Raphson method, namely
\[
a_{k+1} = a_k - \frac{\hat{g}(a_k)}{\hat{g}'(a_k)}, \quad k = 0, 1, \ldots
\]
(3.7)

where
\[
\hat{g}(a) = a^2(\tilde{x}^T M \tilde{x} - 2\tilde{x}^T CB + \langle \mathbf{j}, \mathbf{j} \rangle) - (\tilde{x} + a\mathbf{1})^2a,
\]
\[
\hat{g}'(a) = 2a^2 - 1(\tilde{x}^T M \tilde{x} - 2\tilde{x}^T CB + \langle \mathbf{j}, \mathbf{j} \rangle)
\]
\[
- a^2[\tilde{x}^T M (a + M)^{-1}\tilde{x} - \tilde{x}^T (a + M)^{-1}MC \tilde{x} - 2\tilde{x}^T (a + M)^{-1}CB],
\]
(3.8)

with
\[
\mathbf{j} = \{x_1, x_2, \ldots, x_m\},
\]
\[
\{B\}_i = \{e_i, \mathbf{j}\}, \quad i = 1, 2, \ldots, m,
\]
\[
\{M\}_j = \sum_{r=1}^{2m} (e_r, A\mathbf{r})\{e_j, A\mathbf{r}\}, \quad i, j = 1, 2, \ldots, 2^n
\]
(3.9)
\[
\{C\}_j = \{\phi, \phi\}, \quad \phi_i = P_{2n} T^* e_i, \quad \phi = P_{2n} T^* \mathbf{e},
\]
\[
i \in \{2^{\mathbb{L}-1}, 2^{\mathbb{L}}\}, \quad \mathbb{L} = 1, 2, \ldots, n.
Here we used the notation \([A]_{ij}\) for the \(ij\)th entry of an \(n \times n\) matrix \(A\) and \([B]_i\) for the \(i\)th entry of an \(n \times 1\) (column) matrix \(B\).

In the following examples, we take the perturbed data \(\tilde{y}\) as

\[
\tilde{y}(s) = y(s) + \delta, \quad 0 \leq s \leq 1.
\]  

\((3.10)\)

For the a posteriori case, we take \(p\) and \(q\) such that \(p/(q + 1) = 2/3\), and \(a_n = (2^{-n})^{1/2}\) with \(\lambda = 2/3\). As per Corollary 2.6, the rate is

\[
O((\delta + a_n)^{p/2} + 1),
\]

We will use the notation \(\tilde{e}_{\alpha,n}\) for the computed value of \(\|\hat{x} - \tilde{x}_{\alpha,n}\|\).

**Example 3.1.** Let \(y(s) = (1/6)(s - s^3)\). In this case, it can be seen that \(\hat{x}(t) = t\), \(t \in [0, 1]\). It is known (cf. [2]) that \(\hat{x} \in R(T^*T)^\nu\) for all \(\nu < 1/8\). In the following two cases we take \(\nu = 1/9\).

**A priori case**

<table>
<thead>
<tr>
<th>(\delta)</th>
<th>(n)</th>
<th>(m)</th>
<th>(\tilde{e}_{\alpha,n})</th>
<th>(A)</th>
<th>(B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2^{-1.2})</td>
<td>2</td>
<td>4</td>
<td>0.9059731</td>
<td>0.7371346</td>
<td>1.220947</td>
</tr>
<tr>
<td>(2^{-1.2})</td>
<td>3</td>
<td>8</td>
<td>0.7722685</td>
<td>0.6328782</td>
<td>1.220246</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>16</td>
<td>0.4068352</td>
<td>0.5433674</td>
<td>0.7487295</td>
</tr>
</tbody>
</table>

**A posteriori case**

<table>
<thead>
<tr>
<th>(p)</th>
<th>(q)</th>
<th>(\delta)</th>
<th>(n)</th>
<th>(m)</th>
<th>(\tilde{e}_{\alpha,n})</th>
<th>(A)</th>
<th>(B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p = 1)</td>
<td>(q = 1/2)</td>
<td>(2^{-1.2})</td>
<td>2</td>
<td>4</td>
<td>0.5102194</td>
<td>0.8945074</td>
<td>0.5709195</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>3</td>
<td>8</td>
<td>0.4906685</td>
<td>0.8196771</td>
<td>0.5966055</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>4</td>
<td>16</td>
<td>0.3504178</td>
<td>0.7517244</td>
<td>0.4616520</td>
</tr>
<tr>
<td>(p = 2)</td>
<td>(q = 2)</td>
<td>(2^{-1.2})</td>
<td>2</td>
<td>4</td>
<td>0.4009030</td>
<td>0.8945074</td>
<td>0.4821355</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>3</td>
<td>8</td>
<td>0.566437</td>
<td>0.8196771</td>
<td>0.447047</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>4</td>
<td>16</td>
<td>0.3294871</td>
<td>0.7517244</td>
<td>0.4580837</td>
</tr>
<tr>
<td>(p = 1)</td>
<td>(q = 1/2)</td>
<td>(10^{-3})</td>
<td>2</td>
<td>4</td>
<td>0.57549441</td>
<td>0.8414794</td>
<td>0.6839856</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>3</td>
<td>8</td>
<td>0.5400543</td>
<td>0.7719075</td>
<td>0.703708</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>4</td>
<td>16</td>
<td>0.2978588</td>
<td>0.7187710</td>
<td>0.4302669</td>
</tr>
<tr>
<td>(p = 2)</td>
<td>(q = 2)</td>
<td>(10^{-3})</td>
<td>2</td>
<td>4</td>
<td>0.46468603</td>
<td>0.7719075</td>
<td>0.6022228</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>3</td>
<td>8</td>
<td>0.28503888</td>
<td>0.7187710</td>
<td>0.3965642</td>
</tr>
</tbody>
</table>

**Example 3.2.** Let \(y(s) = (1/24)(s - 2s^3 - s^4)\). In this case, \(\tilde{x}(t) = (1/2)(t - s^3), t \in [0, 1]\) and \(\hat{x} \in R(T^*T)^\nu\) for all \(\nu < 1/8\) (cf. [2]).
Arcangeli’s type discrepancy principles

### A priori case

<table>
<thead>
<tr>
<th>Δ</th>
<th>n</th>
<th>m</th>
<th>ε(Δ)</th>
<th>Δ</th>
<th>ε(Δ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2^{-2n}/2</td>
<td>2</td>
<td>4</td>
<td>0.2362887</td>
<td>0.1787366</td>
<td>1.3386617</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>8</td>
<td>0.04444126</td>
<td>0.0838854</td>
<td>1.0641567</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>16</td>
<td>0.04338350</td>
<td>0.04419417</td>
<td>0.98063492</td>
</tr>
</tbody>
</table>

### A posteriori case

<table>
<thead>
<tr>
<th>p, q</th>
<th>ℓ</th>
<th>n</th>
<th>m</th>
<th>ε(ℓ)</th>
<th>Δ</th>
<th>ε(Δ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2^{-2n}/2</td>
<td>2</td>
<td>4</td>
<td>0.08553568</td>
<td>0.54195173</td>
<td>0.16125629</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>8</td>
<td>0.08927489</td>
<td>0.37696366</td>
<td>0.23682611</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>16</td>
<td>0.08501988</td>
<td>0.26366060</td>
<td>0.32861129</td>
<td></td>
</tr>
<tr>
<td>2^{-2n}/2</td>
<td>2</td>
<td>4</td>
<td>0.07940677</td>
<td>0.54195173</td>
<td>0.14650240</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>8</td>
<td>0.07740040</td>
<td>0.37696366</td>
<td>0.20532590</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>16</td>
<td>0.06635354</td>
<td>0.26366060</td>
<td>0.25936980</td>
<td></td>
</tr>
<tr>
<td>10^{-50}</td>
<td>2</td>
<td>4</td>
<td>0.09125593</td>
<td>0.50347777</td>
<td>0.18125116</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>8</td>
<td>0.08918976</td>
<td>0.55724853</td>
<td>0.25420012</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>16</td>
<td>0.06553270</td>
<td>0.55724853</td>
<td>0.34135620</td>
<td></td>
</tr>
<tr>
<td>10^{-50}</td>
<td>2</td>
<td>4</td>
<td>0.00045663</td>
<td>0.50347777</td>
<td>0.17566684</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>8</td>
<td>0.00787900</td>
<td>0.55724853</td>
<td>0.24013831</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>16</td>
<td>0.07340813</td>
<td>0.55724853</td>
<td>0.28957690</td>
<td></td>
</tr>
</tbody>
</table>

### References


M. T. Nair and M. P. Rajan


M. T. Nair: Department of Mathematics, Indian Institute of Technology Madras, Chennai 600 036, India
E-mail address: mtnair@iitm.ac.in

M. P. Rajan: Department of Mathematics, Indian Institute of Technology Madras, Chennai 600 036, India