EXISTENCE AND REGULARITY OF WEAK SOLUTIONS TO THE PRESCRIBED MEAN CURVATURE EQUATION FOR A NONPARAMETRIC SURFACE

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1. Introduction

The prescribed mean curvature equation with Dirichlet condition for a nonparametric surface \( X : \Omega \to \mathbb{R}^3, X(u, v) = (u, v, f(u, v)) \) is the quasilinear partial differential equation

\[
(1 + f_v^2) f_{uu} + (1 + f_u^2) f_{vv} - 2 f_u f_v f_{uv} = 2h(u, v, f)\left(1 + |\nabla f|^2\right)^{3/2} \quad \text{in} \; \Omega, \quad f = g \quad \text{in} \; \partial \Omega, \tag{1.1}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^2 \), \( h : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \) is continuous and \( g \in H^1(\Omega) \).

We call \( f \in H^1(\Omega) \) a weak solution of (1.1) if \( f \in g + H^1_0(\Omega) \) and for every \( \varphi \in C^1_0(\Omega) \)

\[
\int_{\Omega} \left(((1 + |\nabla f|^2)^{-1/2} \nabla f \nabla \varphi + 2h(u, v, f)\varphi\right) du dv = 0. \tag{1.2}
\]

It is known that for the parametric Plateau’s problem, weak solutions can be obtained as critical points of a functional (see [2, 6, 7, 8, 10, 11]).

The nonparametric case has been studied for \( H = H(x, y) \) (and generally \( H = H(x_1, \ldots, x_n) \) for hypersurfaces in \( \mathbb{R}^{n+1} \)) by Gilbarg, Trudinger, Simon, and Serrin, among other authors. It has been proved [5] that there exists a solution for any smooth boundary data if the mean curvature \( H' \) of \( \partial \Omega \) satisfies

\[
H'(x_1, \ldots, x_n) \geq \frac{n}{n-1} \left| H(x_1, \ldots, x_n) \right| \tag{1.3}
\]

for any \( (x_1, \ldots, x_n) \in \partial \Omega \), and \( H \in C^1(\overline{\Omega}, \mathbb{R}) \) satisfying the inequality

\[
\left| \int_{\Omega} H \varphi \right| \leq \frac{1-\epsilon}{n} \int_{\Omega} |D\varphi| \tag{1.4}
\]

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for any $\varphi \in C^1_0(\Omega, \mathbb{R})$ and some $\epsilon > 0$. They also proved a non-existence result (see [5, Corollary 14.13]): if $H'(x_1, \ldots, x_n) < (n/(n-1))|H(x_1, \ldots, x_n)|$ for some $(x_1, \ldots, x_n)$ and the sign of $H$ is constant, then for any $\epsilon > 0$ there exists $g \in C^\infty(\overline{\Omega})$ such that $\|g\|_\infty \leq \epsilon$ and that Dirichlet’s problem is not solvable.

We remark that the solutions obtained in [5] are classical. In this paper, we find weak solutions of the problem by variational methods.

We prove that for prescribed $h$ there exists an associated functional to $h$, and under some conditions on $h$ and $g$ we find that this functional has a global minimum in a convex subset of $H^1(\Omega)$, which provides a weak solution of (1.1). We denote by $H^1(\Omega)$ the usual Sobolev space, [1].

2. The associated variational problem

Given a function $f \in C^2(\Omega)$, the generated nonparametric surface associated to this function is the graph of $f$ in $\mathbb{R}^3$, parametrized as $X(u, v) = (u, v, f(u, v))$.

The mean curvature of this surface is

$$h(u, v, f) = \frac{1}{2} \frac{E f_{vv} - 2 F f_{uv} + G f_{uu}}{(1 + f_u^2 + f_v^2)^{3/2}},$$

where $E$, $F$, and $G$ are the coefficients of the first fundamental form [4, 9].

For prescribed $h$, weak solutions of (1.1) can be obtained as critical points of a functional.

Proposition 2.1. Let $J_h : H^1(\Omega) \to \mathbb{R}$ be the functional defined by

$$J_h(f) = \int_\Omega \left( (1 + |\nabla f|^2)^{1/2} + H(u, v, f) \right) dudv,$$

where $H(u, v, z) = \int_0^z 2h(u, v, t)dt$. Then (1.1) is the Euler Lagrange equation of (2.2).

Remark 2.2. If $f \in T = g + H^1_0(\Omega)$ is a critical point of $J_h$, then $f$ is a weak solution of (1.1).

Proof. For $\varphi \in C^1_0(\Omega)$, integrating by parts we obtain

$$d J_h(f)(\varphi) = 2 \int_\Omega \left( \frac{1}{2} \frac{E f_{vv} - 2 F f_{uv} + G f_{uu}}{(1 + f_u^2 + f_v^2)^{3/2}} - h(u, v, f) \right) \varphi dudv.$$  \hfill (2.3)

\[ \square \]

3. Behavior of the functional $J_h$

In this section, we study the behavior of the functional $J_h$ restricted to $T$. For simplicity we write $J_h(f) = A(f) + B(f)$, with

$$A(f) = \int_\Omega \left( (1 + |\nabla f|^2)^{1/2} \right) dudv, \quad B(f) = \int_\Omega H(u, v, f) dudv.$$  \hfill (3.1)

We will assume that $h$ is bounded.
Lemma 3.1. The functional $A : T \to \mathbb{R}$ is continuous and convex.

Proof. Continuity can be proved by a simple computation. Let $a, b \geq 0$ such that $a + b = 1$. By Cauchy inequality, it follows that

$$\sqrt{1 + \left| \nabla (af + bf_0) \right|^2} \leq a\sqrt{1 + |\nabla f|^2} + b\sqrt{1 + |\nabla f_0|^2}$$

(3.2)

and convexity holds. \qed

Remark 3.2. As $A$ is continuous and convex, then it is weakly lower semicontinuous in $T$.

Lemma 3.3. The functional $B$ is weakly lower semicontinuous in $T$.

Proof. Since $h$ is bounded, we have

$$|H(u, v, z)| \leq c|z| + d.$$  

(3.3)

From the compact immersion $H^1_0(\Omega) \hookrightarrow L^1(\Omega)$ and the continuity of Nemytskii operator associated to $H$ in $L^1(\Omega)$, we conclude that $B$ is weakly lower semicontinuous in $T$ (see [3, 12]). \qed

4. Weak solutions as critical points of $J_h$

Let us assume that $g \in W^{1,\infty}$, and consider for each $k > 0$, the following subset of $T$:

$$\overline{M}_k = \{ f \in T : \| \nabla (f - g) \|_\infty \leq k \}. $$

(4.1)

$\overline{M}_k$ is nonempty, closed, convex, bounded, then it is weakly compact.

Remark 4.1. As $g \in W^{1,\infty}$, taking $p > 2$ we obtain, for any $f \in \overline{M}_k$:

$$\| f - g \|_p \leq c\| \nabla (f - g) \|_p.$$  

(4.2)

Then, by Sobolev imbedding, $\| f - g \|_\infty \leq c_1\| f - g \|_1, p \leq \tilde{c}k$ for some constant $\tilde{c}$. We deduce that $f \in W^{1,\infty}$ and $f(\Omega) \subset K$ for some fixed compact $K \subset \mathbb{R}$. Thus, the assumption $\| h \|_\infty < \infty$ is not needed.

Let $\rho$ be the slope of $J_h$ in $\overline{M}_k$ defined by

$$\rho(f_0, \overline{M}_k) = \sup \{ dJ_h(f_0)(f_0 - f) ; f \in \overline{M}_k \}$$

(4.3)

(see [7, 11]), then the following result holds.

Lemma 4.2. If $f_0 \in \overline{M}_k$ verifies

$$J_h(f_0) = \inf \{ J_h(f) : f \in \overline{M}_k \},$$

(4.4)

then $\rho(f_0, \overline{M}_k) = 0$. 


Proof. 

\[
d J_h(f_0)(f - f_0) = \lim_{\varepsilon \to 0} \frac{J_h(f_0 + \varepsilon(f - f_0)) - J_h(f_0)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{J_h((1 - \varepsilon)f_0 + \varepsilon f) - J_h(f_0)}{\varepsilon}.
\]  

(4.5)

When \(0 < \varepsilon < 1\) we have that \((1 - \varepsilon)f_0 + \varepsilon f \in \overline{M}_k\), and then \(d J_h(f_0)(f_0 - f) \leq 0\) for all \(f \in \overline{M}_k\). As \(d J_h(f_0)(f_0 - f) = 0\), we conclude that \(\rho(f_0, \overline{M}_k) = 0\). \(\square\)

Remark 4.3. Let \(J_h\) be weakly semicontinuous and let \(\overline{M}_k\) be a weakly compact subset of \(T\), then \(J_h\) achieves a minimum \(f_0\) in \(\overline{M}_k\). By Lemma 4.2, \(\rho(f_0, \overline{M}_k) = 0\).

As in [7], if \(f_0\) has zero slope, we call it a \(\rho\)-critical point. The following result gives sufficient conditions to assure that if \(f_0\) is a \(\rho\)-critical point, then it is a critical point of \(J_h\).

Theorem 4.4. Let \(f_0 \in \overline{M}_k\) such that \(\rho(f_0, \overline{M}_k) = 0\), and assume that one of the following conditions holds:

(i) \(d J_h(f_0)(f_0 - g) \geq 0\)

(ii) \(\|\nabla(f_0 - g)\|_\infty < k\).

Then \(d J_h(f_0) = 0\).

Proof. As \(\rho(f_0, \overline{M}_k) = 0\), we have that \(d J_h(f_0)(f_0 - f) \leq 0\), and then \(d J_h(f_0)(f_0 - g) \leq d J_h(f_0)(f_0 - g)\) for any \(f \in \overline{M}_k\).

We will prove that \(d J_h(f_0)(\varphi) = 0\) for any \(\varphi \in C^0_0\). Let \(\tilde{\varphi} = k\varphi/2\|\nabla\varphi\|_\infty\), then \(\pm \tilde{\varphi} + g \in \overline{M}_k\), and then \(d J_h(f_0)(f_0 - g) \leq \pm d J_h(f_0)(\tilde{\varphi})\).

Suppose that \(d J_h(f_0)(\tilde{\varphi}) \neq 0\), then \(d J_h(f_0)(f_0 - g) < 0\).

If (i) holds, we immediately get a contradiction. On the other hand, if (ii) holds, there exists \(r > 1\) such that \(g + r(f_0 - g) \in \overline{M}_k\). Then \(d J_h(f_0)(f_0 - g) \leq r d J_h(f_0)(f_0 - g)\), a contradiction. \(\square\)

Examples

Let us assume that \(\int_{\Omega} ((\nabla(f - g)\nabla g)/\sqrt{1 + |\nabla f|^2}) du dv \geq 0\) for any \(f \in \overline{M}_k\). Then condition (i) of Theorem 4.4 is fulfilled for example if

(a) \(|h(u, v, z)| \leq c(z - g(u, v))_+\) for every \((u, v) \in \Omega\), \(z \in \mathbb{R}^3\), for some constant \(c\) small enough.

(b) \(\int_{\Omega} h(u, v, f)(f - g) du dv \geq 0\) for every \(f \in \overline{M}_k\). As a particular case, we may take \(h(u, v, z) = c(z - g(u, v))\) for any \(c \geq 0\).

(c) \(h(u, v, z) = -c(z - g(u, v))\) for some \(c > 0\) small enough.
Indeed, in all the examples the inequality \( dJ_h(f)(f - g) \geq 0 \) holds for any \( f \in \overline{M}_k \), since

\[
dJ_h(f)(f - g) = \int_{\Omega} \left( \frac{\nabla f \nabla (f - g)}{\sqrt{1 + |\nabla f|^2}} + 2h(u, v, f)(f - g) \right) \, du \, dv \\
= \int_{\Omega} \left( \frac{|\nabla (f - g)|^2}{\sqrt{1 + |\nabla f|^2}} + 2h(f - g) \right) \, du \, dv + \int_{\Omega} \nabla (f - g) \nabla g \, du \, dv \\
\geq \int_{\Omega} \left( \frac{|\nabla (f - g)|^2}{\sqrt{1 + |\nabla f|^2}} + 2h(f - g) \right) \, du \, dv.
\]

(4.6)

Then the result follows immediately in example (b). In examples (a) and (c), being \( \|\nabla (f - g)\|_{\infty} \leq k \) we can choose \( \tilde{k} \) such that \( \sqrt{1 + \|\nabla f\|_{\infty}^2} \leq \tilde{k} \). Then

\[
\int_{\Omega} \left( \frac{|\nabla (f - g)|^2}{\sqrt{1 + |\nabla f|^2}} + 2h(u, v, f)(f - g) \right) \, du \, dv \geq \int_{\Omega} \left( \frac{|\nabla (f - g)|^2}{k} - 2c(f - g)^2 \right) \, du \, dv \\
\geq \frac{1}{k} \|\nabla (f - g)\|_2^2 - 2c_1^2 \|\nabla (f - g)\|_2^2 \\
= \left( \frac{1}{k} - 2c_1^2 \right) \|\nabla (f - g)\|_2^2.
\]

(4.7)

where \( c_1 \) is the Poincaré’s constant associated to \( \Omega \).

Thus, the result holds for \( c \leq 1/2\tilde{k}c_1^2 \).

Remark 4.5. As in the preceding examples, it can be proved that if \( dJ_h(f)(f - g) \geq 0 \) for any \( f \in \overline{M}_k \), then \( g \) is a weak solution of (1.1). Indeed, if \( dJ_h(g) \neq 0 \), from Theorem 4.4 it follows that \( \rho(g, \overline{M}_k) > 0 \). As \( J_h \) achieves a minimum in every \( \overline{M}_k \), we may take \( k \geq k_n \to 0 \), and \( f_n \) such that \( \rho(f_n, \overline{M}_{k_n}) = 0 \). As \( \overline{M}_{k_n} \subset \overline{M}_k \), condition (i) in Theorem 4.4 holds, and then \( dJ_h(f_n) = 0 \). It is immediate that \( f_n \to g \) in \( W^{1,\infty} \), and then it follows easily that \( dJ_h(g) = 0 \).

Furthermore, for constant \( g \) we can see that if \( dJ_h(f)(f - g) \geq 0 \) for any \( f \in \overline{M}_k \), then \( g \) is a global minimum of \( J_h \) in \( \overline{M}_k \): let us define \( \varphi(t) = J_h(tf + (1 - t)g) \), then \( \varphi'(t) = dJ_h(tf + (1 - t)g)(f - g) \). As \( 0 \leq dJ_h(tf + (1 - t)g)(tf + (1 - t)g - g) = tdJ_h(tf + (1 - t)g)(f - g) \) it follows that \( J_h(f) - J_h(g) = \varphi(1) - \varphi(0) = \varphi'(c) \geq 0 \).

5. Multiple solutions

In this section, we study the multiplicity of weak solutions of (1.1). Consider

\[
\overline{N}_k = \left\{ f \in \overline{M}_k \cap H^2 : \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_2 \leq k \right\},
\]

(5.1)
\( \Omega_k \) is a nonempty, closed, bounded, and convex subset of \( T \), therefore \( \Omega_k \) is weakly compact.

Then we obtain the following theorem, which is a variant of the mountain pass lemma.

**Theorem 5.1.** Let \( f_0 \in \Omega_k \) be a local minimum of \( J_h \) and assume that \( J_h(f_1) < J_h(f_0) \) for some \( f_1 \in \Omega_k \). Let

\[
c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J_h(\gamma(t)),
\]

where \( \Gamma = \{ \gamma \in C([0,1], \Omega_k) : \gamma(0) = f_0, \gamma(1) = f_1 \} \). Then there exists \( f \in \Omega_k \) such that \( J_h(f) = c \) and \( \rho(f, \Omega_k) = 0 \).

We remark that \( f \) is not a local minimum of \( J_h \). This kind of \( f \) is called an unstable critical point.

The proof of Theorem 5.1 follows from Theorem 3 in [7] and Lemmas 5.2, 5.3, and 5.4 below.

**Lemma 5.2.** The functional \( J_h \) is \( C^1(\Omega_k) \).

**Proof.** Let \( f, f_0 \in \Omega_k \). Then

\[
\left| d J_h(f)(\varphi) - d J_h(f_0)(\varphi) \right| \\
\leq \| \varphi \|_{H^1_0} \left( \left\| \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} - \frac{\nabla f_0}{\sqrt{1 + |\nabla f_0|^2}} \right\|_2^2 + \| N_h(f_0) - N_h(f) \|_2^2 \right),
\]

where \( N_h \) is the Nemytskii operator associated to \( h \). Let

\[
\left\| \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} - \frac{\nabla f_0}{\sqrt{1 + |\nabla f_0|^2}} \right\|_2 \leq \left( \sqrt{1 + |\nabla f|^2} \nabla f - \sqrt{1 + |\nabla f_0|^2} \nabla f_0 \right)_2 \\
\leq \kappa \| f_0 - f \|_{H^1_0}
\]

and \( N_h : L^2 \to L^2 \) continuous, the result holds. \( \square \)

**Lemma 5.3.** The slope \( \rho \) is \( H^1 \)-continuous.

**Proof.** Let \( f_n \in \Omega_k \) such that \( f_n \to f_0 \) in \( H^1_0 \). For \( \epsilon > 0 \) we take \( g_n \in \Omega_k \) such that \( \rho(f_n, \Omega_k) - \epsilon/2 < d J_h(f_n)(f_n - g_n) \). Then

\[
\rho(f_n, \Omega_k) - \rho(f_0, \Omega_k) \\
\leq d J_h(f_n)(f_n - g_n) + \frac{\epsilon}{2} - d J_h(f_0)(f_0 - g_n) \\
\leq \| d J_h(f_n) \|_{(H^1_0)^*} \| (f_n - f_0) \|_{H^1_0} \\
+ \| d J_h(f_n) - d J_h(f_0) \|_{(H^1_0)^*} \| (f_0 - g_n) \|_{H^1_0} + \frac{\epsilon}{2} < \epsilon
\]

for \( n \geq n_0 \). Operating in the same way with \( \rho(f_0, \Omega_k) - \rho(f_n, \Omega_k) \), we conclude that \( \rho(f_n, \Omega_k) \to \rho(f_0, \Omega_k) \). \( \square \)
Lemma 5.4 (Palais Smale condition). Let \((f_n)_{n \in \mathbb{N}} \subset \overline{N}_k\) such that \(\lim_{n \to \infty} \rho(f_n, \overline{N}_k) = 0\). Then \((f_n)_{n \in \mathbb{N}}\) has a convergent subsequence in \(H_0^1(\Omega)\).

Proof. As \(f_n \in \overline{N}_k\), we may suppose that \(f_n \to f\) weakly. Let \(\Psi_n = f_n - f\). We will see that \(\Psi_n \to 0\). Indeed,

\[
d J_h(f_n)(\Psi_n) = \int_\Omega \left( \frac{\nabla f_n}{\sqrt{1 + |\nabla f_n|^2}} \nabla \Psi_n + 2h(u,v,f_n) \Psi_n \right) du \, dv
\]

\[
= \int_\Omega \frac{1}{\sqrt{1 + |\nabla f_n|^2}} |\nabla \Psi_n|^2 du \, dv + \int_\Omega \frac{\nabla \Psi_n}{\sqrt{1 + |\nabla f_n|^2}} \nabla f du \, dv (5.6)
\]

\[
+ \int_\Omega 2h(u,v,f_n) \Psi_n du \, dv.
\]

Then for some constant \(c\)

\[
c \|\nabla \Psi_n\|_2^2 \leq \rho(f_n, \overline{N}_k) - \int_\Omega \frac{\nabla \Psi_n}{\sqrt{1 + |\nabla f_n|^2}} \nabla f du \, dv - \int_\Omega 2h(u,v,f_n) \Psi_n du \, dv. (5.7)
\]

By Rellich-Kondrachov theorem \(\Psi_n \to 0\) in \(L^2(\Omega)\), and then

\[
\left| \int_\Omega 2h(u,v,f_n) \Psi_n du \, dv \right| \leq 2 \|h\|_\infty \|\Psi_n\|_2 \to 0, (5.8)
\]

\[
\left| \int_\Omega \frac{\nabla \Psi_n}{\sqrt{1 + |\nabla f_n|^2}} \nabla f du \, dv \right|
\]

\[
= \left| - \int_\Omega \frac{\Delta f}{\sqrt{1 + |\nabla f_n|^2}} \Psi_n du \, dv - \int_\Omega \Psi_n \nabla (1 + |\nabla f_n|^2)^{-1/2} \nabla f du \, dv \right| (5.9)
\]

\[
\leq \|\Delta f\|_2 \|\Psi_n\|_2 + \|\nabla f_n\|_\infty \|\nabla f\|_\infty \|D^2 f_n\|_2 \|\Psi_n\|_2 \to 0. \quad \square
\]

Example 5.5. Now we will show with an example that problem (1.1) may have at least three \(\rho\)-critical points in \(N_k\).

Let \(g = g_0\) be a constant, and \(h(u,v,z) = -c(z - g_0)\) for some constant \(c > 0\). Then, \(g_0\) is a minimum of \(J_h\) in \(\overline{M}_{k_1}\) for \(k_1\) small enough, and a local minimum in \(M_k\) for any \(k \geq k_1\).

Moreover, taking \(\Omega = B_R\), \(f(u,v) = g_0 + R^2 - (u^2 + v^2)\), it follows that

\[
J_h(f) - J_h(g_0) = 2\pi \left( a(R^3) - \frac{c}{6} R^6 \right), (5.10)
\]

and taking \(k = 2\sqrt{\pi} R\) it holds that \(f \in \overline{N}_k\). Hence, if \(R\) is big enough, it follows that \(g_0\) is not a global minimum in \(\overline{N}_k\). Furthermore, we see that the proof of Lemma 4.2 may be repeated in \(\overline{N}_k\), and then the minimum of \(J_h\) in \(\overline{N}_k\) is a \(\rho\)-critical point. From Theorem 5.1 there is a third \(\rho\)-critical point which is not a local minimum of \(J_h\).
6. Regularity

As we proved, problem (1.1) admits (for an appropriate \( k > 0 \)) a weak solution in a subset \( M(k) = \{ f \in T / \| \nabla (f - g) \|_\infty \leq k \} \).

Consider \( p > 2 \), and \( f_0 \in W^{2,p}(\Omega) \hookrightarrow C(\overline{\Omega}) \) a weak solution of (1.1). Then \( L_{f_0} f_0 = 2h(u, v, f_0)(1 + \nabla f_0^2)^{3/2} \) in \( \Omega \) where for any \( f \in C^1(\overline{\Omega}) \) \( L_f : W^{2,p} \rightarrow L^p \) is the strictly elliptic operator given by

\[
L_f \phi = (1 + f^2_v)\phi_{uu} + (1 + f^2_u)\phi_{vv} - 2f_u f_v \phi_{uv}.
\]

(6.1)

In order to prove the regularity of \( f_0 \), we study equation (6.2)

\[
L_{f_0} \phi = 2h(u, v, f_0)(1 + \nabla f_0^2)^{3/2} \text{ in } \Omega, \; \phi = g \text{ in } \partial\Omega.
\]

(6.2)

**Proposition 6.1.** Let us assume that \( \partial\Omega \in C^{2,\alpha}, g \in C^{2,\alpha}, \) and \( h \in C^\alpha \) for some \( 0 < \alpha \leq 1 - 2/p \). Then, if \( \phi \in W^{2,p} \) is a strong solution of (6.2), \( \phi \in C^{2,\alpha}(\overline{\Omega}) \).

**Proof.** By Sobolev imbedding \( \phi \in C^{1,\alpha}(\overline{\Omega}) \). Then \( L_{f_0} \phi \in C^\alpha(\overline{\Omega}) \) and the coefficients of the operator \( L_{f_0} \) belong to \( C^\alpha \). By Theorem 6.14 in [5], the equation \( Lw = L_{f_0} \phi \) in \( \Omega \), \( w = g \) in \( \partial\Omega \) is uniquely solvable in \( C^{2,\alpha}(\overline{\Omega}) \), and the result follows from the uniqueness in Theorem 9.15 in [5]. \( \square \)

**Remark 6.2.** As a simple consequence, we obtain that \( f_0 \in C^{2,\alpha}(\overline{\Omega}) \), by the uniqueness in \( W^{2,p} \) given by [5, Theorem 9.15].

**Corollary 6.3.** Let us assume that \( \partial\Omega \in C^{k+2,\alpha}, g \in C^{k+2,\alpha}, \) and \( h \in C^{k,\alpha} \) for some \( 0 < \alpha \leq 1 - 2/p \). Then \( f_0 \in C^{k+2,\alpha}(\overline{\Omega}) \).

**Proof.** It is immediate from Proposition 2.1 and Theorem 6.19 in [5]. \( \square \)

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