We determine the class of functions, the divided difference of which, at $n$ distinct numbers, is a continuous function of the product of these numbers.

1. Introduction

We first introduce the needed terminology. The divided differences of a function $f$ at distinct points are defined recursively as follows:

$$f[x_1] := f(x_1),$$

$$f[x_1, \ldots, x_n] := \frac{f[x_2, \ldots, x_n] - f[x_1, \ldots, x_{n-1}]}{x_n - x_1}, \quad n \geq 2. \tag{1.1}$$

The following formula is well known, [2],

$$f[x_1, \ldots, x_n] = \sum_{j=1}^{n} \frac{f(x_j)}{\omega_n'(x_j)}, \tag{1.2}$$

where $\{x_j\}_{j=1}^n \subset \mathbb{C}$ are distinct numbers and $\omega_n(x) := \prod_{k=1}^{n} (x - x_k)$.

Furthermore, if $f \in \mathcal{P}_{n-1}$, the set of algebraic polynomials of degree at most $n - 1$, then, [2], $f[x_1, \ldots, x_{n+1}] = 0$. This fact can be proved by induction in $n$.

The more general definition of divided differences allowing repeated points is that $f[x_1, \ldots, x_n] := a_{n-1}$, where $p_{n-1}(x) = \sum_{k=0}^{n-1} a_k x^k$ is the polynomial that interpolates $f$ at $\{x_j\}_{j=1}^n$ in the Hermite sense, [2].

The main results of this paper are the following theorems.

**Theorem 1.1.** Let $n \in \mathbb{N}$, $n \geq 2$, be fixed and $f$ be a real-valued function defined on an open set $D$ of the real line $\mathbb{R}$. Assume that the divided difference $f[x_1, \ldots, x_n]$ satisfies
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the functional equation

\[ f[x_1, \ldots, x_n] = G \left( \prod_{j=1}^{n} x_j \right) \]  \hspace{1cm} (1.3)

for every set of \( n \) distinct numbers \( \{x_j\}_{j=1}^{n} \subset D \), and \( G \) is a continuous function on the set of products \( P_n(D) := \{ \prod_{j=1}^{n} x_j : \{x_j\}_{j=1}^{n} \subset D \} \). Then

\[ f(x) = a_{-1} \frac{x}{x} + \sum_{k=0}^{n-1} a_k x^k. \]  \hspace{1cm} (1.4)

Furthermore, \( G(t) = (-1)^{n+1} a_{-1}/t + a_{n-1} \).

**Theorem 1.2.** Let \( f \) be a complex-valued function defined on an open set \( D \) of the complex plane \( \mathbb{C} \). Assume that \( f \) satisfies the conditions of Theorem 1.1 on \( D \). Then \( f \) and \( G \) have the forms given by Theorem 1.1.

A similar characterization was obtained in [1] for functions the divided difference of which at \( n \) distinct points is a function of the sum of the points.

2. Proofs of the theorems

**Proof of Theorem 1.1.** First assume that \( 0 \not\in D \). We claim that \( f \in C^\infty(D) \), that is, \( f \) has continuous derivatives of arbitrary order in \( D \). Let \( \{x_j\}_{j=1}^{n} \subset D \) be distinct. From (1.2) and (1.3) for \( f[x_1, \ldots, x_n] \) we get

\[ \left( \frac{f(x_2) - f(x_1)}{x_2 - x_1} \right) \left( \frac{x_2 - x_1}{\omega'_n(x_2)} \right) + f(x_1) \left( \frac{1}{\omega'_n(x_2)} + \frac{1}{\omega'_n(x_1)} \right) + \sum_{j=3}^{n} \frac{f(x_j)}{\omega'_n(x_j)} = G(x_1 \cdots x_n). \]  \hspace{1cm} (2.1)

We let \( x_2 \to x_1 \) in (2.1). All terms on the left-hand side of (2.1) beginning with the second term which tends to \( f(x_1)(1/\prod_{j=3}^{n} (x - x_j))|_{x=x_1} \), and the right-hand side \( G \) have finite limits. Then the first term on the left-hand side of (2.1) has a finite limit, that is, \( f'(x_1) \) exists. Since the points \( x_1, x_3, \ldots, x_n \) are distinct, all terms in the equation obtained from (2.1) after taking the limit \( x_2 \to x_1 \), except possibly the first one, are continuous at \( x_1 \) for fixed \( \{x_j\}_{j=3}^{n} \). We obtain \( f \in C^1(D) \). Then (1.2) and (1.3) imply \( G \in C^1(P_n(D)) \). This can be seen by observing that the derivative of the right-hand side of (1.2) with respect to \( x_1 \) is continuous at \( x_1 \) if the points \( \{x_j\}_{j=1}^{n} \) are distinct and \( \{x_j\}_{j=2}^{n} \) are fixed. Then with \( a = \prod_{j=3}^{n} x_j \), \( dG(ax_1)/dx_1 \) exists and is continuous at \( x_1 \). Therefore, \( G(t) \in C(P_n(D)) \) exists because every \( t \in P_n(D) \) can be written as a product of \( n \) distinct numbers from \( D \) and \( P_n(D) \) is an open set.
Next, from (1.2) we have

$$\lambda^{n-1} f[\lambda x_1, \ldots, \lambda x_n] = \sum_{j=1}^{n} \frac{f(\lambda x_j)}{\omega'_n(x_j)} = \lambda^{n-1} G(\lambda^n x_1 \cdots x_n). \quad (2.2)$$

Differentiating (2.2) with respect to $\lambda$ and setting $\lambda = 1$, we obtain

$$\left(xf'(x)\right)[x_1, \ldots, x_n] = \sum_{j=0}^{n} \frac{x_j f'(x_j)}{\omega'_n(x_j)} = (n-1)G(x_1 \cdots x_n) + nx_1 \cdots x_n G'(x_1 \cdots x_n). \quad (2.3)$$

Equation (2.3) for $f_1(x) = xf'(x)$ has the same form as (1.3) for $f(x)$, and $G_1(t) = (n-1)G(t) + ntG'(t) \in C(P_n(D))$. Using the same argument and induction, we can show that for every $k \in \mathbb{N}$,

$$f_k(x) \in C^1(D)$$

and $G_k(t) \in C^1(P_n(D))$, where the functions $f_k$ and $G_k$ are defined recursively by $f_{k+1}(x) = xf'_k(x)$, $k \geq 0$, $f_0(x) = f(x)$, and $G_{k+1}(t) = (n-1)G_k(t) + ntG'_k(t)$, $k \geq 0$, $G_0(t) = G(t)$. From the definition of $f_k$, we get

$$f_k(x) = \sum_{j=1}^{k} a_{k,j}x^j f^{(j)}(x), \quad (2.4)$$

where $a_{k,1} = a_{k,k} = 1$ and $a_{k+1,j} = a_{k,j-1} + ja_{k,j}$, $j = 2, \ldots, k$. Since $0 \notin D$, we get $f^{(k)}(x) \in C(D)$ for every $k \geq 0$, that is, $f \in C^\infty(D)$.

We proceed by induction with respect to $n \in \mathbb{N}$, $n \geq 2$. For $n = 2$ we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = G(x_1 x_2), \quad x_1, x_2 \in D, \ x_1 \neq x_2, \quad (2.5)$$

and $f \in C^\infty(D)$. Set $x_1 = x$ and let $x_2 \to x$. We get

$$f'(x) = G(x^2) = \frac{f(x^2) - f(1)}{x^2 - 1}, \quad x \in D \setminus \{1\}. \quad (2.6)$$

We may assume that $1 \in D$, otherwise we consider $f(ax)$ instead of $f(x)$ for some $a \in D$. Set $g(t) = f(1+t)$. Since $0 \in D(g)$, the domain of $g$, and $g \in C^\infty(D(g))$, $g$ has a power series representation

$$g(t) = \sum_{k=0}^{\infty} g_k t^k, \quad t \in (-r, r), \quad (2.7)$$

for small $r > 0$. Relations (2.6), with $x = t+1$, and (2.7) yield

$$\sum_{k=1}^{\infty} g_k t^{k-1} = g'(t) = \frac{g(t^2 + 2t) - g(0)}{t^2 + 2t} = \sum_{s=0}^{\infty} g_{s+1} t^s (t+2)^s \quad (2.8)$$
Comparing the coefficients of $t^j$ in (2.8) we obtain

$$(j + 1)g_{j+1} = \sum_{s=0}^{j} \binom{s}{j-s} 2^{2s-j} g_{s+1}, \quad j \geq 0, \quad (2.9)$$

or, equivalently, (with $v = s + 1$)

$$(j + 1 - 2^j)g_{j+1} = \sum_{v=2}^{j} \binom{v-1}{j-v+1} 2^{2v-j-2} g_v, \quad j \geq 2. \quad (2.10)$$

Equation (2.10) is nontrivial only if $j \geq 2$. This is so because adding to $g$ (or $f$) a linear function and multiplying $g$ (or $f$) by a constant does not change its properties. The binomial coefficients in (2.10) are not zero only if $\nu - 1 \geq j - \nu + 1$, that is, if $\nu \geq j/2 + 1 \geq 2$. Then (2.10) implies that $g_j$ is a multiple of $g_2$ for every $j \geq 2$. In particular, if $g_2 = 0$ then $g$ (and hence $f$) is a linear function.

If $g_2 \neq 0$ we may assume that $g_2 = 1$. In this case (2.10) implies $g_j = (-1)^j$ for every $j \geq 2$. This follows from (2.9) and the identity

$$\sum_{s=1}^{j} \binom{s}{j-s} 2^{2s-j} (-1)^{s-j} = j + 1, \quad j \geq 2, \quad (2.11)$$

which is the special case $\alpha = 2$, $\beta = -1$ of the formula

$$A_j(\alpha, \beta) := \sum_{s=0}^{j} \binom{s}{j-s} \alpha^{2s-j} \beta^{j-s} = \sum_{k=0}^{j} \lambda_1^k \lambda_2^{j-k}, \quad j \geq 0, \quad (2.12)$$

where $\alpha, \beta \in \mathbb{C}$ and $\lambda_{1,2}$ are the zeros of $\lambda^2 - \alpha \lambda - \beta$. Hence, it is enough to verify (2.12).

For $j = 0$ and $j = 1$, (2.12) is obvious. Next, for $j \geq 1$

$$A_{j+1}(\alpha, \beta) = \sum_{s=1}^{j+1} \binom{s}{j+1-s} \alpha^{2s-j-1} \beta^{j+1-s} = \alpha \sum_{s=1}^{j+1} \binom{s-1}{j-(s-1)} \alpha^{2(s-1)-j} \beta^{j-(s-1)}$$

$$+ \beta \sum_{s=1}^{j} \binom{s-1}{j-1-(s-1)} \alpha^{2(s-1)-(j-1)} \beta^{(j-1)-(s-1)}$$

$$= \alpha A_j(\alpha, \beta) + \beta A_{j-1}(\alpha, \beta)$$

$$= \alpha \sum_{k=0}^{j} \lambda_1^k \lambda_2^{j-k} + \beta \sum_{k=0}^{j-1} \lambda_1^k \lambda_2^{j-1-k}$$

$$= \sum_{k=0}^{j} \left( \lambda_1^{k+1} \lambda_2^{j-k} + \lambda_1^k \lambda_2^{j+1-k} \right) - \sum_{k=0}^{j-1} \lambda_1^k \lambda_2^{j-k}$$

$$= \sum_{k=0}^{j} \lambda_1^k \lambda_2^{j+1-k}, \quad (2.13)$$
where we used the identities
\[
\binom{s}{j + 1 - s} = \binom{s - 1}{j - s + 1} + \binom{s - 1}{j - s}, \quad s = 1, \ldots, j.
\] (2.14)

\[\lambda_1 + \lambda_2 = \alpha, \quad \lambda_1 \lambda_2 = -\beta,\] and induction with respect to \( j \).

Since \( g_j = (-1)^j, \ j \geq 2 \), it follows from (2.7) that \( g(t) = g_2/(1 + t) + At + B \) for \( t \in (-r, r) \). We have proved that for every \( x_0 \in D \) there exists \( r(x_0) > 0 \) such that \( f(x) = c/x + ax + b, \ |x - x_0| < r(x_0) \). We have to show that the coefficients \( a, b, \) and \( c \) are independent of \( x_0 \in D \). Let \( I_1 \) and \( I_2 \) be two open subinterval of \( D \) such that

\[f(x) = \frac{c_v}{x} + a_v x + b_v, \quad x \in I_v, \ v = 1, 2.\] (2.15)

From (2.5) we get

\[f[x, y] = \frac{(c_2 + a_2 y^2 + b_2 y)x - (c_1 + a_1 x^2 + b_1 x)y}{(y - x)xy} = G(xy), \quad x \in I_1, \ y \in I_2.\] (2.16)

Let \( C = x_0 y_0 \) for some \( x_0 \in I_1 \) and \( y_0 \in I_2 \), and let \( \gamma_C = (I_1 \times I_2) \cap \{(x, y) : xy = C\} \). Then

\[f[x, y] = \frac{(c_2 + a_2 y^2 + b_2 y)C - (c_1 y^2 + a_1 C^2 + b_1 Cy)}{(y^2 - C)C} = G(C), \quad (x, y) \in \gamma_C.\] (2.17)

Since \( \gamma_C \) is a continuous curve, (2.17) implies that \((b_2 - b_1)C = 0, a_2 C - c_1 = G(C)C, \) and \((c_2 - a_1)C = -G(C)C^2\). Using that \( C \neq 0 \) we get \( b_1 = b_2 \) and \( a_1 C - c_2 = G(C)C = a_2 C - c_1, \) hence, \( (a_2 - a_1)C = c_1 - c_2. \) Unless \( a_1 = a_2 \) we can choose \( C \neq (c_1 - c_2)/(a_2 - a_1) \) and that choice would give us a contradiction. Thus \( a_1 = a_2 \) and then \( c_1 = c_2. \)

Now assume that Theorem 1.1 is true for some \( n \geq 2 \) and consider the case \( n + 1 \).

Let \( x_1 \in D \) be fixed. We define \( \tilde{f}(x) := f[x_1, x]. \) Let \( \{x_j\}_{j=2}^{n+1} \subset D \setminus \{x_1\} \) be distinct numbers. From (1.2) and (1.3) with \( \tilde{\omega}_n(x) := \prod_{k=2}^{n+1}(x - x_k), \) we obtain

\[
\tilde{f}[x_2, \ldots, x_{n+1}] = \sum_{j=2}^{n+1} \frac{f(x_j) - f(x_1)}{x_j - x_1} \tilde{\omega}'_n(x_j) = \sum_{j=1}^{n+1} \frac{f(x_j) - f(x_1)}{\tilde{\omega}'_n(x_j)} \omega'_n(x_j) = f[x_1, \ldots, x_{n+1}] = G\left(x_1 \prod_{j=2}^{n+1} x_j\right).\] (2.18)

where we also used that \( f(x_1)[x_1, \ldots, x_{n+1}] = 0. \) Hence, \( \tilde{f} \) satisfies the conditions of Theorem 1.1 for \( n \) points. By the induction assumption

\[f[x_1, x] = \tilde{f}(x) = \sum_{j=-1}^{n-1} \tilde{a}_j(x_1)x^j, \quad x \in D \setminus \{x_1\}.\] (2.19)
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or, equivalently,

\[ f(x) = f(x_1) + (x - x_1) \sum_{j=-1}^{n-1} \tilde{a}_j(x_1)x^j =: \sum_{j=-1}^{n} a_j(x_1)x^j, \quad x \in D \setminus \{x_1\}. \quad (2.20) \]

Equation (2.20) is true for \( x = x_1 \) as well. Furthermore, (2.20) is unique in the sense that the coefficients \( a_j(x_1) \) are independent of \( x_1 \in D \). Indeed,

\[ xf(x) = \sum_{j=0}^{n+1} a_{j-1}(x_1)x^j = \sum_{j=0}^{n+1} a_{j-1}(x_2)x^j, \quad x \in D \quad (2.21) \]

implies \( a_j(x_1) = a_j(x_2), \quad j = -1, \ldots, n, \) since a nonzero polynomial has finitely many zeros.

Now assume that \( 0 \in D \). Then \( D_1 := D \setminus \{0\} \) is an open set. For every \( n - 1 \) distinct numbers \( \{x_j\}_{j=2}^{n} \subset D_1 \), from (2.18) with \( \tilde{f}(x) := f[0, x] \) and (1.3) we obtain

\[ \tilde{f}[x_2, \ldots, x_n] = f[0, x_2, \ldots, x_n] = G(0). \quad (2.22) \]

Then \( \tilde{f}(x) \) satisfies the conditions of Theorem 1.1 on the set \( D_1 \) and \( 0 \notin D_1 \). Therefore, \( \tilde{f}(x) = (f(x) - f(0))/x = a_{-1}/x + p(x), \quad p(x) \in \mathbb{P}_{n-2} \). Since \( G \in C(D) \), (1.3) implies \( f \in C(D) \) and \( a_{-1} = 0 \). Hence \( f(x) = f(0) + xp(x) \in \mathbb{P}_{n-1} \).

The formula for \( G(t) \) follows from the identities \( x^k[x_1, \ldots, x_n] = \delta_{n-1,k}, \quad k = 0, \ldots, n - 1, \) and

\[ \frac{1}{x}[x_1, \ldots, x_n] = \frac{(-1)^{n+1}}{x_1 \cdots x_n}, \quad n \geq 1. \quad (2.23) \]

The proof of Theorem 1.1 is complete.

Proof of Theorem 1.2. The proof of Theorem 1.2 follows the same arguments as the proof of Theorem 1.1 except that after verifying \( f \in C^1(D) \) as in the proof of Theorem 1.1, we automatically obtain that \( f \in C^\infty(D) \) via Cauchy’s integral formula.

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References


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