We give necessary and sufficient conditions for an operator on the space $C(T, X)$ to be $(r, p)$-absolutely summing. Also we prove that the injective tensor product of an integral operator and an $(r, p)$-absolutely summing operator is an $(r, p)$-absolutely summing operator.

For $X$ and $Y$ Banach spaces we denote by $L(X, Y)$ the Banach space of all linear and continuous operators from $X$ to $Y$ equipped with the operator norm, and by $X \otimes \varepsilon Y$ the injective tensor product of $X$ and $Y$, that is, the completion of the algebraic tensor product $X \otimes Y$ with respect to the injective cross-norm $\varepsilon(u) = \sup \{ \langle x^* \otimes y^*, u \rangle | \|x^*\| \leq 1, \|y^*\| \leq 1 \}, u \in X \otimes Y$. If $T$ is a compact Hausdorff space and $X$ is a Banach space, we denote by $C(T, X)$ the Banach space of all continuous $X$-valued functions defined on $T$, equipped with the supremum norm and by $C(T) = C(T, \mathbb{R})$ for $X = \mathbb{R}$ or $\mathbb{C}$. It is well known that $C(T, X) = C(T) \otimes X$. Also if $T$ is a compact space and $X$ is a Banach space, we denote by $\Sigma$ the $\sigma$-field of Borel subsets of $T$, $\Sigma(X)$ the space of $X$-valued $\Sigma$-simple functions on $T$, and by $B(\Sigma, X)$ we denote the uniform closure of the space $\Sigma(X)$. If $X = \mathbb{R}$ or $\mathbb{C}$, we also use that $B(\Sigma, X) \hookrightarrow C(T, X)^{**}$.

For the representing theorems of the linear and continuous operators on the space $C(T, X)$, see [1, 3]. Recall only that to each $U \in L(C(T, X), Y)$ correspond a representing measure $G : \Sigma \rightarrow L(X, Y^{**})$ and $G(E)x = U^{**}(\chi_E x)$. Also if $U \in L(X, Y)$, $V \in L(X_1, Y_1)$, by $U \otimes V : X \otimes Y \rightarrow X_1 \otimes Y_1$ we denote the injective tensor product of the operators $U$ and $V$. If $U \in L(X_1 \otimes Y, Z)$, for each $x \in X$ we consider the operator $U^x : Y \rightarrow Z$, $(U^x y)(y) = U(x \otimes y)$, $y \in Y$, and evidently $U^x : L(Y, Z)$ is linear and continuous. For $1 \leq r < \infty$ and $x_1, \ldots, x_n \in X$ we write, $l(x_i) = 1, n = (\sum_{j=1}^n |x_j|^r)^{1/r}$ and $w_i(x_i) = 1, n = \sup \{ \sum_{j=1}^n |x^*(x_j)|^{r/r'} | x^* \in X^*, \|x^*\| \leq 1 \}$. Let $1 \leq p \leq r < \infty$, $|x^*| \leq 1$.
summing operators we call absolutely summing and As
that is, $(r,p)$-absolutely summing norm of $U$ in $(X,Y)$
into $Y$ equipped with the $(r,p)$-absolutely summing norm.

The following theorem is an extension of [1, Proposition 2.2(ii)], [8, Theorem 3].

Theorem 1. If $U \in A_{(r,p)}(X \otimes Y, Z)$, then $U^*x \in A_{(r,p)}(Y, Z)$ for each $x \in X$ and $U^* : X \to A_{(r,p)}(Y, Z)$ is an $(r,p)$-absolutely summing operator with respect to the $(r,p)$-absolutely summing norm on $A_{(r,p)}(Y, Z)$. In addition, $\|U^*x\|_{l\psi,p} \leq \|U\|_{l\psi,p}$.

Proof. For $x \in X$, let $V_x : Y \to X \otimes Y$, $V_x(y) = x \otimes y$. Then by the hypothesis and the ideal property of the $(r,p)$-absolutely summing operators it follows that $U^*x = V_x$ is an $(r,p)$-absolutely summing operator. Now let $x_1, \ldots, x_n \in X$ with $\|U^*x_i\|_{l\psi,p} > 0$ and $0 < \epsilon < \|U^*x_i\|_{l\psi,p}$ for each $i = 1, n$. By the definition of the $(r,p)$-absolutely summing norm it follows that there is $(y_j)_{j \in \sigma_i}$, $\sigma_i$ finite, $\alpha \subseteq N$ such that $\|U^*x_i\|_{l\psi,p} - \epsilon < \ell_U(U^*x_i(y_j))$ and $w_{\psi}(y_j) \geq \alpha \sigma_i \leq 1$ for each $i = 1, n$. Hence $\ell_U(U^*x_i \otimes y_j) \geq \epsilon \alpha \sigma_i \leq 1$ for each $i = 1, n$. As $U$ is an $(r,p)$-absolutely summing operator we obtain

$$\ell_U(U^*x_i \otimes y_j) \geq \epsilon \alpha \sigma_i \leq 1$$

(1)

But we claim that $w_{\psi}(y_j) \geq \alpha \sigma_i \leq 1$ and thus we obtain

$$\ell_U(U^*x_i \otimes y_j) \geq \epsilon \alpha \sigma_i \leq 1$$

(2)

that is, $\ell_U(U^*x_i \otimes y_j) \geq \epsilon \alpha \sigma_i \leq 1$ and $w_{\psi}(y_j) \geq \alpha \sigma_i \leq 1$. Now for $x_1, \ldots, x_n \in X$, if we denote by $I = \{i \in \alpha \mid U^*x_i \otimes y_j \geq \epsilon \alpha \sigma_i \leq 1\}$, then from (2) we have

$$\ell_U(U^*x_i \otimes y_j) \geq \epsilon \alpha \sigma_i \leq 1$$

(3)

and the proof of the theorem will be finished. Now let $\phi \in (X \otimes Y)^*$, $\|\phi\| \leq 1$. Then, as it is well known, there is a regular Borel measure $\mu$ on $U_X \times U_Y = T$
such that for $x \in X$ and $y \in Y$, \( \psi(x, y) = \int x^*(s(x))y^*(y) \, d\mu(x^*, y^*) \). \( \| \psi \| = |\mu|_T \leq 1 \) (see [2] or [3]). Then using the Hölder inequality and the fact that \( \| \psi \| = |\mu|_T \leq 1 \) we have

\[
|\langle x \otimes y, \psi \rangle| \leq \left( \int |x^*(s(x))|^p |y^*(y)|^p \, d\mu(x^*, y^*) \right)^{1/p},
\]

for $x \in X$, $y \in Y$. (4)

Thus

\[
\sum_{i=1}^n \sum_{j \in \sigma_i} |\langle x_i \otimes y_{ij}, \psi \rangle|^p \leq \int \sum_{i=1}^n |x_i^*(x_i)|^p \sum_{j \in \sigma_i} |y_{ij}^*(y_{ij})|^p \, d\mu(x^*, y^*)
\]

\[
\leq \left( \sum_{i=1}^n |x_i^*(x_i)|^p |y_{ij}^*(y_{ij})|^p \right)^{1/p} \mu(T),
\]

since $w_p(x_{ij})$ $j \in \sigma_i$) $\leq 1$, for each $i = 1, n$. Hence $w_p(x_i \otimes y_{ij} | j \in \sigma_i, i = 1, n) \leq w_p(x_i | i = 1, n)$ and the claim is proved. □

In [5, 7], examples of operators are given which show that the converse of Theorem 1 is not true.

The next theorem is an extension of [1, Theorem 2.5] and the result of Swartz [8, Theorem 2].

**Theorem 2.** Let $U : C(T, X) \to Y$ be a linear and continuous operator, $G$ its representing measure. If $U$ is an $(r, p)$-absolutely summing operator, then $G(E) \in As_{r,p}(X, Y)$, for each $E \in \Sigma$ and $G : \Sigma \to As_{r,p}(X, Y)$ has the property that $|G(E)|_r = \sup \{ \sum_{i=1}^n |G(E_i)|_r^{1/r} | E_1, \ldots, E_n \subset E \}$.

Proof. As it is well known, if $V$ is an $(r, p)$-absolutely summing operator then its bidual $V^{**}$ is also $(r, p)$-absolutely summing (see [6]). As $U$ is an $(r, p)$-absolutely summing operator we obtain, using Theorem 1, that $V = U^*$.

On $C(T) \to As_{r,p}(X, Y)$ is $(r, p)$-absolutely summing and hence $V^{**}$ is also $(r, p)$-absolutely summing. But on $C(T)$, $(r, p)$-absolutely summing operators are weakly compact. This follows easily using [3, Theorem 15, page 159]. Hence the representing measure $G$ of $U$ which coincides with that of $V = U^*$.

As $G(E) = V^{**}(E)$, from (6) and (7) we obtain the theorem. □
The following lemmas show that in the inequality from Theorem 2, we can have both equality and strict inequality.

**Lemma 3.** For $1 \leq p < r < \infty$, $X$ and $Y$ Banach spaces, there is $U : C([0, 1], X) \to Y$ an $(r, p)$-absolutely summing operator whose representing measure has the properties $\|G\|_{r,p}([0, 1]) = (2^r + 1)^{1/r}$. $\|U\|_{r,p} = 3$ and hence if $r \neq 1$, $\|G\|_{r,p}([0, 1]) < \|U\|_{r,p}$.

**Proof.** Let $x^* \in X^*$ with $\|x^*\| = 1$, $y \in Y$, $\|y\| = 1$. For $t \in [0, 1]$, $t$ fixed, we denote $\nu = 2\delta_t - \mu$, where $\delta_t$ is the Dirac measure and $\mu$ is the Lebesgue measure. Let $U : C([0, 1], X) \to Y$, $U(f) = \left( \int_0^1 x^* f \, d\nu \right) y$. Evidently $G(E) = (x^* \otimes y)\nu(E)$ is the representing measure of $U$ and $\|G\|_{r,p}([0, 1]) = \|G\|_{r,p}$, from where

\[
\|G\|_{r,p}([0, 1]) = \sup \left\{ \left( \sum_{i=1}^n \|G(E_i)\|_{r,p}^{1/r} \right)^{1/r} \mid E_1, \ldots, E_n \subset \Sigma \text{ a finite partition of } T \right\} = (2^r + 1)^{1/r}. \tag{8}
\]

On the other hand,

\[
L_r(Uf_i \mid i = 1, n) = \left( \sum_{i=1}^n \left( \int_0^1 \left| \int_0^1 x^* f_i \, d\nu \right|^r \right)^{1/r} \right)^{1/r} \leq \|G\|_{r,p}([0, 1]) \leq \|U\|_{r,p}, \tag{9}
\]

hence, $\|U\|_{r,p} \leq 3$. Also, $3 = \|G\|_{r,p} \leq \|U\|_{r,p}$ and the lemma is proved. \[\square\]

**Lemma 4.** For $1 \leq r < \infty$, $X$ and $Y$ Banach spaces, $T$ a compact Hausdorff space, $\mu$ a regular positive finite Borel measure on $T$, there is $U : C(T, X) \to L_r(\mu, Y)$, an $r$-absolutely summing operator, whose representing measure $G$ has the property $\|G\|_{r,r}(T) = \|U\|_{r,r}$.

**Proof.** Let $J : C(T) \to L_r(\mu)$ be the canonical inclusion. As it is well known and easy to prove (cf. \cite{[2, 6]}) $J$ is an $r$-absolutely summing operator with $\|J\|_{r} = \|\mu(T)\|^{1/r}$. Also, $F(E) = f_E$ is the representing measure of $F$ and
Proof. We consider $U$ that is, $(\text{absolutely summing norm on } As)$ is an extension of $U$ on $As$. Since $G$ is dense in $S(T,H)$, there is $U$ such that $\|U\|_r = \|U\|_r$. Then $G(E) = (\text{absolutely summing norm of } As)$ is an extension of $U$ on $As$. But $G$ is an $r$-absolutely summing operator with $|G|_r(T) = |U|_r$. This shows that $G$ has finite variation with respect to the $r$-p- absolutely summing norm on $As$. Let $fi, x = 1, n$. Then $l_i(Uf_i | i = 1, n) = \sum_{j=1}^{\infty} |G(E_j)x_i | i = 1, n) \leq \sum_{j=1}^{\infty} l_j(G(E_j)x_i) | i = 1, n) \leq \sum_{j=1}^{\infty} G(E_j)|_{l_p} w_p(x_i) | i = 1, n)$, since $G$ takes its values in $As$. But $w_p(x_i) | i = 1, n) \geq \max_{1 \leq i \leq k} w_p(x_i) | i = 1, n)$, because if $|x_i|^p | i = 1, n)$, then $w_p(x_i) | i = 1, n) \geq (\sum_{i=1}^{\infty} |x_i|^p | i = 1, n)^{1/p}$, thus, $l_i(Uf_i | i = 1, n) \leq \left( \sum_{i=1}^{\infty} |G(E_j)|_{l_p} \right) w_p(x_i) | i = 1, n)$, since $G$ has finite variation with respect to the $(r,p)$-absolutely summing norm on $As$. This shows that $U$ is $(r,p)$-absolutely summing and $\|U\|_r \leq |G|_r(T)$ and the proof is finished.

The following theorem is an extension of a result from [1, Proposition 3].

Theorem 5. Let $U : C(T, X) \to Y$ be a linear and continuous operator. $G$ its representing measure. If $G(E) \in As_p (X,Y)$ for each $E \subseteq \sum$ and $G : \sum \to As_p (X,Y)$ has finite variation with respect to the $(r,p)$-absolutely summing norm on $As_p (X,Y)$, then $U$ is an $(r,p)$-absolutely summing operator.

Proof. We consider $\hat{U} : B(\sum) \to Y, \hat{U}(f) = \int f \, dG(f, e \in B(\sum X))$. Since $\hat{U}$ is an extension of $U$ to $B(\sum X)$ and $S(T, X)$ is dense in $B(\sum X)$, it suffices to prove that $\hat{U}$ is $(r,p)$-absolutely summing on $S(T, X)$. Let $f_1, ..., f_n \in S(T, X)$. Then there is $\{E_1, ..., E_k \} \subseteq \sum$, a finite partition of $T$ and $x_i \in X$ such that $f_i = \sum_{i=1}^{\infty} \chi_{E_i} x_i$, for each $i = 1, ..., n$. Then

$$l_i(\hat{U}f_i | i = 1, n) \leq \sum_{j=1}^{\infty} l_j(G(E_j)x_i) | i = 1, n) \leq \sum_{j=1}^{\infty} l_j(G(E_j)x_i) | i = 1, n) \leq \sum_{j=1}^{\infty} G(E_j)|_{l_p} w_p(x_i) | i = 1, n),$$

since $G$ takes its values in $As_p (X,Y)$, but $w_p(x_i) | i = 1, n) \geq \max_{1 \leq i \leq k} w_p(x_i) | i = 1, n)$, because if $|x_i|^p | i = 1, n)$, then $w_p(x_i) | i = 1, n) \geq (\sum_{i=1}^{\infty} |x_i|^p | i = 1, n)^{1/p}$, thus, $l_i(\hat{U}f_i | i = 1, n) \leq \left( \sum_{i=1}^{\infty} |G(E_j)|_{l_p} \right) w_p(x_i) | i = 1, n) \leq \|G|_r(T)w_p(f_i | i = 1, n).$$

The proof is finished.
Theorem 6. Let \( U : C(T) \to Y \) be an absolutely summing operator, \( V \in \text{As}_{r,p}(X, Z) \). Then the injective tensor product \( U \otimes V \) is an element of \( \text{As}_{r,p}(C(T, X), Y \otimes Z) \).

Proof. Let \( F \in \text{rcabv}(\sum, Y) \) be the representing measure of \( U \), (see [3]). Then \( G(E)x = F(E) \otimes V(x) \), \( x \in X \) is the representing measure of \( U \otimes \epsilon V \). In addition, \( G(E) \in \text{As}_{r,p}(X, Y \otimes Z) \) and \( \|G(E)\|_{r,p} \leq \|F(E)\|_{r,p} \|V\|_{r,p} \) for \( E \in \sum \). Indeed, for \( E \in \sum \), let \( S_E : Z \to Y \otimes Z \), \( S_E(z) = F(E) \otimes z \). Then \( G(E) = S_E \otimes V \), hence, because \((\text{As}_{r,p}, \|\|_{r,p})\) is a normed ideal of operators and \( V \in \text{As}_{r,p}(X, Z) \), we obtain that \( G(E) \in \text{As}_{r,p}(X, Y \otimes Z) \) and \( \|G(E)\|_{r,p} \leq \|S_E\|_{r,p} \|V\|_{r,p} \). But \( \|S_E\|_{r,p} \leq \|F(E)\|_{r,p} \) and hence \( \|G(E)\|_{r,p} \leq \|F(E)\|_{r,p} \|V\|_{r,p} \).

Now \( F \) has bounded variation and hence \( G \) satisfies the properties from Theorem 6. Thus, \( U \otimes \epsilon V \in \text{As}_{r,p}(C(T, X), Y \otimes Z) \).

In [2, Chapter 34], various results concerning tensor stability and tensor instability of some operator ideals are given. In the next theorem, we prove a result of the same type.

Theorem 7. Let \( U : X \to X_1 \) be an integral operator, \( V \in \text{As}_{r,p}(Y, Y_1) \). Then \( U \otimes \epsilon V \in \text{As}_{r,p}(X \otimes Y_1, X_1 \otimes Y) \) and \( \|U \otimes \epsilon V\|_{r,p} \leq \|U\|_{\text{int}} \|V\|_{r,p} \).

Proof. As \( U \) is an integral operator, we have the factorization
\[
\begin{align*}
\begin{array}{c}
X \xrightarrow{U} X_1 \xrightarrow{S} X_1^{**} \\
\xrightarrow{C(T)} \\
\end{array}
\end{align*}
\] (12)

where \( S \) is an absolutely summing operator (\( T \) being a compact Hausdorff space), (see [2, 3]).

Hence we have the following factorization of \( U \otimes \epsilon V \)
\[
\begin{align*}
\begin{array}{c}
X \otimes Y \xrightarrow{U \otimes \epsilon V} X_1 \otimes Y_1 \xrightarrow{S} X_1^{**} \otimes Y \\
\xrightarrow{C(T, Y)} \\
\end{array}
\end{align*}
\] (13)

(For the inclusion \( X_1^{**} \otimes Y_1 \hookrightarrow (X_1 \otimes Y_1)^{**} \), see [4, Lemma 1].) Using Theorem 6 it follows that \( S \otimes \epsilon V \in \text{As}_{r,p}(C(T, Y), X_1^{**} \otimes Y_1) \), hence by the ideal property of \( \text{As}_{r,p} \) we obtain that \( J(U \otimes \epsilon V) \in \text{As}_{r,p}(X \otimes Y, X_1 \otimes Y) \).
\[ X_1 \otimes Y_1 \] is the canonical embedding into the bidual, and hence \( U \otimes V \in A_{r,p}(X \otimes Y, X_1 \otimes Y_1) \).

The inequality \( \|U \otimes V\|_{r,p} \leq \|U\|_{int} \|V\|_{r,p} \) is also clear. \( \square \)

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