UNIFORM ASYMPTOTIC NORMAL STRUCTURE,
THE UNIFORM SEMI-OPIAL PROPERTY AND
FIXED POINTS OF ASYMPTOTICALLY REGULAR
UNIFORMLY LIPSCHITZIAN SEMIGROUPS. PART I

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Abstract. In this paper we introduce the uniform asymptotic normal structure and the uniform semi-Opial properties of Banach spaces. This part is devoted to a study of the spaces with these properties. We also compare them with those spaces which have uniform normal structure and with spaces with $WCS(X) > 1$.

1. Introduction

Normal structure is one of the basic concepts in metric fixed point theory. It was introduced by Brodskii and Milman [6] and applied in Kirk’s well-known fixed point theorem [24]. Asymptotic normal structure appeared for the first time in a paper by Baillon and Schöneberg [4] in which they generalized Kirk’s theorem. The semi-Opial property was considered in the context of the fixed point property in product spaces [25]. To study more carefully the geometric structure of Banach spaces Bynum [9] introduced the normal structure coefficient $N(X)$ which was applied by Casini and Maluta [10] to obtain a fixed point theorem for uniformly lipschitzian mappings. This result has been recently improved by Domínguez Benavides [15]. In his paper he used both $N(X)$ and the weakly convergent sequence coefficient $WCS(X)$ [9]. In the first part of the present paper we introduce new geometric coefficients: the asymptotic normal structure and the semi-Opial coefficients. In the second part of our paper we apply them to the fixed point theory of uniformly lipschitzian nonlinear semigroups.

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2. The asymptotic normal structure and the semi-Opial coefficients

Let \((X, \|\cdot\|)\) be a Banach space. As we mentioned in the Introduction, Bynum [9] introduced the coefficient \(N(X)\) related to normal structure. Namely, he defined \(N(X)\) as the biggest constant \(k\) such that

\[
k \cdot r(C) \leq \text{diam}(C)
\]

for each nonempty bounded convex set \(C \subset X\), where \(\text{diam}(C)\) denotes the diameter of \(C\) and \(r(C)\) is the Chebyshev radius of \(C\) with respect to itself, i.e.,

\[
r(C) = \inf_{y \in C} \sup_{x \in C} \|x - y\|.
\]

If \(\{x_n\}_{n \geq 1}\) is a bounded sequence in \((X, \|\cdot\|)\) and \(\{x_{n_i}\}_{i \geq 1}\) is a subsequence, then we denote by \(r_a(\{x_{n_i}\}_{i \geq 1})\) the asymptotic radius for the norm \(\|\cdot\|\) of this subsequence with respect to the set \(\text{conv} \left( \{x_n\}_{n \geq 1} \right)\) (the closure in the norm \(\|\cdot\|\) of the convex hull of the whole sequence \(\{x_n\}_{n \geq 1}\)), i.e.,

\[
r_a(\{x_{n_i}\}_{i \geq 1}) = \inf \left\{ r_a(x, \{x_{n_i}\}_{i \geq 1}) : x \in \text{conv} \left( \{x_n\}_{n \geq 1} \right) \right\}.
\]

Throughout this paper we will use the following notation:

\[
\alpha_k = \text{diam}_{\|\cdot\|} \left( \{x_n\}_{n \geq k} \right), \quad \text{diam}_a \left( \{x_n\} \right) = \lim_{k} \alpha_k = \alpha.
\]

One can consider (see [9] and [2]) the following weakly convergent sequence coefficient:

\[
\text{WCS}(X) = \sup \left\{ k : k \cdot r_a(\{x_n\}) \leq \text{diam} \left( \{x_n\} \right) \right\}
\]

for every weakly convergent sequence \(\{x_n\}\) in \(X\) =

\[
= \sup \left\{ k : k \cdot \limsup_{n} \|x_n\| \leq \text{diam} \left( \{x_n\} \right) \right\}
\]

for every weakly null sequence \(\{x_n\}\) in \(X\).

Let us observe that in the above definition of \(\text{WCS}(X)\), \(\text{diam} \left( \{x_n\} \right)\) can be replaced by \(\text{diam}_a \left( \{x_n\} \right)\) and that our definition is a little different from the one in common use.

We always have

\[
1 \leq N(X) \leq \text{WCS}(X),
\]

and for some Banach spaces (see e.g. [15]) the strict inequalities

\[
1 < N(X) < \text{WCS}(X)
\]

are valid.

Recall that a bounded sequence \(\{x_n\}_{n \geq 1}\) with \(x_n - x_{n+1} \to 0\) is called asymptotically regular.
We say that $X$ has asymptotic normal structure (with respect to the weak topology) \cite{4}, $\text{ANS}$ (respectively, $w\text{-ANS}$) for short, if for each bounded closed (weakly compact) and convex subset $C$ of $X$ consisting of more than one point and each asymptotically regular sequence $\{x_n\}$ in $C$, there is a point $x \in C$ such that

$$\liminf_n \|x - x_n\| < \text{diam}(C)$$

(see also \cite{1, 2, 7, 8, 19, 20, 26, 30, 36}).

Recall that a Banach space is said to have the semi-Opial (weak semi-Opial) property \cite{8, 25}, $\text{SO}$ ($w\text{-SO}$) for short, if for each bounded nonconstant asymptotically regular sequence $\{x_n\}$ (with a weakly compact convex hull), there exists a subsequence $\{x_{n_i}\}$, weakly convergent to $x$, such that

$$\liminf_i \|x - x_{n_i}\| < \text{diam}(\{x_n\}).$$

Let us observe that in Examples 1 and 5 on page 461 in \cite{25} the authors use, in fact, the weak semi-Opial property. Similarly in Theorem 4 in \cite{25} we can assume that $(X_2, \|\|)$ has the weak semi-Opial property.

A Banach space $X$ is said to satisfy the Opial condition \cite{32} (respectively, the nonstrict Opial condition \cite{22}) if whenever a sequence $\{x_n\}$ in $X$ converges weakly to $x$, then

$$\liminf_n \|x - x_n\| < \liminf_n \|y - x_n\|$$

$$\left(\liminf_n \|x - x_n\| \leq \liminf_n \|y - x_n\|\right)$$

for every $y \in X \setminus \{x\}$.

For more information about the connections between the above mentioned geometric properties of Banach spaces (and other ones) see \cite{1, 2, 3, 13, 14, 18, 19, 20, 27, 29, 33, 34, 35, 37, 38, 39, 40}.

We now define the asymptotic normal structure coefficient by

$$\sup \left\{ k : k \cdot \inf \left\{ \left. r_a \left( \{x_n\}_{i \geq 1} \right) \right\mid \{x_n\}_{i \geq 1} \leq \text{diam}_a (\{x_n\}) \right\} \right\}$$

for each bounded sequence $\{x_n\}_{n \geq 1}$ with $x_n - x_{n+1} \to 0$.

We denote it by $\text{AN}(X)$.

If in the definition of $\text{AN}(X)$ we add the condition that the sequence $\{x_n\}_{n \geq 1}$ has a weakly compact convex $\text{conv} \left( \{x_n\}_{n \geq 1} \right)$, then we get the asymptotic normal structure coefficient with respect to the weak topology, $w\text{-AN}(X)$, for short. In other words,

$$w\text{-AN}(X) = \sup \left\{ k : k \cdot \inf \left\{ \left. r_a \left( \{x_n\}_{i \geq 1} \right) \right\mid \{x_n\}_{i \geq 1} \leq \text{diam}_a (\{x_n\}) \right\} \right\}$$

for each sequence $\{x_n\}_{n \geq 1}$ such that
The semi-Opial coefficient with respect to the weak topology, \( w-SOC \) for short, is defined as follows:

\[
w-SOC (X) = \sup \left\{ k : k \cdot \inf_{\{x_n\}_{i \geq 1}, x_n \rightharpoonup y} r_a(y, \{x_n\}_{i \geq 1}) \leq \text{diam}_a(\{x_n\}) \right\}
\]

for each sequence \( \{x_n\}_{n \geq 1} \) such that \( \text{conv}(\{x_n\}_{n \geq 1}) \) is weakly compact and \( x_n - x_{n+1} \to 0 \).

If \( AN(X) > 1 \), then we say that \( (X, \|\cdot\|) \) has uniform asymptotic normal structure, \( UAN \) for short. If \( w-AN(X) > 1 \), then we say that \( (X, \|\cdot\|) \) has uniform asymptotic normal structure with respect to the weak topology \( (w-UAN) \). Similarly, if \( w-SOC(X) > 1 \), then \( (X, \|\cdot\|) \) has the uniform semi-Opial property with respect to the weak topology \( (w-USO) \).

Directly from the above definitions we get

\[
1 \leq AN(X) \leq w-AN(X) \leq WCS(X) \leq w-SOC(X) \leq w-AN(X),
\]

(1)

We do not know if \( w-AN(X) \) is different from \( w-SOC(X) \), but we will present an example of a Banach space with \( 1 < WCS(X) < w-SOC(X) \) (Example 6.2). There are Banach spaces which have asymptotic normal structure but lack \( UAN \), and there are also Banach spaces with \( 1 = AN(X) < w-AN(X) \) (Example 6.1).

**Proposition 2.1.** In the definitions of \( w-AN(X) \) and \( w-SOC(X) \) we can replace \( \text{diam}_a(\{x_n\}) \) by \( \text{diam}(\{x_n\}) \).

Proof. Let us observe that in the above definition every asymptotically regular sequence \( \{x_n\} \) can be replaced by \( \{x_n\}_{n \geq m} \) with arbitrary \( m \). This yields the claimed statement. \( \blacksquare \)

**Theorem 2.1.** If a Banach space \( (X, \|\cdot\|) \) has \( AN(X) > 1 \), then it is reflexive.

Proof. It is sufficient to recall the following result of D.P. Milman and V.D. Milman [31]: If a Banach space \( (X, \|\cdot\|) \) is not reflexive, then for each \( \epsilon > 0 \) there exists a sequence \( \{y_n\} \) with the following properties:

1. \( \|y_n\| = 1 \) for \( n = 1, 2, ...; \)
2. \( 1 + \epsilon \geq \|z_{1j} - z_{j\omega}\| \geq 1 - \epsilon \) for each \( j = 1, 2, ... \) and for each \( z_{1j} \in \text{conv}(\{y_n\}_{n=1}^j) \) and \( z_{j\omega} \in \text{conv}(\{y_n\}_{n=j+1}^\infty) \);
3. \( 1 - \epsilon \leq \|z_{1j}\| \leq 1 \) and \( 1 - \epsilon \leq \|z_{j\omega}\| \leq 1 \) for each \( z_{1j} \in \text{conv}(\{y_n\}_{n=1}^j) \) and \( z_{j\omega} \in \text{conv}(\{y_n\}_{n=j+1}^\infty) \).
Hence, if a Banach space \((X, \|\cdot\|)\) is not reflexive and \(\epsilon > 0\), then we can choose elements \(y_n\) which satisfy the above conditions 1.-3. and next we construct an asymptotically regular sequence \(\{x_n\}\) by dividing every segment \([y_n, y_{n+1}]\) into \(2^n\) equal subsegments and taking their endpoints as subsequent elements of \(\{x_n\}\). For this bounded sequence \(\{x_n\}_{n \geq 1}\) we have \(x_n - x_{n+1} \to 0\) and \(\frac{1+\epsilon}{1-\epsilon} r_a(y, \{x_n\}_{i \geq 1}) \geq \text{diam}_a(\{x_n\})\) for each subsequence \(\{x_{n_i}\}_{i \geq 1}\) and \(y \in \overline{\text{conv}} \{x_n\}\).

**Remark 2.1.** The condition \(w-AN(X) > 1\) implies the weak fixed point property for nonexpansive mappings as a consequence of the Baillon-Schöenberg theorem (see also [7]).

**Theorem 2.2.** i) If a Banach space \((X, \|\cdot\|)\) has \(N(X) > 1\), then \(w-SOC(X) > 1\).

ii) If a Banach space \((X, \|\cdot\|)\) has the nonstrict Opial property, then \(w-SOC(X) = w-AN(X)\).

*Proof. i)* If \(N(X) > 1\), then \(X\) is reflexive [29], and

\[1 < N(X) \leq WCS(X) \leq w-SOC(X)\]

(see (1) and [33]).

ii) We get this equality directly from the definition of the nonstrict Opial property.

**Remark 2.2.** There exist \(w-USO\) spaces without the nonstrict Opial property. For example, \(L^p([0, 2\pi])\) with \(1 < p < \infty\) and \(p \neq 2\) is such a space. It has uniform normal structure, and thus (see point i) in the above theorem) it is \(w-USO\), but it does not satisfy the nonstrict Opial condition [32].

We finish this section by showing the stability of the uniform asymptotic normal structure and the uniform semi-Opial properties.

**Theorem 2.3.** Let \((X_1, \|\cdot\|_1)\) and \((X_2, \|\cdot\|_2)\) be isomorphic Banach spaces and let \(d(X_1, X_2)\) be the Banach-Mazur distance between them. Then we have

\[AN(X_1) \leq d(X_1, X_2) \cdot AN(X_2),\]

\[w-AN(X_1) \leq d(X_1, X_2) \cdot w-AN(X_2),\]

and

\[w-SOC(X_1) \leq d(X_1, X_2) \cdot w-SOC(X_2).\]

*Proof. All the inequalities have similar proofs. For example, we prove the third one:

\[w-SOC(X_1) \leq d(X_1, X_2) \cdot w-SOC(X_2).\]

Let \(\{x_n\}\) be asymptotically regular in \(X_2\), and let \(\overline{\text{conv}} \{x_n\}\) be weakly compact. Let \(T : X_2 \to X_1\) be an isomorphism and assume \(0 < k < w-SOC(X_1)\). Then there exists a weakly convergent to \(y\) subsequence \(\{Tx_{n_i}\}\) such that

\[kr_a(T^{-1}y, \{x_{n_i}\}_{i \geq 1}) \leq k \|T^{-1}\| r_a(y, \{Tx_{n_i}\}_{i \geq 1}).\]
\[ \leq \|T^{-1}\| \cdot \text{diam}_a(\{Tx_n\}) \leq \|T^{-1}\| \cdot \|T\| \cdot \text{diam}_a(\{x_n\}). \]

Hence we get

\[ \frac{k}{\|T^{-1}\| \cdot \|T\|} \leq w-\text{SOC}(X_2) \]

which yields the claimed inequality. \(\blacksquare\)

**Remark 2.3.** Theorem 2.3 can be understood as a stability result for the weak fixed point property for nonexpansive mappings. This means that if \(w-\text{AN}(X_1) > 1\) and \(d(X_1,X_2) < w-\text{AN}(X_1)\), then \(w-\text{AN}(X_2) > 1\), and by Remark 2.1 the space \(X_2\) also has the weak fixed point property for nonexpansive mappings.

### 3. Connections between asymptotically regular sequences and properties of Banach spaces

It is natural to ask, when either \(\text{AN}(X)\) or \(w-\text{SOC}(X)\) is equal to \(\infty\). The following theorem gives the answer.

**Theorem 3.1.** i) \(\text{AN}(X) = \infty\) if and only if \((X,\|\cdot\|)\) is finite dimensional.

ii) \(w-\text{SOC}(X) = \infty\) if and only if \((X,\|\cdot\|)\) is a Schur space.

iii) \(w-\text{AN}(X) = \infty\) if and only if \((X,\|\cdot\|)\) is a Schur space.

**Proof.** i) The equality \(\text{AN}(X) = \infty\) is equivalent to the following

\[
\sup \left\{ \inf_{\{x_{n_i}\}_{i \geq 1}} r_a(\{x_{n_i}\}_{i \geq 1}) : \{x_n\}_{n \geq 1} \text{ is bounded and } x_n - x_{n+1} \to 0 \right\} = 0
\]

If \(X\) is finite dimensional, then the above equality is obvious.

When \(X\) is infinite dimensional, then by the Riesz Lemma [12] there exists a sequence \(\{y_n\}\) such that

\[ \|y_n\| = 1 \quad \text{for } n=1,2,... \]

and for \(n = 1, 2, ...\)

\[ \|y - y_{n+1}\| \geq 1 - \frac{1}{n+1} \quad \text{for every } y \in \text{lin} \{y_1, y_2, ..., y_n\}. \]

Let us observe that

\[ \lim_{n} \inf \|x - y_n\| \geq 1 \]

for each \(x \in \text{lin} \{y_n\}\). Now we construct a new sequence \(\{x_n\}\) in the following way. We divide each segment \([y_n, y_{n+1}]\) into \(2^n\) equal parts and take the endpoints as subsequent elements of \(\{x_n\}\). This sequence satisfies \(x_n - x_{n+1} \to 0\). We will show that for each \(x \in \text{lin} \{x_n\}\) we have

\[ \inf_{\{x_{n_i}\}_{i \geq 1}} r_a(x,\{x_{n_i}\}_{i \geq 1}) \geq \frac{1}{4}. \]

Indeed, every \(x_n\) can be written in the following way:

\[ x_n = \alpha_n y_{k(n)} + (1 - \alpha_n) y_{k(n)+1}, \]
where $0 \leq \alpha_n \leq 1$. If we choose any subsequence $\{x_{n_i}\}_{i \geq 1}$, then without loss of generality we may assume that $\alpha_{n_i} \to \alpha$. It is obvious that $k(n) \to \infty$ and therefore $k(n_i) \to \infty$ too.

First we claim that for $0 \leq \alpha \leq \frac{3}{4}$ and for each $x \in \text{lin} \{x_n\} = \text{lin} \{y_n\}$ we have

$$\liminf \|x - x_{n_i}\| \geq \frac{1}{4}. \quad (3)$$

Indeed, for such an $\alpha$ we get

$$\liminf \|x - x_{n_i}\| = \liminf \|x - \alpha_{n_i} y_{k(n_i)} - (1 - \alpha_{n_i}) y_{k(n_i)+1}\|$$

$$\geq \lim \left(1 - \alpha_{n_i}\right) \left(1 - \frac{1}{k(n_i) + 1}\right) = 1 - \alpha \geq \frac{1}{4}. \quad (4)$$

Next we obtain

$$\liminf \|x - x_{n_i}\| = \liminf \|x - \alpha_{n_i} y_{k(n_i)} - (1 - \alpha_{n_i}) y_{k(n_i)+1}\|$$

$$\geq \liminf \left|\|x - \alpha_{n_i} y_{k(n_i)}\| - \lim \|1 - \alpha_{n_i}\) y_{k(n_i)+1}\| \right|$$

$$\geq \lim \alpha_{n_i} \left(1 - \frac{1}{k(n_i)}\right) = \lim (1 - \alpha_{n_i}) = 2\alpha - 1 \geq \frac{1}{2}$$

for $\frac{3}{4} \leq \alpha \leq 1$ and for each $x \in \text{lin} \{x_n\}$.

Hence (3) and (4) imply that the inequality (2) is valid and therefore

$$\sup \left\{ \inf_{\{x_{n_i}\}_{n \geq 1}} r_a \left(\{x_{n_i}\}_{i \geq 1}\right) : \{x_n\}_{n \geq 1} \text{ is bounded and } x_n - x_{n+1} \to 0 \right\}$$

$$\geq \frac{1}{4}. \quad \text{(5)}$$

This means that $AN(X) < \infty$.

ii) If $(X, \|\|)$ is a Schur space [12], then the following equality

$$\sup \left\{ \inf_{\{x_{n_i}\}_{n \geq 1}} r_a \left(y, \{x_{n_i}\}_{i \geq 1}\right) : \{x_{n_i}\}_{i \geq 1} \text{ is weakly convergent and } \right. \quad \text{(6)}$$

$$y = \text{w-lim } x_{n_i} : \{x_n\}_{n \geq 1} \text{ with a weakly compact } \overline{\text{conv}} \left(\{x_n\}_{n \geq 1}\right) \text{ and }$$

$$x_n - x_{n+1} \to 0 \right\} = 0$$

is obvious.

Let us assume that $(X, \|\|)$ is not a Schur space. We will show

$$\sup \left\{ \inf_{\{x_{n_i}\}_{n \geq 1}} r_a \left(y, \{x_{n_i}\}_{i \geq 1}\right) : \{x_{n_i}\}_{i \geq 1} \text{ is weakly convergent and } \right. \quad \text{(7)}$$

$$y = \text{w-lim } x_{n_i} : \{x_n\}_{n \geq 1} \text{ with a weakly compact } \overline{\text{conv}} \left(\{x_n\}_{n \geq 1}\right) \text{ and }$$

$$x_n - x_{n+1} \to 0 \right\} \geq \frac{1}{8}. \quad \text{(8)}$$
In $X$ there exists a weakly null sequence $\{y_n\}$ with $\|y_n\| = 1$, $n = 1, 2, \ldots$. Therefore, we can choose a subsequence $\{y_{n_k}\}$ such that for every $y \in [y_{n_k}, y_{n_k+1}]$ we have $\|y\| \geq \frac{1}{2}$. Indeed, we take $y_{n_1} = y_0$ and next if we have chosen $y_{n_1}, \ldots, y_{n_k}$, then we take $n_{k+1} > n_k$ so large that

$$\|(1 - \alpha) y_{n_k} + \alpha y_{n_k+1}\| \geq \frac{1}{8}$$

for every $0 \leq \alpha \leq \frac{3}{4}$. This is possible because by the lower semicontinuity of $\|\cdot\|$ with respect to the weak topology we get

$$\liminf_n \|(1 - \alpha) y_{n_k} + \alpha y_n\| \geq (1 - \alpha) \|y_n\| \geq \frac{1}{4}.$$

Now for this $y_{n_k+1}$ and each $\frac{3}{4} \leq \alpha \leq 1$ we also have

$$\|(1 - \alpha) y_{n_k} + \alpha y_{n_k+1}\| \geq \|\alpha y_{n_k+1}\| - \|(1 - \alpha) y_{n_k}\| = 2\alpha - 1 \geq \frac{1}{2}.$$ 

Now we construct an asymptotically regular sequence $\{x_n\}$ by dividing every segment $[y_{n_k}, y_{n_k+1}]$ into $2^n$ equal parts and then taking the endpoints as subsequent elements of $\{x_n\}$. It is obvious that

$$\inf_{\{x_n\}_{i \geq 1}} r_a\left(0, \{x_n\}_{i \geq 1}\right) \geq \frac{1}{8}$$

and the proof is complete.

iii) Assume that $X$ is not Schur. We will show that

$$\sup \left\{ \inf_{\{x_n\}_{i \geq 1}} r_a\left(\{x_n\}_{i \geq 1}\right) : \{x_n\}_{n \geq 1} \text{ with a weakly compact} \right\} > 0.$$ 

We use the asymptotically regular sequence $\{x_n\}$ constructed in the proof of ii). We know that $\|x_n\| \geq \frac{1}{8}$ for each $n$ and that $w\text{-}\lim x_n = 0$. Now we prove that

$$\liminf_n \|x - x_n\| \geq \frac{1}{16}$$

for every $x \in X$. Indeed, if $\|x\| \geq \frac{1}{16}$, then by the lower semicontinuity of $\|\cdot\|$ with respect to the weak topology we get

$$\liminf_n \|x - x_n\| \geq \|x\| \geq \frac{1}{16}.$$ 

On the other hand for $\|x\| \leq \frac{1}{16}$ we obtain

$$\liminf_n \|x - x_n\| \geq \liminf_n (\|x_n\| - \|x\|) \geq \frac{1}{16},$$

and this completes the proof. \(\blacksquare\)

We end this section with a characterization of reflexive spaces by asymptotically regular sequences.
Theorem 3.2. A Banach space \((X, \|\cdot\|)\) is reflexive if and only if every asymptotically regular sequence has a weakly convergent subsequence.

Proof. It is known [11] that in reflexive spaces each bounded sequence has a weakly convergent subsequence.

Let us now assume that in the Banach space \((X, \|\cdot\|)\) every asymptotically regular sequence has a weakly convergent subsequence. To get the reflexivity of \((X, \|\cdot\|)\) it is sufficient to prove ([11]) that each decreasing sequence \(\{C_n\}\) of nonempty, bounded, closed and convex sets has a nonempty intersection. Without loss of generality we can assume that \(\text{diam} (C_n) > 0\) for each \(n\). Now we choose \(y_n\) from each set \(C_n\) and next we construct an asymptotically regular sequence \(\{x_n\}\) by dividing every segment \([y_n, y_{n+1}]\) into \(2^n\) equal parts and the taking the endpoints as subsequent elements of \(\{x_n\}\). This sequence contains a weakly convergent subsequence \(\{x_{n_i}\}\). Its weak limit is a common element of \(C_n\) for \(n = 1, 2, \ldots\) 

Remark 3.1. A proof similar to the above one was used in [8] to prove that every Banach space with the SO property is reflexive.

4. ON THE 3-SPACE PROBLEM

In this section we consider the following problem: When can the uniform asymptotic normal structure property or the uniform semi-Opial property be extended from a subspace to the whole space? Two slightly different approaches to the solution of this problem will be demonstrated in the following theorems.

Theorem 4.1. Suppose that \(X = W \oplus Z\), where \(W\) is a closed subspace of \(X\), \(Z\) is a Schur space, and the projection onto \(W\) has norm 1. Then we have \(w\text{-SOC} (X) = w\text{-SOC} (W)\).

Proof. Suppose \(\{x_n\} = \{w_n + z_n\}\) is an asymptotically regular sequence, \(w_n \in W, z_n \in Z\) for \(n = 1, 2, \ldots\) and \(\text{conv} \{x_n\}\) is weakly compact. For each \(k < w\text{-SOC} (W)\) we find a subsequence \(\{x_{n_i}\}\) such that

\[
x_{n_i} = w_{n_i} + z_{n_i} \rightharpoonup w + z, \quad w \in W, z \in Z
\]

and

\[
k \lim_i \|w_{n_i} - w\| \leq \text{diam}_{a} \{w_n\}.
\]

Then we have \(w + z \in \text{conv} \{x_n\}, z_{n_i} \rightarrow z\) and

\[
k \lim_i \|w_{n_i} + z_{n_i} - w - z\| = k \lim_i \|w_{n_i} - w\| \leq \text{diam}_{a} \{w_n\}.
\]

Now, since the projection on \(W\) is of norm 1, we obtain

\[
k \lim_i \|w_{n_i} + z_{n_i} - w - z\| \leq \text{diam}_{a} \{w_n\} \leq \text{diam}_{a} \{x_n\}
\]

and therefore \(w\text{-SOC} (X) = w\text{-SOC} (W)\). 

Now we consider the Cartesian product of two spaces.
Theorem 4.2. Let \((X_1, \|\cdot\|_1)\) and \((X_2, \|\cdot\|_2)\) be Banach spaces. If \((X_1, \|\cdot\|_1)\)
is w-USO and \((X_2, \|\cdot\|_2)\) has WCS \((X_2) > 1\), then \(X_1 \times X_2\) equipped with
the \(l_p\)-norm \(\|\cdot\| = (\|\cdot\|_1^p + \|\cdot\|_2^p)^{\frac{1}{p}}\) \(1 \leq p < \infty\) is also w-USO.

Proof. Let \(0 < \theta < 1\) be such that \(\frac{1}{\theta} < \min (w-SOC (X_1), \text{WCS} (X_2))\). Let
us take an arbitrary asymptotically regular sequence \(\{x_n\} = \{(x_{1n}, x_{2n})\}\) in
\((X_1 \times X_2, \|\cdot\|)\) with a weakly compact \(\text{cl} \{x_n\}\). Then \(\{x_{1n}\}\) is also asymptotically
regular in \((X_1, \|\cdot\|_1)\) and we can choose a subsequence \(\{x_{ni}\}\) such that
\(\{x_{ni}\}\) tends weakly to \((x_1, x_2)\) (see [13, 17, 33, 40]) and

\[
\begin{align*}
\bar{d} &= \text{diam}_{a} \{x_n\} \\
&\geq \text{diam}_{a} \{x_{ni}\} = \lim_{i, k \to \infty} \|x_{ni} - x_{nk}\| = \lim_{i \to \infty} \lim_{k \to \infty} \|x_{ni} - x_{nk}\| \, , \\
r &= \lim_{i \to \infty} \|x_{ni} - x\| , \\
d_1 &= \text{diam}_{a} \{x_{1n}\} \\
\bar{d}_1 &= \text{diam}_{a} \{x_{1ni}\} = \lim_{i, k \to \infty} \|x_{1ni} - x_{1nk}\| = \lim_{i \to \infty} \lim_{k \to \infty} \|x_{1ni} - x_{1nk}\|_1 \, , \\
r_1 &= \lim_{i \to \infty} \|x_{1ni} - x_{1ni}\|_1 \leq \theta d_1 , \\
\bar{d}_2 &= \text{diam}_{a} \{x_{2ni}\} = \lim_{i, k \to \infty} \|x_{2ni} - x_{2nk}\|_2 = \lim_{i \to \infty} \lim_{k \to \infty} \|x_{2ni} - x_{2nk}\|_2 \, , \\
r_2 &= \lim_{i \to \infty} \|x_{2ni} - x_{2ni}\|_2 .
\end{align*}
\]

Let us observe that

\[
\begin{align*}
d_1 &\leq d , \\
r^p &= r_1^p + r_2^p \leq \bar{d}_1^p + \bar{d}_2^p \leq d^p , \\
r_1 &\leq \theta d_1 \leq \theta d \\
\text{and} \\
r_2 &\leq \theta d_2 .
\end{align*}
\]

Now we have to consider two possibilities: either

\[
r_1^p + \bar{d}_2^p \leq \frac{1 + 3\theta^p}{4} d^p \\
or
\]

\[
r_1^p + \bar{d}_2^p \geq \frac{1 + 3\theta^p}{4} d^p .
\]

For the first possibility we obtain

\[
(6) \quad r^p = r_1^p + r_2^p \leq r_1^p + \bar{d}_2^p \leq \frac{1 + 3\theta^p}{4} d^p .
\]

For the second possibility we have

\[
\bar{d}_2^p \geq \frac{1 + 3\theta^p}{4} d^p - r_1^p \geq \frac{1 + 3\theta^p}{4} d^p - \theta d^p = \frac{1 - \theta^p}{4} d^p .
\]
by (5) and therefore we get
\begin{align}
    r^p &= r_1^p + r_2^p \\
    &\leq r_1^p + \theta^p r_2^p \\
    &\leq \overline{d}_1^p + \overline{d}_2^p - (1 - \theta^p) \overline{d}_2^p \\
    &\leq d^p - \frac{(1 - \theta^p)^2}{4} d^p \\
    &= \left[ 1 - \frac{(1 - \theta^p)^2}{4} \right] d^p.
\end{align}

Finally, inequalities (6) and (7) imply
\begin{align}
    r &\leq \max \left\{ \left[ \frac{\frac{1}{p} + 3\theta^p}{4} \right]^\frac{1}{p}, \left[ 1 - \frac{(1 - \theta^p)^2}{4} \right]^\frac{1}{p} \right\} d \\
    &\leq \frac{1}{2} \left[ 1 - \frac{(1 - \theta^p)^2}{4} \right]^\frac{1}{p} d.
\end{align}

This completes the proof. \(\blacksquare\)

5. The space \(X^p_\beta\) and its w-SOC

In this section we give an example of a space with \(N(X) < AN(X) < w-SOC(X)\). To this end, let us consider \(l^p\) with the norm
\[\|x\| = \max \left\{ \|x\|, \frac{\|x\|_p}{\beta} \right\},\]
where \(p > 1, 1 < \beta < +\infty, \|x\|_\infty = \max \{|x(j)| : j = 1, 2, \ldots\}\) and
\[\|x\|_p = \left( \sum_{j=1}^{\infty} |x(j)|^p \right)^{\frac{1}{p}}.\]
We denote this space by \(X^p_\beta\). The space \(X^2_\beta\) was introduced by R.C. James [5]. This is essentially the space which has been discussed in various places in the literature, e.g., [1, 2, 4, 5, 7, 8, 10, 15, 16, 19, 20, 21, 22, 23, 25, 26, 28, 39].

For the convenience of the reader we recall the notations from Section 2. If \(\{x_n\}_{n \geq 1}\) is a bounded sequence in \((l^p, \|\cdot\|)\) and \(\{x_{n_i}\}_{i \geq 1}\) is a subsequence, then \(r_d(\{x_{n_i}\}_{n_i \geq 1})\) denote the asymptotic radius of this sequence with respect to the set \(\text{conv}(\{x_n\}_{n \geq 1})\) in the norm \(\|\cdot\|\). We also have
\[\alpha_k = \text{diam}_{\|\cdot\|}(\{x_n\}_{n \geq k})\quad \text{and}\quad \text{diam}_a(\{x_n\}) = \lim_k \alpha_k = \alpha.\]

Let us observe that for each \(n \in \mathbb{N}\) and for each \(y \in C\) there exists an index \(j_{n,y}\) (we fix it here for every pair \(n,y\)) such that
\[\|x_n - y\|_\infty = \max \{|x_n(j) - y(j)| : j = 1, 2, \ldots\}\]
\((x_n = (x_n(j)))_{j \geq 1}\) and \(y = (y(j))_{j \geq 1}\).

The space \(X^p_\beta\) has the nonstrict Opial property [22].
Theorem 5.1. If a sequence \( \{x_n\}_{n \geq 1} \) is bounded and \( x_n - x_{n+1} \to 0 \), then

\[
\inf_{x \in \text{conv}(\{x_n\}_{n \geq 1})} \left( \liminf_n \|x_n - x\| \right) = \inf_{\{x_n\}} \left[ r_a(\{x_n\}_{i \geq 1}) \right] \\
\leq \min \left[ 1, \max \left( 2^{-\frac{1}{p}}, \frac{\beta}{4^\frac{1}{p}} \right) \right] \cdot \text{diam}_a(\{x_n\})
\]

and this constant is the best possible. Therefore

\[
w \cdot \text{SOC} \left( X_\beta^p \right) = \max \left[ 1, \min \left( 2^\frac{1}{p}, \frac{4^\frac{1}{p}}{\beta} \right) \right].
\]

Proof. We begin our proof in the case \( 1 < \beta < \frac{1}{2^p} \).

Let

\[
C = \text{conv}(\{x_n\}_{n \geq 1}) \quad \text{and} \quad C_k = \text{conv}(\{x_n\}_{n \geq k}), \quad k = 1, 2, \ldots.
\]

Clearly \( \text{diam}C_k = \alpha_k \). Let us observe the following fact. For every subsequence \( \{x_{n_l}\} \) which is weakly convergent to \( y \) we have (\([2, 39]\))

\[
\lim_i \|y - x_{n_l}\|_p \leq \limsup_i \|y - x_{n_l}\|_p \leq \frac{1}{2^p} \text{diam}_a(\|\cdot\|_p)(\{x_{n_l}\}) \leq \lim_k \frac{1}{2^p} \text{diam}_a(\|\cdot\|_p)C_k \leq \frac{\beta}{2^p} \text{diam}_a(\|\cdot\|_p)C_k = \frac{\beta}{2^p} \alpha.
\]

Next choosing in an arbitrary way a subsequence \( \{x_{n_l}\} \) such that

\[
\lim_i \|y - x_{n_l}\|_\infty = \lim_l \|y - x_{n_l}\|_\infty,
\]

we get

\[
\lim_l \inf \|y - x_{n_l}\| \leq \liminf_l \max \left\{ \|y - x_{n_l}\|_\infty, \frac{\|y - x_{n_l}\|_p}{\beta} \right\} \leq \max \left\{ \lim_l \|y - x_{n_l}\|_\infty, \limsup_l \frac{\|y - x_{n_l}\|_p}{\beta} \right\} \leq \max \left\{ \lim_l \|y - x_{n_l}\|_\infty, \frac{\alpha}{2^p} \right\} = \max \left\{ \liminf_i \|y - x_{n_i}\|_\infty, \frac{\alpha}{2^p} \right\}.
\]
Now we can begin the proof of our inequality (9). For $1 < \beta < 4^{\frac{1}{p}}$ it reduces to
\[
\inf_{x \in \overline{\text{conv}}(\{x_n\}_{n \geq 1})} \left( \liminf_n \|x_n - x\| \right) \leq \max \left( 2^{\frac{1}{p}}, \frac{\beta}{4^{\frac{1}{p}}} \right) \cdot \text{diam}_a (\{x_n\}).
\]

Without loss of generality we can assume that $\alpha > 0$, and this implies that for each $k$ we have $\alpha_k \geq \alpha > 0$. Suppose that for some asymptotically regular sequence $\{x_n\}$ and for some $t$ with $\max \left( \frac{1}{2^{\frac{1}{p}}}, \frac{\beta}{4^{\frac{1}{p}}} \right) < t < 1$ the following inequality is valid:
\[
\inf_{x \in C} \left( \liminf_n \|x_n - x\| \right) > t \alpha.
\]
We will try to reach a contradiction. For our $t$ there exists $\epsilon > 0$ such that the inequalities
\[
\epsilon < t \alpha \quad \text{and} \quad \frac{\beta^p}{2^{p} \alpha^p} + \epsilon < 2 \left( t \alpha - \epsilon \right)^p
\]
are valid.

Let us take an arbitrary subsequence $\{x_{n_i}\}$ which converges weakly to some $y$. Directly from the definition of the norm $\|\cdot\|$, by (11) and by $t > \frac{1}{2^{\frac{1}{p}}}$, we have
\[
t \alpha < \liminf_n \|x_n - y\| \leq \liminf_i \|x_{n_i} - y\|
\leq \max \left\{ \liminf_i \|x_{n_i} - y\|_{\infty}, \frac{\alpha}{2^{\frac{1}{p}}} \right\} = \liminf_i \|x_{n_i} - y\|_{\infty}
\leq \liminf_i \max \left\{ \|x_{n_i} - y\|_{\infty}, \frac{\|x_{n_i} - y\|_p}{\beta} \right\} = \liminf_i \|x_{n_i} - y\|,
\]
which implies
\[
\liminf_i \|x_{n_i} - y\|_{\infty} = \liminf_i \|x_{n_i} - y\| > t \alpha.
\]

By formulas (8) and (14) and because $\{x_{n_i}\}$ tends weakly to $y$ we get
\[
\lim_{i} j_{n_i, y} = +\infty.
\]

Using (8, 14, 15) and $\lim_n \|x_n - x_{n+1}\| = 0$ we can find $\tilde{n}$ and $\tilde{i}$, $\tilde{i} \geq \tilde{n}$, such that for $n \geq \tilde{n}$ and $i \geq \tilde{i}$ we have
\[
\|x_n - x_{n+1}\|_{\infty} \leq \frac{\epsilon}{3},
\]
\[
\|x_{n_i} - y\|_{\infty} = |x_{n_i} (j_{n_i, y}) - y (j_{n_i, y})| > t \alpha,
\]
and
\[
|y (j)| \leq \frac{\epsilon}{3}
\]
for each \( j \geq \min_{i \geq 1} j_{n_i,y} \). Therefore (17) and (18) yield
\[
\|x_{n_i}\| \geq \|x_{n_i}\|_{\infty} \geq |x_{n_i}(j_{n_i,y})| \geq t\alpha - \frac{\epsilon}{3}
\]
for \( i \geq \tilde{i} \).

Now let us return to the sequence \( \{x_{n_i}\} \) and set
\[
j_n = \begin{cases} \max \{ j : |x_n(j)| \geq t\alpha - \frac{\epsilon}{3} \} & \text{if there exists } j \\ \max \{ j : |x_n(j)| = \|x_n\|_{\infty} \} & \text{otherwise.} \end{cases}
\]

We claim that
\[
\lim_n j_n = +\infty.
\]
If this were false, then there would exist a subsequence \( \{j_{n_i}\} \). We could then choose a subsequence \( \{x_{n_i}\} \) which tends weakly to its weak limit \( y \). In this case (see formulas (19) and (20)) we have
\[
j_{n_i,y} \leq j_{n_i}
\]
for all \( l \) greater than or equal to some \( \tilde{l} \) and therefore (see (15)) \( \lim j_{n_i} = +\infty \) which contradicts our assumption.

Since \( \lim_n j_n = +\infty \) (see (21)), there exists a subsequence \( \{n_i\} \) such that \( \{x_{n_i}\} \) converges weakly to some \( y \) and
\[
j_i < j_{i+1}
\]
for \( i = 1, 2, \ldots \). Since \( \|x_n - x_{n+1}\| \to 0 \) we get \( x_{n+1} \rightharpoonup y \) and therefore for \( i \geq \tilde{i} \) (see formulas (16, 17, 18, 19, 20)) we get
\[
|\alpha_{n+1}(j_{n+1,y}) - y(j_{n+1,y})| = \|x_{n+1} - y\|_{\infty}
\]
\[
\geq \|x_{n_i} - y\|_{\infty} - \|x_{n_i} - x_{n+1}\|_{\infty} \geq t\alpha - \frac{\epsilon}{3},
\]
\[
|\alpha_{n+1}(j_{n+1,y}) - y(j_{n_i,y})|
\]
\[
\geq |\alpha_{n_i}(j_{n_i,y}) - y(j_{n_i,y})| - |\alpha_{n_i}(j_{n_i,y}) - \alpha_{n+1}(j_{n_i,y})|
\]
\[
\geq \|x_{n_i} - y\|_{\infty} - \|x_{n_i} - x_{n+1}\|_{\infty} \geq t\alpha - \frac{\epsilon}{3},
\]
and
\[
|\alpha_{n+1}(j_{n+1}) - y(j_{i+1})| \geq |\alpha_{n+1}(j_{n+1})| - |y(j_{n+1})|
\]
\[
\geq \min \left\{ t\alpha - \frac{\epsilon}{3}, \|x_{n+1}\|_{\infty} \right\} - |y(j_{n+1})|
\]
\[
\geq \min \left\{ t\alpha - \frac{\epsilon}{3}, \|x_{n_i}\|_{\infty} - \|x_{n_i} - x_{n+1}\|_{\infty} \right\} - |y(j_{n+1})|
\]
\[
\geq t\alpha - \frac{2\epsilon}{3} - \frac{\epsilon}{3} = t\alpha - \epsilon.
\]

We will now show that there exists \( \tilde{i} > \tilde{i} \) such that for \( i \geq \tilde{i} \) we have
\( j_{n+1,y} = j_{n,y} \).
Indeed, let us take $\tilde{i} > i$ such that

\begin{equation}
\|y - x_{n+i+1}\|_p^p \leq \frac{\beta p}{2} \alpha^p + \epsilon
\end{equation}

is satisfied for $i \geq \tilde{i}$ (this is possible by (10)). If $j_{n+i,y} \neq j_{n,y}$, then by (13, 23, 24) we would obtain

\[
2 (t\alpha - \epsilon)^p \leq |x_{n+i+1} (j_{n,y}) - y (j_{n,y})|^p + |x_{n+i+1} (j_{n+1,y}) - y (j_{n+1,y})|^p
\]

\[
\leq \|x_{n+i+1} - y\|_p^p \leq \frac{\beta p}{2} \alpha^p + \epsilon < 2 (t\alpha - \epsilon)^p .
\]

But this is impossible.

Therefore for every $i \geq \tilde{i}$ we have

\begin{equation}
j_{n,y} = j_{n+i,y} .
\end{equation}

Next by (16, 22, 27) for $i \geq \tilde{i}$ we have

\[
j_{n+i,y} = j_{n,y} \leq j_{n+1} < j_{n+i+1} \text{ and } \|x_{n+i+1} - x_{n+1}\|_\infty \leq \frac{\epsilon}{3} .
\]

Hence by (13, 23, 25, 26) we get the following contradiction

\[
2 (t\alpha - \epsilon)^p \leq |x_{n+i+1} (j_{n,y}) - y (j_{n,y})|^p + |x_{n+i+1} (j_{n+1,y}) - y (j_{n+1,y})|^p
\]

\[
\leq \|x_{n+i+1} - y\|_p^p \leq \frac{\beta p}{2} \alpha^p + \epsilon < 2 (t\alpha - \epsilon)^p .
\]

Thus the sequence $\{j_{n,i}\}$ cannot be strictly increasing, contrary to (22).

Hence the inequality (12) is false and therefore the claimed inequality

\[
\inf_{x \in C} \left( \liminf_n \|x_n - x\| \right) \leq t\alpha
\]

is valid for arbitrary $t$ satisfying $\max \left( \frac{1}{2^p}, \frac{\beta}{4^p} \right) < t < 1$. We conclude that

\[
w-SOC \left( X^p_\beta \right) \geq \max \left( 2^{\frac{1}{2^p}}, \frac{4^\frac{1}{p}}{\beta} \right)
\]

for $1 < \beta < 4^\frac{1}{p}$.

To show that the constant

\[
\min \left[ 1, \max \left( 2^{\frac{1}{2^p}}, \frac{4^\frac{1}{p}}{\beta} \right) \right]
\]

is the best possible in $X^p_\beta$ ($1 < p < \infty$ and $1 < \beta < \infty$), let us consider two sequences. We define the sequence $\{x_n\}$ by (see [4])

\[
x_n = \begin{cases} 
(2k+1)^2 - n & \text{if } (2k)^2 < n \leq (2k+1)^2 , \\
\frac{4k+1}{4k+1} e_k + e_k+1 & \text{if } (2k+1)^2 < n \leq (2k+2)^2 , \\
\frac{n-(2k+1)^2}{4k+3} e_{k+2} & \text{if } (2k+2)^2 < n \leq (2k+3)^2 .
\end{cases}
\]
where \( \{e_k\} \) is the standard basis in \( l^p \). Then we have \( x_n \to 0 \), \( x_{n+1} - x_n \to 0 \),

\[
diam_{a\|\cdot\|} \{x_n\} = \max \left( \frac{4^\frac{1}{p}}{\beta}, 1 \right)
\]

and

\[
\|x_{(2k+1)^2}\| = 1.
\]

This yields

\[
\max \left( \frac{4^\frac{1}{p}}{\beta}, 1 \right) \cdot \inf_{x \in \text{conv} \{x_n\}} \left( \liminf_n \|x_n - x\| \right)
\]

\[
= \max \left( \frac{4^\frac{1}{p}}{\beta}, 1 \right) \cdot r_{a} (\{x_{(2k+1)^2}\}) = \max \left( \frac{4^\frac{1}{p}}{\beta}, 1 \right) = diam_{a\|\cdot\|} (\{x_n\}).
\]

The second sequence is defined as follows:

\[
x_n = \left[ r_n \cos \left( \frac{n - (2k)^2}{8k + 4} \cdot \frac{\pi}{2} \right) \right] e_k + \left[ r_n \sin \left( \frac{n - (2k)^2}{8k + 4} \cdot \frac{\pi}{2} \right) \right] e_{k+1},
\]

where

\[
r_n = \left\{ \left[ \cos \left( \frac{n - (2k)^2}{8k + 4} \cdot \frac{\pi}{2} \right) \right]^p + \left[ \sin \left( \frac{n - (2k)^2}{8k + 4} \cdot \frac{\pi}{2} \right) \right]^p \right\}^{-\frac{1}{p}}
\]

for \( (2k)^2 < n \leq (2k + 2)^2 \) and \( k = 1, 2, \ldots \). For this sequence we get \( x_n \to 0 \), \( x_{n+1} - x_n \to 0 \),

\[
diam_{a\|\cdot\|} \{x_n\} = \max \left( 1, \frac{2^\frac{1}{p}}{\beta} \right)
\]

and

\[
\|x_{4k^2+4k+2}\| = \max \left( \frac{1}{2^\frac{1}{p}}, \frac{1}{\beta} \right) = \frac{1}{2^\frac{1}{p}} \max \left( 1, \frac{2^\frac{1}{p}}{\beta} \right).
\]

Hence we obtain

\[
2^\frac{1}{p} \cdot \inf_{x \in \text{conv} \{x_n\}} \left( \liminf_n \|x_n - x\| \right)
\]

\[
= 2^\frac{1}{p} \cdot r_{a} (\{x_{4k^2+4k+2}\}) = \max \left( 1, \frac{2^\frac{1}{p}}{\beta} \right) = diam_{a\|\cdot\|} (\{x_n\}).
\]

This completes the proof. \( \blacksquare \)

**Remark 5.1.** A Banach space \( Y \) which is isomorphic to \( X^p_\beta \) with \( 1 < \beta < \frac{4^\frac{1}{p}}{2} \) has the weak fixed point property for nonexpansive mappings if

\[
d \left( Y, X^p_\beta \right) < \min \left( \frac{2^\frac{1}{p}}{\beta}, \frac{4^\frac{1}{p}}{\beta} \right)
\]
(see Remark 2.3), but recently T. Domínguez Benavides and M.Á. Japón Pineda obtained a better result for $X^2_\beta$. Namely, if
\[
 d \left( Y, X^2_\beta \right) < M \left( X^2_\beta \right) = \begin{cases} 
 \sqrt{3} & \text{for } 1 < \beta \leq \sqrt{3}, \\
 \sqrt{2} \beta + \left( 1 + \sqrt{\beta^2 - 1} \right) & \text{for } \frac{\sqrt{3}}{2} < \beta < \sqrt{2}, \\
 1 + \frac{1}{\sqrt{2}} & \text{for } \sqrt{2} \leq \beta,
\end{cases}
\]
then $Y$ has the weak fixed point property [16]. But we have to mention that in the second part of our paper the coefficient $w$-SOC $(X)$ is applied to the problem of existence of fixed points of asymptotically regular uniformly lipschitzian semigroups, where till now we have not been able to use the coefficient $M \left( X \right)$.

6. Comparison of the basic geometric coefficients of Banach spaces

As we mentioned in Section 2 the following inequalities
\[
 AN \left( X \right) \leq w$-AN \left( X \right),
\]
and
\[
 WCS \left( X \right) \leq w$-SOC \left( X \right)
\]
are always valid. The following examples show that for particular spaces strict inequalities may occur.

**Example 6.1.** If we take the Cartesian product $X^2_{\sqrt{2}} \times l^1$ equipped with the $l^1$ norm, then this space is nonreflexive and therefore by Theorem 2.1, $AN \left( X^2_{\sqrt{2}} \times l^1 \right) = 1$, but after applying Theorems 2.2, 4.2 and 5.1 we obtain $w$-AN $\left( X^2_{\sqrt{2}} \times l^1 \right) = w$-SOC $\left( X^2_{\sqrt{2}} \times l^1 \right) = \sqrt{2}$.

**Example 6.2.** Taking $X^2_\beta$ with $1 < \beta < \sqrt{2}$ and applying Theorem 5.1 we obtain
\[
 1 < WCS \left( X^2_\beta \right) = \frac{\sqrt{2}}{\beta} < w$-SOC $\left( X^2_\beta \right) = \sqrt{2}.
\]

**Example 6.3.** Let us consider the product space
\[
 X = \left\{ x = \{x_p\}_{p=1}^\infty : x_p \in l^p, \ \left( \sum_{p=1}^\infty \|x_p\|^2_p \right)^{\frac{1}{2}} = \|x\| \right\}.
\]

Since $X$ contains isometric copies of $l^p$ for every $p$ and since both the semi-Opial coefficient and the asymptotic normal structure coefficient satisfy
\[
 w$-SOC $\left( l^p \right) = AN \left( l^p \right) = 2^\frac{1}{p},
\]
the space $X$ has neither the uniform asymptotic normal structure nor the uniform semi-Opial properties. But it easy to observe that $X$ is still SO. In fact, it has the Opial property [26].
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