We find a lower estimation for the projection constant of the projective tensor product $X \otimes \wedge Y$ and the injective tensor product $X \otimes \vee Y$, we apply this estimation on some previous results, and we also introduce a new concept of the projection constants of operators rather than that defined for Banach spaces.

1. Introduction

If $Y$ is a closed subspace of a Banach space $X$, then the relative projection constant of $Y$ in $X$ is defined by

$$\lambda(Y, X) := \inf \{ \|P\| : P \text{ is a linear projection from } X \text{ onto } Y \}.$$  \hspace{1cm} (1.1)

And the absolute projection constant of $Y$ is defined by

$$\lambda(Y) := \sup \{ \lambda(Y, X) : X \text{ contains } Y \text{ as a closed subspace} \}.$$  \hspace{1cm} (1.2)

It is well known that any Banach space $Y$ can be isometrically embedded into $l_\infty(\Gamma_1)$ for some index set $\Gamma_1$ ($\Gamma_1$ is usually taken to be $\mathcal{F}(Y^*)$, where $Y^*$ denotes the dual space of $Y$ and $\mathcal{F}(Y^*)$ denotes the set $\{f : f \in Y^*, \|f\| \leq 1\}$) and that if $Y$ is complemented in $l_\infty(\Gamma_1)$, then it is complemented in every Banach space containing it as a closed subspace, that is, $Y$ is injective. We also know that for any such embedding the supremum in (1.2) is attained, that is, $\lambda(Y) = \lambda(\Gamma_1, l_\infty(\Gamma_1))$ (see [1, 4]). For each finite-dimensional space $Y_n$ with dim $Y_n = n$, Kadets and Snobar [6] proved that $\lambda(Y_n) \leq \sqrt{n}$. König [7] showed that for each prime number $n$ the space $l_\infty^n$ contains an $n$-dimensional subspace $Y_n$ with projection constant

$$\lambda(Y_n) = \sqrt{n} - \frac{1}{\sqrt{n}} - \frac{1}{n}.$$  \hspace{1cm} (1.3)
König and Lewis [9] verified the strict inequality $\lambda(Y_n) < \sqrt{n}$ in case $n \geq 2$. Lewis [14] showed that

$$\lambda(Y_n) \leq \sqrt{n} \left[ 1 - \frac{1}{n} + O\left(\frac{1}{n^{1/4}}\right)\right].$$

(1.4)

König and Tomczak-Jaegermann [11] also showed that there is a sequence $\{X_n\}_{n \in \mathbb{N}}$ of Banach spaces $X_n$ with $\dim X_n = n$ such that

$$\lim_{n \to \infty} \frac{\lambda(X_n)}{\sqrt{n}} = 1.$$ (1.5)

In fact, it is shown in [9] that for each Banach space $Y_n$ with dimension $n$,

$$\lambda(Y_n) \leq \sqrt{n} - \frac{c}{\sqrt{n}},$$

where $c > 0$ is a numerical constant and the $n$-dimensional spaces $X_n$ satisfy $\sqrt{n} - \frac{c}{\sqrt{n}} \leq \lambda(X_n)$. The improvement of these results was given in [12], where an upper estimate for $\lambda(Y_n)$ was found in the form

$$\lambda(Y_n) \leq \begin{cases} \sqrt{n} - \frac{1}{\sqrt{n}} + O\left(\frac{1}{n^{1/4}}\right), & \text{in the real field,} \\ \sqrt{n} - \frac{1}{2\sqrt{n}} + O\left(\frac{1}{n^{1/4}}\right), & \text{in the complex field.} \end{cases}$$

(1.6)

The precise values of $l_1^n$, $l_2^n$, and $l_p^n$, $1 < p < \infty$, $p \neq 2$, have been calculated by Grünbaum [4], Rutovitz [15], Gordon [3], and Garling and Gordon [2]. In the case of $1 < p < 2$, the improvement of these results was given by König, Schütt, and Tomczak-Jaegermann in [10], they showed that

$$\lim_{n \to \infty} \frac{\lambda(l_p^n)}{\sqrt{n}} = \begin{cases} \sqrt{n} - \frac{1}{\sqrt{n}} + O\left(\frac{1}{n^{1/4}}\right), & \text{in the real field,} \\ \sqrt{n} - \frac{1}{2\sqrt{n}} + O\left(\frac{1}{n^{1/4}}\right), & \text{in the complex field.} \end{cases}$$

(1.7)

Some other results are mentioned in [2, 3, 13, 15].

2. Notations and basic definitions

The sets $X$, $Y$, $Z$, and $E$ denote Banach spaces, $X^*$ denotes the conjugate space of $X$ and $U_X$ denotes the unit ball of the space $X$. Elements of $X$, $Y$, $X^*$, and $Y^*$ will be denoted by $x, u, y, v, f, h, \ldots$, and $g, k, \ldots$, respectively.
injective tensor product $X \otimes^\vee Y$ between the normed spaces $X$ and $Y$ is defined as the completion of the smallest cross norm on the space $X \otimes Y$ and the norm on the space $X \otimes Y$ is defined by

$$\left\| \sum_{i=1}^n x_i \otimes y_i \right\|_{X \otimes Y} = \sup \left\{ \sum_{i=1}^n |f(x_i)| |g(y_i)| \right\},$$

(2.1)

where the supremum is taken over all functionals $f \in U_X^*$ and $g \in U_Y^*$.

The projective tensor product $X \otimes^\wedge Y$ between the normed spaces $X$ and $Y$ is defined as the completion of the largest cross norm on the space $X \otimes Y$ and the norm on $X \otimes Y$ is defined by

$$\left\| \sum_{i=1}^n x_i \otimes y_i \right\|_{X \otimes^\wedge Y} = \inf \left\{ \sum_{j=1}^m \| u_j \| \| v_j \| \right\},$$

(2.2)

where the infimum is taken over all equivalent representations $\sum_{j=1}^m u_j \otimes v_j \in X \otimes Y$ of $\sum_{i=1}^n x_i \otimes y_i$ (see [5]).

If $X$ is a Banach space on which every linear bounded operator from $X$ into any Banach space $Y$ is nuclear (this is the case in all finite-dimensional Banach spaces $X$), then for any Banach space $Y$ the space $X \otimes^\vee Y$ is isomorphically isometric to $X \otimes^\wedge Y$ (see [16]).

The set $\Omega_1 = \{(f,g) : f \in U_X^*, g \in U_Y^*\} = U_X^* \times U_Y^*$. We start with the following two lemmas.

**Lemma 2.1.** For Banach spaces $X$ and $Y$ there is a norm one projection from $l_\infty(U_X^*) \otimes^\vee l_\infty(U_Y^*)$ onto $l_\infty(\Omega_1)$.

**Proof.** Since the space $l_\infty(\Omega_1)$ has the 1-extension property, it is sufficient to show that $l_\infty(\Omega_1)$ can be isometrically embedded in the space $l_\infty(U_X^*) \otimes^\vee l_\infty(U_Y^*)$. In fact, every nonzero element $0 \neq \Phi = [\Phi((f,g))]_{(f,g) \in U_X^* \times U_Y^*}$ in the space $l_\infty(\Omega_1)$, (note that the norm in this Banach space is given by $\| \Phi \|_{l_\infty(\Omega_1)} = \sup_{(f,g) \in U_X^* \times U_Y^*} \| \Phi((f,g)) \|$) defines two scalar-valued functions $F \in l_\infty(U_X^*)$ and $G \in l_\infty(U_Y^*)$ by the following formulas:

$$F(f) = \sup_{g \in U_Y^*} \| \Phi((f,g)) \|, \quad G(g) = \sup_{f \in U_X^*} \| \Phi((f,g)) \|. \quad (2.3)$$

Clearly the element $\tilde{\Phi} = (1/\| \Phi \|_{l_\infty(\Omega_1)}) \times (F \otimes G)$ is an element of the space $l_\infty(U_X^*) \otimes^\vee l_\infty(U_Y^*)$. Since both the injective and the projective tensor products are cross norms, $\| \tilde{\Phi} \|_{l_\infty(\Omega_1)} = \| \Phi \|_{l_\infty(\Omega_1)}$. The mapping $J$ defined by the formula $J(\tilde{\Phi}) = \tilde{\Phi}$ is the required isometric embedding. □

**Lemma 2.2.** Let $X$ and $Y$ be two Banach spaces. Then $\lambda(X \otimes^\wedge Y) = \lambda(X \otimes^\vee Y, l_\infty(\Omega_1))$. 
Proof. It is also sufficient to show that the space $X \otimes Y$ can be isometrically embedded in $l_\infty(\Omega_1)$. In fact, every element $F = \sum_{i=1}^{m} x_i \otimes y_i \in X \otimes Y$ defines a scalar-valued bounded function $\hat{F} \in l_\infty(\Omega_1)$ by the formula $\hat{F}(f, g) = \sum_{i=1}^{m} f(x_i)g(y_i)$. Using definition (2.1) for the injective tensor product, we have $\|\hat{F}\| = \|\hat{F}\|_{l_\infty(\Omega_1)}$. The mapping $i$ defined by the formula $i(F) = \hat{F}$ is the required isometric embedding.

We have the following theorem.

Theorem 2.3. (1) If $Y_1$ and $Y_2$ are complemented subspaces of Banach spaces $X_1$ and $X_2$, respectively, then the injective (resp., projective) tensor product $Y_1 \otimes Y_2$ (resp., $Y_1 \otimes X_2$) of the spaces $Y_1$ and $Y_2$ is complemented in the injective (resp., projective) tensor product $X_1 \otimes X_2$ (resp., $X_1 \otimes X_2$) of the spaces $X_1$ and $X_2$ and

$$\lambda\left(Y_1 \otimes (\vee \text{or } \wedge) Y_2, X_1 \otimes (\vee \text{or } \wedge) X_2\right) \leq \lambda(Y_1, X_1)\lambda(Y_2, X_2).$$

(2.4)

(2) If $X$ and $Y$ are injective spaces, then the space $X \otimes Y$ is injective. Moreover,

$$\lambda(X \otimes Y) \leq \lambda(X)\lambda(Y).$$

(2.5)

Proof. Let $P_1$ and $P_2$ be any projections from $X_1$ onto $Y_1$ and from $X_2$ onto $Y_2$, respectively. Then the operator $P$ from the space $X_1 \otimes X_2$ onto the space $Y_1 \otimes Y_2$ (resp., from the space $X_1 \otimes X_2$ onto the space $Y_1 \otimes Y_2$) defined by

$$P\left(\sum_{i=1}^{m} x_i \otimes y_i\right) = \sum_{i=1}^{m} P_1(x_i) \otimes P_2(y_i)$$

(2.6)

is a projection and its norm $\|P\|$ is not exceeding $\|P_1\|\|P_2\|$. In fact, let $\sum_{i=1}^{m} x_i \otimes y_i$ be any element of the space $X_1 \otimes (\vee \text{or } \wedge) X_2$. Then, in the case of projective tensor product we have

$$\left\|P\left(\sum_{i=1}^{m} x_i \otimes y_i\right)\right\|_{Y_1 \otimes Y_2} = \left\|\sum_{i=1}^{m} P_1(x_i) \otimes P_2(y_i)\right\|_{Y_1 \otimes Y_2} = \sum_{i=1}^{m} \|P_1(x_i)\|\|P_2(y_i)\| \leq \|P\|\|P_1\|\|P_2\|\sum_{i=1}^{m} \|x_i\|\|y_i\|.$$
for all equivalent representations $\sum_{i=1}^{n} u_i \otimes v_i$ of $\sum_{i=1}^{n} x_i \otimes y_i$. So

$$
\left\| P \left( \sum_{i=1}^{n} u_i \otimes v_i \right) \right\|_{T_1 \otimes T_2} \leq \left\| P_1 \right\| \left\| P_2 \right\| \left\| \sum_{i=1}^{n} u_i \otimes v_i \right\|_{X \otimes Y}.
$$

(2.8)

And in the case of injective tensor product we have

$$
\left\| P \left( \sum_{i=1}^{n} u_i \otimes v_i \right) \right\|_{T_1 \otimes T_2} = \sup \left\{ \left\| P \left( \sum_{i=1}^{n} u_i \otimes v_i \right) \right\|_{X \otimes Y} : f \in U_{T_1}, g \in U_{T_2} \right\} = \left\| P_1 \right\| \left\| P_2 \right\| \left\| \sum_{i=1}^{n} u_i \otimes v_i \right\|_{X \otimes Y}.
$$

(2.9)

Thus in both cases, $\left\| P \right\| \leq \left\| P_1 \right\| \left\| P_2 \right\|$. Taking the infimum of each side with respect to all such $P_1$ and $P_2$, we get inequality (2.4). To prove inequality (2.5), we apply inequality (2.4) and get in particular

$$
\lambda(X \otimes Y, l_{\infty}(U_{X'}) \otimes l_{\infty}(U_{Y'})) \geq \lambda(X, l_{\infty}(U_{X})) \lambda(Y, l_{\infty}(U_{Y})).
$$

(2.10)

Using Lemma 2.2 and definition (1.2), we get $\lambda(X \otimes Y, l_{\infty}(\Omega_1)) \geq \lambda(X \otimes Y, l_{\infty}(U_{X'}) \otimes l_{\infty}(U_{Y'}))$. We claim that the sign $\geq$ is an equal sign. In fact, if $P$ is any projection from $l_{\infty}(U_{X'}) \otimes l_{\infty}(U_{Y'})$ onto $X \otimes Y$ and $J$ is the embedding given in Lemma 2.1, then $P = PJ$ is a projection from $l_{\infty}(\Omega_1)$ onto $X \otimes Y$ with $\left\| P \right\| \leq \left\| P \right\|$. This is the sufficient condition for the two infimums.
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\[ \lambda(X \otimes^\oplus Y, I_{\lambda}) = \lambda(X \otimes^\oplus Y, I_{\lambda}(U_X \otimes^\oplus U_Y)) \]

Therefore

\[ \lambda(X \otimes^\oplus Y) = \lambda(X \otimes^\oplus Y, I_{\lambda}(U_X \otimes^\oplus U_Y)) \]

(2.11)

Using inequality (2.10), we get (2.5).

Remark 2.4. Since \( \lambda(l^\infty(\Gamma_1)) = 1 \) for any index set \( \Gamma_1 \), we conclude that \( \lambda(l^\infty(\Gamma_1) \otimes (\oplus \text{ or } \otimes) l^\infty(\Lambda_1), X \otimes (\oplus \text{ or } \otimes) Y) = 1 \) for every \( X \supset l^\infty(\Gamma_1) \) and \( Y \supset l^\infty(\Lambda_1) \).

We have the following two corollaries.

Corollary 2.5. For any finite sequence \( \{X_i\}_{i=1}^n \) of Banach spaces with complemented subspaces \( \{Y_i\}_{i=1}^n \), the relative projection constant of the injective (resp., projective) tensor product \( \bigotimes_{i=1}^n Y_i \) of the spaces \( Y_i \) in the space \( \bigotimes_{i=1}^n X_i \) satisfies

\[ \lambda \left( \bigotimes_{i=1}^n Y_i, \bigotimes_{i=1}^n X_i \right) \leq \prod_{i=1}^n \lambda(Y_i, X_i) \]  

(2.12)

Corollary 2.6. Let \( \{Y_i\}_{i=1}^n \) be a finite sequence of finite-dimensional Banach spaces. Then the relation between the absolute projection constant of the projective (or injective) tensor product \( \bigotimes_{i=1}^n Y_i \) and the direct sum \( \sum_{i=1}^n Y_i \) (with the supremum norm) is as follows:

\[ \lambda \left( \bigotimes_{i=1}^n Y_i \right) \leq 
\left( \lambda \left( \sum_{i=1}^n Y_i \right) \right)^n \]  

(2.13)

Proof. In fact, the proof is a combination of Corollary 2.5 and the results of [3, Theorem 4].

3. Applications

In this section, using Theorem 2.3, we obtain new results.

1. For finite-dimensional Banach spaces \( X \) and \( Y \) with dimensions \( n \) and \( m \), respectively, we have

\[ \lambda(X \otimes Y) \leq \sqrt{nm} - \frac{1}{\sqrt{nm}} + O(\text{min}^{-3/4}) \]

\[ - \left( \sqrt{m} - \frac{1}{\sqrt{m}} \right) \left( \sqrt{n} - O(\text{min}^{-3/4}) \right) \]

(3.1)

\[ + \left( \sqrt{n} - \frac{1}{\sqrt{n}} \right) \left( \frac{1}{\sqrt{nm}} - O(n^{-3/4}) \right) \].
in the real field and
\[ \lambda(X \otimes Y) \leq \sqrt{\frac{m}{2}} + O\left(\frac{m^{-3/4}}{\sqrt{m}}\right) \]
\[ - \left(\sqrt{m} - \frac{1}{2\sqrt{m}}\right) \left(\frac{1}{2\sqrt{m}} - O\left(\frac{m^{-3/4}}{\sqrt{m}}\right)\right) \]
\[ + \left(\sqrt{m} - \frac{1}{2\sqrt{m}}\right) \left(\frac{1}{2\sqrt{m}} - O\left(\frac{m^{-3/4}}{\sqrt{m}}\right)\right) \].

(3.2)
in the complex field. Compare this result with the result in (1.6).

(2) For any positive integer \( m \) (not necessarily prime) with a prime factorization \( m = \prod_{i=1}^{n} q_i \) where the numbers \( q_i \) are distinct prime numbers, the space \( \bigotimes_{i=1}^{n} l^p_{q_i} \) contains a subspace \( Y \) of dimension \( m \) with
\[ \lambda(Y) \leq \prod_{i=1}^{n} \left(\sqrt{q_i} - \frac{1}{2\sqrt{q_i}}\right) - C(m), \]
where \( C(m) \) is a positive number depending on \( m \) (in case of \( m = q_1 q_2 \), \( C(m) = (1/\sqrt{q_2} - 1/q_2) + (1/\sqrt{q_1} - 1/q_1) \)). Comparing this result with (1.3), we mention that the \( m^2 \)-dimension of the space \( \bigotimes_{i=1}^{n} l^p_{q_i} \) is not a square of a prime number, so it gives a new subspace \( Y \) with a new projection constant.

(3) For numbers \( p, q \) with \( 1 \leq p, q \leq 2 \), we have
\[ \lim_{n,m \to \infty} \frac{\lambda_{l}(ln \otimes lm)}{\sqrt{nm}} \leq \begin{cases} \frac{2}{n} & \text{in the real field}, \\ \frac{\pi}{4} & \text{in the complex field}. \end{cases} \]

(3.4)

4. The projection constants of operators

Now we start with our basic definitions of the projection constants of operators.

Definition 4.1. (1) A linear bounded operator \( A \) from a Banach space \( X \) into a Banach space \( Y \) is said to be left complemented with respect to a Banach space \( Z \) (\( Z \) contains \( Y \) as a closed subspace) if and only if there exists a linear bounded operator \( B \) from \( Z \) into \( X \) such that the composition \( AB \) is a projection from \( Z \) onto \( Y \). In this case \( Z \) is said to be a left complementation of \( A \).

If \( P_Z(A) \) denotes the convex set of all operators \( B \) from \( Z \) into \( X \) such that the composition \( AB \) is a projection, then
\[ \lambda(\mathcal{L}(A)) := \inf \left\{ \|AB\| : B \in P_Z(A) \right\}. \]
(4.1)
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(3) And the left absolute projection constant of \( A \) is defined as

\[
\lambda_l(A) := \sup \{ \lambda_l(A, Z) : Z \text{ is a left complementation of the operator } A \}.
\]

We define the same analogy from the right.

Remark 4.2. We notice the following.

(1) From the definition of \( \lambda_l(A, Z) \), the infimum in (4.1) is taken only with respect to the projections that are factored (through \( X \)) into two operators one of them is \( A \) and the other is an operator from \( Z \) into \( X \), so

\[
1 \leq \lambda(Y, Z) \leq \lambda_l(A, Z)
\]

for every left complementation \( Z \) of \( A \).

(2) If \( A \) is a projection from \( X \) onto \( Y \), then \( A \) is left complemented with respect to \( Y \). In fact \( AJ \) is a projection for any embedding \( J \) from \( Y \) into \( X \).

(3) If \( I_Y \) is the identity operator on \( Y \) and \( X \) contains \( Y \) as a complemented subspace, then \( I_Y P = P \) for every projection \( P \) from \( X \) onto \( Y \) and hence \( I_Y \) is left complemented with respect to \( X \). Moreover, \( \lambda_l(I_Y, X) = \lambda(Y, X) \), that is, the relative projection constant of the identity operator on the space \( Y \) with respect to the space \( X \) is the relative projection constant of the space \( Y \) in the space \( X \).

(4) If \( Z \) is a left complementation of the linear bounded operator \( A : X \to Y \), then \( Y \) is complemented in \( Z \) and the operator \( A \) is onto.

(5) If \( Z \) is a separable or reflexive Banach space and \( X \) is a Banach space, then for any index set \( \Gamma \) the space \( Z \) is not a right complementation of any linear bounded operator from \( l_\infty(\Gamma) \) into \( X \). In particular, if \( X \) is a Banach space, then for any index set \( \Gamma \), the space \( l_\infty(\Gamma) \) is not a left complementation of any linear bounded operator from \( X \) into the space \( c_0 \).

The following lemma is parallel to that lemma mentioned in [8] for Banach spaces and we omit the proof since the proof is nearly similar.

Lemma 4.3. Let \( \Gamma \) be an index set such that \( Y \) is isometrically embedded into \( l_\infty(\Gamma) \) and let \( A \) be a linear bounded operator from \( X \) onto \( Y \) such that \( l_\infty(\Gamma) \) is one of its left complementation. Then for a given \( B \in P_{l_\infty(\Gamma)}(A) \),

(1) For all Banach spaces \( E, Z, E \subseteq Z \) and every linear bounded operator \( T \) from \( E \) into \( Y \) there is an operator \( T' \) from \( Z \) into \( Y \) extending the operator \( T \) with \( \| T' \| \leq \| AB \| \| T \| \), that is, the space \( Y \) has \( \| AB \| \)-extension property, and in particular, if \( Z \supseteq X \), the operator \( A \) has a linear extension \( \hat{A} \) from \( Z \) into \( Y \) with \( \| \hat{A} \| \leq \| AB \| \| A \| \). That is, the extension constant \( c(A) \) of the operator \( A \) defined by \( c(A) := \sup_{E \subseteq Z } \inf \{ \| \hat{A} \| : \hat{A} \text{ is an extension of } A \text{ and } \hat{A} : Z \to Y \} \) satisfies \( c(A) \leq \| AB \| \| A \| \).

(2) For every Banach space \( Z \supseteq Y \), there exists a projection \( P \) from \( Z \) onto \( Y \) such that \( \| P \| \leq \| AB \| \).

The following theorem is also parallel to that given in (1.3) for Banach spaces.
Theorem 4.4. Let $Y$ be isometrically embedded in $l_\infty(\Gamma)$ and let $A$ be a linear bounded operator from $X$ onto $Y$ such that $l_\infty(\Gamma)$ is a left complementation of $A$. Then $A$ is left complemented with respect to any other Banach space $Z$ containing $Y$ as a closed subspace. Moreover,

$$\lambda_1(A, Z) \leq \lambda_1(A, l_\infty(\Gamma))$$

(4.3)

for every Banach space $Z$ containing $Y$ as a closed subspace, that is, $\lambda_1(A)$ attains its supremum at $l_\infty(\Gamma)$. Therefore,

$$\lambda_1(A) = \lambda_1(A, l_\infty(\Gamma)),$$

(4.4)

References


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