Research Article

Some Inequalities for Multiple Integrals on the \(n\)-Dimensional Ellipsoid, Spherical Shell, and Ball

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The authors establish some new inequalities of Pólya type for multiple integrals on the \(n\)-dimensional ellipsoid, spherical shell, and ball, in terms of bounds of the higher order derivatives of the integrands. These results generalize the main result in the paper by Feng Qi, Inequalities for a multiple integral, Acta Mathematica Hungarica (1999).

1. Introduction

In [1], it was obtained that if \(f\) is differentiable and if \(f(a) = f(b) = 0\), then

\[
f'(\tau) > \frac{4}{(b-a)^2} \int_a^b f(t) \, dt,
\]

for a certain \(\tau\) between \(a\) and \(b\). This inequality can be found in \([2–4]\) and many other textbooks. It can be reformulated as follows. If \(f(x)\) is differentiable and not identically constant, such that \(f(a) = f(b) = 0\) and \(|f'(x)| \leq M\) on \([a,b]\), then

\[
\left| \int_a^b f(x) \, dx \right| \leq \frac{(b-a)^2}{4M}.
\]

In the literature, the inequalities (1) or (2) is called the Pólya integral inequality.

In [5], the inequality (1), or say (2), was generalized as

\[
\left| \int_a^b f(x) \, dx - \frac{1}{2} (b-a) [f(a) + f(b)] \right| \leq \frac{M(b-a)^2}{4} - \frac{[f(b) - f(a)]^2}{4M},
\]

where \(f : [a, b] \to \mathbb{R}\) is a differentiable function and \(|f'(x)| \leq M\).

In [6–9], the above inequalities were refined and generalized as follows.

**Theorem 1** (see [9, Proposition 1]). Let \(f(x)\) be continuous on \([a, b]\) and differentiable in \((a, b)\). Suppose that \(f(a) = f(b) = 0\), and that \(m \leq f'(x) \leq M\) in \((a, b)\). If \(f(x)\) is not identically zero, then \(m < 0 < M\) and

\[
\left| \int_a^b f(x) \, dx \right| \leq -\frac{(b-a)^2}{2} \frac{mM}{M-m}.
\]

**Theorem 2** (see \([6, 7, 9]\)). Let \(f(x)\) be continuous on \([a, b]\) and differentiable in \((a, b)\). Suppose that \(f(x)\) is not identically a constant, and that \(m \leq f'(x) \leq M\) in \((a, b)\). Then,

\[
\left| \frac{1}{b-a} \int_a^b f(x) \, dx - \frac{f(a) + f(b)}{2} \right|
\]

\[
\leq \left[ f(b) - f(a) - m(b-a) \right]
\]

\[
\times \left[ M(b-a) - f(b) + f(a) \right]
\]

\[
\times \left[ 2(M-m)(b-a) \right]^{-1}
\]

\[
= -\frac{M - S_0(a,b)}{2(M-m)} \frac{m - S_0(a,b)}{b-a},
\]
where
\[
S_3(a,b) = \frac{f(b) - f(a)}{b - a}.
\] (6)

**Theorem 3** (see [8]). For \(a = (a_1, \ldots, a_m) \in \mathbb{R}^m\) and \(b = (b_1, \ldots, b_m) \in \mathbb{R}^m\) with \(a_i < b_i\) for \(i = 1, 2, \ldots, m\), denote the \(m\)-rectangles by

\[
Q_m = \prod_{i=1}^m [a_i, b_i], \quad Q_m(t) = \prod_{i=1}^m [a_i, c_i(t)],
\] (7)

\[
Q_m = \prod_{i=1}^m [a_i, b_i],
\] (8)

where \(c_i(t) = (1-t)a_i + tb_i\) for \(i = 1, 2, \ldots, m\) and \(t \in (0,1)\). Let \(\nu = (\nu_1, \ldots, \nu_m)\) be a multi-index; that is, \(\nu_i\) is a nonnegative integer, with \(|\nu| = \sum_{i=1}^m \nu_i\). Let \(f \in C^{m+1}(Q_m)\) be a function of \(m\) variables on \(Q_m\), and let its partial derivatives of \((n+1)\)th order remain between \(M_{n+1}(\nu)\) and \(N_{n+1}(\nu)\) in \(Q_m\); that is,

\[
N_{n+1}(\nu) \leq D^\nu f(x) \leq M_{n+1}(\nu), \quad x \in Q_m,
\] (9)

where \(|\nu| = n + 1\) and

\[
D^\nu f(x) = \frac{\partial^{|\nu|} f(x)}{\prod_{i=1}^m \partial x_i^{\nu_i}}.
\] (10)

Let

\[
A(\nu) = \prod_{i=1}^m \left( \frac{(b_i - a_i)^{\nu_i+1}}{(\nu_i + 1)!} M_{n+1}(\nu) \right),
\]

\[
B(\nu, f(x)) = \prod_{i=1}^m \left( \frac{(b_i - a_i)^{\nu_i+1}}{(\nu_i + 1)!} \left( \frac{\partial}{\partial x_i} \right)^{\nu_i} f(x) \right),
\]

\[
C(\nu) = \prod_{i=1}^m \left( \frac{(b_i - a_i)^{\nu_i+1}}{(\nu_i + 1)!} N_{n+1}(\nu) \right),
\]

\[
T(\nu, t) = \prod_{i=1}^m \left( 1 - (1-t)^{\nu_i+1} \right) - 1,
\] (11)

for \(t \in (0,1)\). Then, for any \(t \in (0,1)\),

(1) when \(n\) is even, one has

\[
\sum_{|\nu|=n+1} C(\nu) t^{|\nu|+n+1} + \sum_{|\nu|=m+1} A(\nu) T(\nu, t)
\]

\[
\leq \int_{Q_m} f(x) \, dx - \sum_{k=0}^{m} \sum_{|\nu|=k} B(\nu, f(a)) t^{m+k}
\]

\[
+ \sum_{k=0}^{m} (-1)^k \sum_{|\nu|=k} B(\nu, f(b)) T(\nu, t)
\]

\[
\leq \sum_{|\nu|=n+1} A(\nu) t^{|\nu|+n+1} + \sum_{|\nu|=m+1} C(\nu) T(\nu, t).
\] (12)

(2) When \(n\) is odd, one has

\[
\sum_{|\nu|=n+1} C(\nu) \left[ t^{|\nu|+n+1} + T(\nu, t) \right]
\]

\[
\leq \int_{Q_m} f(x) \, dx - \sum_{k=0}^{n} \sum_{|\nu|=k} B(\nu, f(a)) t^{m+k}
\]

\[
+ \sum_{k=0}^{n} (-1)^k \sum_{|\nu|=k} B(\nu, f(b)) T(\nu, t)
\]

\[
\leq \sum_{|\nu|=n+1} A(\nu) t^{|\nu|+n+1} + \sum_{|\nu|=m+1} C(\nu) T(\nu, t).
\] (13)

Moreover, let \(f : I \subseteq \mathbb{R} \to \mathbb{R}\) be an \((m+1)\)-times differentiable function, and let

\[
g_1(x) = \sqrt{\sum_{i=1}^n (x_i - a_i)^2}, \quad x \in \Omega_1,
\] (14)

\[
g_2(x) = \sqrt{\sum_{i=1}^n x_i^2}, \quad x \in \Omega_2,
\]

\[
g_3(x) = \sqrt{\sum_{i=1}^n (x_i - a_i)^2}, \quad x \in \Omega_3.
\]
In this paper, we will establish some new inequalities of Pólya type for multiple integrals of the composition function \( f \circ g \), on the \( n \)-dimensional ellipsoid \( \Omega_1 \), of the composition function \( f \circ g \), on the spherical shell \( \Omega_2 \), and of the composition function \( f \circ g \), on the \( n \)-dimensional ball \( \Omega_n \). We also obtain a general inequality for the multiple integral \( \int_{\Omega_n} f(x) \, dx \).

2. A Lemma

In order to establish some new inequalities of Pólya type for multiple integrals, we need the following lemma.

**Lemma 4.** For \( b_i, r_i > 0 \), and \( v_i > -1 \), one has

\[
\int_{\Omega_{2r}} \prod_{i=1}^n (x_i - a_i)^{v_i} \, dx = \frac{\prod_{i=1}^n b_i^{v_i+1/r_i} \prod_{i=1}^n \Gamma \left( \frac{(v_i+1)}{2r_i} \right)}{2^{n-1} \sum_{i=1}^n \left( \frac{(v_i+1)}{r_i} \right)} \Gamma \left( \sum_{i=1}^n \left( \frac{(v_i+1)}{2r_i} \right) \right),
\]

where

\[
\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} \, dt, \quad \Re(z) > 0
\]

is the classical Euler gamma function.

**Proof.** Using the spherical coordinates on the region \( \Omega_{2r} \), yields

\[
x_1 = b_1 s^{1/r_1} \cos^{1/r_1} \varphi_1 + a_1,
\]

\[
x_i = b_i \left[ \sin \varphi_1 \prod_{k=1}^{i-1} \sin \varphi_k \right]^{1/r_i} + a_i, \quad 2 \leq i \leq n - 1,
\]

\[
x_n = b_n \left[ \sin \varphi_1 \prod_{k=1}^{n-1} \sin \varphi_k \right]^{1/r_n} + a_n,
\]

where \( 0 \leq s \leq 1 \) and \( 0 \leq \varphi_1, \varphi_2, \ldots, \varphi_{n-1} \leq \pi/2 \), and

\[
F_i = s \prod_{k=1}^{i-1} \sin^2 \varphi_k - \sum_{k=1}^n \left( \frac{x_k - a_k}{b_k} \right)^{2r_k} = 0, \quad 1 \leq i \leq n.
\]

We note that when \( i = 1 \), the empty product in (18) is understood to be 1. It is clear that the expressions in (17) are solutions of (18), and that

\[
J = \frac{Dx}{D(s, \varphi_1, \varphi_2, \ldots, \varphi_{n-1})} = (-1)^n \frac{D(F_1, F_2, \ldots, F_n)}{D(s, \varphi_1, \varphi_2, \ldots, \varphi_{n-1})}.
\]

A straightforward computation gives

\[
J = \prod_{k=1}^n \sum_{r_k} \frac{\sin^{v_k+1/(-1)} \varphi_k}{\sin^{v_k+1/(-1)} \varphi_k} \prod_{k=1}^{n-1} \sin^{v_{k+1}+1/(-1)} \varphi_k \cos^{v_k+1/(-1)} \varphi_k.
\]

Since

\[
\int_0^{\pi/2} \cos^m \varphi \sin^n \varphi \, d\varphi = \frac{\Gamma \left( \frac{(m+1)}{2} \right) \Gamma \left( \frac{(n+1)}{2} \right)}{2 \Gamma \left( \frac{(m+n+2)}{2} \right)}
\]

we obtain

\[
\int_{\Omega_{2r}} \prod_{i=1}^n (x_i - a_i)^{v_i} \, dx = \frac{\prod_{i=1}^n b_i^{v_i+1/r_i} \prod_{i=1}^n \Gamma \left( \frac{(v_i+1)}{2r_i} \right)}{2^{n-1} \sum_{i=1}^n \left( \frac{(v_i+1)}{r_i} \right)} \Gamma \left( \sum_{i=1}^n \left( \frac{(v_i+1)}{2r_i} \right) \right).
\]

The proof of Lemma 4 is complete.

□

3. Main Results

Now, we start out to state and prove our main results.

**Theorem 5.** Let \( f : [0, 1] \to \mathbb{R} \) be an \((m+1)\)-times differentiable function satisfying

\[
N(m) \leq f^{(m+1)}(u) \leq M(m).
\]

Then, one has

\[
\frac{2\pi^{n/2} (n-1)! \prod_{i=1}^n b_i}{\Gamma(n/2) (n+m+1)!} \min \{ (-1)^{m+1} M(m), (-1)^{m+1} N(m) \} \]

\[
\leq \int_{\Omega_1} f(g_1(x)) \, dx
\]

\[
- \sum_{k=0}^m (-1)^k \frac{2\pi^{n/2} (n-1)! \prod_{i=1}^n b_i}{(n+k)! \Gamma(n/2)} f^{(k)}(1)
\]

\[
\leq \frac{2\pi^{n/2} (n-1)! \prod_{i=1}^n b_i}{\Gamma(n/2) (n+m+1)!} \]

\[
\times \max \{ (-1)^{m+1} M(m), (-1)^{m+1} N(m) \}.
\]

(24)

**Proof.** Using the transformation in (17) on \( \Omega_1 \) and letting \( r_i = 1 \) for \( i = 1, 2, \ldots, n \) yield the Jacobian determinant

\[
J = s^{n-1} \prod_{k=1}^n b_k \prod_{k=1}^{n-2} \sin^{n-k-1} \varphi_k,
\]

where

\[
0 \leq \varphi_1, \varphi_2, \ldots, \varphi_{n-1} \leq \pi,
\]

\[
0 \leq \varphi_{n-1} \leq 2\pi.
\]

Because

\[
\int_0^{\pi/2} \sin^r \varphi \, d\varphi = 2 \int_0^{\pi/2} \cos^r \phi \, d\phi = \frac{\sqrt{n} \Gamma \left( \frac{(n+1)}{2} \right)}{\Gamma \left( \frac{(n+2)}{2} \right)}
\]

(27)

we have
\[
\prod_{k=1}^{n-2} \int_0^\pi \sin^{n-k-1} \phi_k \, d\phi_k \int_0^{2\pi} \, d\phi_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{28}
\]

By integration by parts, one has
\[
\int_\alpha^\beta \sin^{n-1} f(s) \, ds = \sum_{k=0}^{m} (-1)^k (n-1)! \left[ \beta^{n+k} f^{(k)}(\beta) - \alpha^{n+k} f^{(k)}(\alpha) \right] (n+k)! + (-1)^{m+1} (n-1)! \int_\alpha^\beta f^{(m+1)}(s) s^{n+m} \, ds.
\]

Choosing \( \alpha = 0 \) and \( \beta = 1 \) in the above equality shows that
\[
\int_{\Omega_1} f(g_1(x)) \, dx = \sum_{k=1}^{n} b_k \int_0^{\frac{\pi}{n-2}} \sin^{n-k-1} \phi_k \, d\phi_k \int_0^{2\pi} \, d\phi_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \sum_{k=0}^{m} (-1)^k (n-1)! f^{(k)}(1) (n+k)! + (-1)^{m+1} \frac{2\pi^{n/2}}{\Gamma(n/2)} \sum_{k=0}^{m} b_k (n-1)! \int_0^{1} f^{(m+1)}(s) s^{n+m} \, ds.
\tag{30}
\]

Further utilizing the condition (23) leads to the inequality (24). The proof of Theorem 5 is completed.

**Theorem 6.** Let \( f : [\rho_1, \rho_2] \to \mathbb{R} \) be an \((m+1)\)-times differentiable function satisfying the inequality (23). Then, one has
\[
\frac{2\pi^{n/2} (\rho_2^{n+m+1} - \rho_1^{n+m+1}) (n-1)!}{\Gamma(n/2) (n+m+1)!} \times \min \left\{ (-1)^{m+1} M(m), (-1)^{m+1} N(m) \right\}
\leq \int_{\Omega_2} f(g_2(x)) \, dx - \frac{2\pi^{n/2}}{\Gamma(n/2)} \sum_{k=0}^{m} (-1)^k (n-1)! \left[ \rho_2^{n+k} f^{(k)}(\rho_2) - \rho_1^{n+k} f^{(k)}(\rho_1) \right] (n+k)! + \frac{2\pi^{n/2} (\rho_2^{n+m+1} - \rho_1^{n+m+1}) (n-1)!}{\Gamma(n/2) (n+m+1)!} \times \max \left\{ (-1)^{m+1} M(m), (-1)^{m+1} N(m) \right\}.
\tag{31}
\]

**Proof.** Using the transformation in (17) on \( \Omega_2 \) and choosing \( r_j = 1, a_i = 0, \) and \( b_i = 1 \) for \( i = 1, 2, \ldots, n \) yield
\[
J = s^{-1-n/2} \prod_{k=1}^{n-2} \sin^{n-k-1} \phi_k,
\]

\( \rho_1 \leq s \leq \rho_2, \quad 0 \leq \phi_1, \phi_2, \ldots, \phi_{n-2} \leq \pi, \)

\[0 \leq \phi_1 \leq 2\pi.\]

Further letting \( \alpha = \rho_1 \) and \( \beta = \rho_2 \) in (29) gives
\[
\int_{\Omega_2} f(g_2(x)) \, dx = \int_{\rho_1}^{\rho_2} \sin^{n-1} f(s) \, ds \sum_{k=0}^{m} (-1)^k (n-1)! \times \left[ \rho_2^{n+k} f^{(k)}(\rho_2) - \rho_1^{n+k} f^{(k)}(\rho_1) \right] (n+k)! + \frac{2\pi^{n/2} (\rho_2^{n+m} - \rho_1^{n+m}) (n-1)!}{\Gamma(n/2) (n+1)!} \int_{\rho_1}^{\rho_2} f^{(m+1)}(s) s^{n+m} \, ds.
\tag{33}
\]

Hence, by virtue of the condition (23), the inequality (31) follows immediately. The proof of Theorem 6 is completed.

**Theorem 7.** Let \( f : [0, \rho] \to \mathbb{R} \) be an \((m+1)\)-times differentiable function satisfying (23). Then, one has
\[
\frac{2\pi^{n/2} (\rho^{n+m+1}) (n-1)!}{\Gamma(n/2) (n+m+1)!} \times \min \left\{ (-1)^{m+1} M(m), (-1)^{m+1} N(m) \right\}
\leq \int_{\Omega_3} f(g_3(x)) \, dx - \sum_{k=0}^{m} (-1)^k \frac{2\pi^{n/2} (n-1)! \rho^{n+k}}{(n+k)!} f^{(k)}(\rho) + \frac{2\pi^{n/2} (\rho^{n+m+1}) (n-1)!}{\Gamma(n/2) (n+m+1)!} \times \max \left\{ (-1)^{m+1} M(m), (-1)^{m+1} N(m) \right\}.
\tag{34}
\]

**Proof.** Similar to the proof of Theorem 5, by choosing \( b_1 = b_2 = \cdots = b_n = \rho \) and \( 0 \leq s \leq \rho, \) we obtain the inequality (34). The proof is complete.
4. A More General Inequality

Let \( \nu = (\nu_1, \nu_2, \ldots, \nu_n) \) be an \( n \)-tuple index; that is, the numbers \( \nu_1, \nu_2, \ldots, \nu_n \) are nonnegative and denote \( |\nu| = \sum_{i=1}^{n} \nu_i \). Let \( f: \Omega \rightarrow \mathbb{R} \) be a function which has an \( m+1 \) times continuous derivative on \( \Omega \), and let

\[
D^\nu f(x) = \frac{\partial^{(|\nu|)}}{\partial x_1^{\nu_1} \cdots \partial x_n^{\nu_n}} f(x),
\]

where \( \nu = (\nu_1, \nu_2, \ldots, \nu_n) \) and \( |\nu| = m+1 \).

**Theorem 9.** Let \( f \in C^{m+1}(\Omega) \) satisfy

\[
N_{m+1}(\nu) \leq D^\nu f(x) \leq M_{m+1}(\nu), \quad x \in \Omega.
\]

Then

\[
H(\nu, b, r) = \frac{\prod_{i=1}^{n} (b_i^{\nu_i+1}/r_i)}{2^{n-1} \sum_{i=1}^{n} ((\nu_i + 1)/r_i)} \Gamma((\nu_i + 1)/2),
\]

where \( b_i \) and \( r_i \) are nonnegative and denote \( |\nu| = m+1 \).

**Proof.** By Taylor’s formula, we obtain

\[
f(x) = \sum_{j=0}^{m} \frac{f^{(j)}(a)}{j!} [x - a]_j f(a) + R_m(x),
\]

where

\[
R_m(x) = \frac{1}{(m+1)!} \left[ \sum_{i=1}^{n} (x_i - a_i) \frac{\partial}{\partial x_i} \right] \left[ x - a \right]_m f(a + \theta(x-a)),
\]

where \( \theta \in (0, 1) \).

Using

\[
\left( \sum_{i=1}^{n} q_i \right)^j = j! \sum_{|\nu| = j} q_1^{\nu_1} q_2^{\nu_2} \cdots q_n^{\nu_n},
\]

we have

\[
f(x) = \sum_{j=0}^{m} \frac{1}{j!} \sum_{|\nu| = j} \frac{n!}{\prod_{i=1}^{n} \nu_i!} \left( \sum_{i=1}^{n} (x_i - a_i) \frac{\partial}{\partial x_i} \right)^{\nu_i} f(a)
\]

Integrating on both sides of the above equality leads to

\[
\int_{\Omega_{\nu}} f(x) \ dx
\]

\[
= \sum_{j=0}^{m} \frac{1}{j!} \sum_{|\nu| = j} \frac{n!}{\prod_{i=1}^{n} \nu_i!} \left( \sum_{i=1}^{n} (x_i - a_i) \frac{\partial}{\partial x_i} \right)^{\nu_i} f(a)
\]

\[
\times \int_{\Omega_{\nu}} \prod_{i=1}^{n} (x_i - a_i)^{\nu_i} \ dx
\]

\[
+ \sum_{|\nu| = m+1} \frac{n!}{\prod_{i=1}^{n} \nu_i!} \left( \sum_{i=1}^{n} (x_i - a_i) \frac{\partial}{\partial x_i} \right)^{\nu_i} f(a) \int_{\Omega_{\nu}} \prod_{i=1}^{n} (x_i - a_i)^{\nu_i} \ dx.
\]

Using

\[
\int_{\Omega_{\nu}} \prod_{i=1}^{n} (x_i - a_i)^{\nu_i} \ dx = I_1 + I_2,
\]

where

\[
I_1 = \sum_{j=0}^{m} \frac{1}{j!} \sum_{|\nu| = j} \frac{n!}{\prod_{i=1}^{n} \nu_i!} \int_{\Omega_{\nu}} \prod_{i=1}^{n} (x_i - a_i)^{\nu_i} \ dx,
\]

\[
I_2 = \sum_{|\nu| = m+1} \frac{n!}{\prod_{i=1}^{n} \nu_i!} \int_{\Omega_{\nu}} \prod_{i=1}^{n} (x_i - a_i)^{\nu_i} \ dx.
\]
By Lemma 4 and (44), one has

\[ I_1 = \sum_{j=0}^{m} \sum_{|\gamma|=j} 1 \frac{\partial^{|\gamma|} f(a)}{\prod_{|\gamma|=j} \gamma!} \prod_{j=1}^{m} (b_j^{|\gamma|}/r_j) \times \frac{1}{2^{n-1}} \frac{\sum_{j=1}^{m} ((r_j + 1)/2r_j)}{\prod_{j=1}^{m} \Gamma} I_1 \]

\[ = \sum_{j=0}^{m} \sum_{|\gamma|=j} \frac{D^\gamma f(a)}{\prod_{|\gamma|=j} \gamma!} H(\gamma, b, r). \] (46)

From (37) and

\[ I_2 = \sum_{|\gamma|=m+1} 1 \frac{D^\gamma f(a)}{\prod_{|\gamma|=m+1} \gamma!} \int_{\Omega_2} \prod_{j=1}^{n} (x_j - a_j)^\gamma D^\gamma f(a + \theta (x - a)) \, dx, \] (47)

we have

\[ \sum_{|\gamma|=m+1} N_{m+1}(\gamma) \frac{D^\gamma f(a)}{\prod_{|\gamma|=m+1} \gamma!} \leq I_2 \leq \sum_{|\gamma|=m+1} M_{m+1}(\gamma) \frac{D^\gamma f(a)}{\prod_{|\gamma|=m+1} \gamma!} \]

\[ \leq H(\gamma, t) \sum_{|\gamma|=m+1} \frac{M_{m+1}(\gamma)}{\prod_{|\gamma|=m+1} \gamma!}, \] (49)

Consequently, the proof of Theorem 9 is complete. \( \square \)

**Corollary 10.** Let \(|\gamma| = m + 1\), and let \( f \in C^{m+1}(\Omega_k) \) with (37). Then, for \( t \in (0, 1] \) one has

\[ H(\gamma, t) \sum_{|\gamma|=m+1} \frac{N_{m+1}(\gamma)}{\prod_{|\gamma|=m+1} \gamma!} \leq I_1 \leq H(\gamma, t) \sum_{|\gamma|=m+1} \frac{M_{m+1}(\gamma)}{\prod_{|\gamma|=m+1} \gamma!}, \]

where

\[ H(\gamma, t) = \frac{\theta^{m+1}}{n+m+1} \prod_{k=1}^{m} \frac{[1 + (-1)^\gamma]}{2^{n-1}} \prod_{j=1}^{m} \Gamma \frac{(r_j + 1)/2r_j}{\prod_{j=1}^{m} \Gamma \frac{(r_j + 1)/2r_j}} \prod_{j=1}^{m+1}. \] (50)

**5. An Application**

Now, we list some special cases of \( \Omega_{2r} \) as follows.

1. If we take \( r_1 = r_2 = \cdots = r_n = 1/2 \), the body \( \Omega_{2r} \) becomes a closed region between the \( n \)-dimensional pyramid and the rectangle \( x_i = a_i \) for \( i = 1, 2, \ldots, n \).

2. If we take \( r_1 = r_2 = \cdots = r_n = 1 \), the body \( \Omega_{2r} \) is a closed region between the \( n \)-dimensional ellipsoid \( \Omega_n(a, b) \) and the rectangle \( x_i = a_i \) for \( i = 1, 2, \ldots, n \).

3. If we take \( r_1 = r_2 = \cdots = r_n = 1 \) and \( b_1 = b_2 = \cdots = b_n = \rho \), the body \( \Omega_{2r} \) is a closed region between the \( n \)-dimensional ball \( \Omega_n(a, \rho) \) and the rectangle \( x_i = a_i \) for \( i = 1, 2, \ldots, n \).

In the calculation of the uniform \( n \)-dimensional volume, static moment, the moment of inertia, the centrifugal moment, and so on, have important applications. See [21, 22].

To show the applicability of the above main results, we now estimate the value of a triple integral

\[ I = \iiint_V \sin \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right)^{5/2} \, dx \, dy \, dz, \] (51)

where \( V \) is the ellipsoid

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1. \] (52)

Choosing \( n = 3, b_1 = a, b_2 = b, \) and \( b_3 = c \) in (25), the Jacobian determinant is

\[ J = abc s^2 \sin \varphi_1, \] (53)

\[ I = \int_0^{2\pi} d\varphi_2 \int_0^{\pi} d\varphi_1 \int_0^1 abc s^2 \sin \varphi_1 \sin s \, ds \]

\[ = 4\pi \int_0^1 s^2 \sin s \, ds. \] (54)

Using Taylor’s formula, it follows that

\[ \sin x = \sum_{k=1}^{m} \frac{(-1)^k x^{2k-1}}{(2k-1)!} + \frac{1}{2m+1} \frac{x^{2m+1}}{2m+1} \cos(\theta x), \]

\[ 0 < \theta < 1. \] (55)

Specially, we have

\[ \sin x = \sum_{k=1}^{3} \frac{(-1)^k x^{2k-1}}{(2k-1)!} + \frac{x^7}{7!} \cos \theta_1 x, \]

\[ \sin x = \sum_{k=1}^{6} \frac{(-1)^k x^{2k-1}}{(2k-1)!} + \frac{x^{13}}{13!} \cos \theta_2 x, \] (56)

where \( 0 < \theta_1, \theta_2 < 1 \) and \( 0 < x < 1 \). Therefore,

\[ \sum_{k=1}^{6} \frac{(-1)^k x^{2k-1}}{(2k-1)!} \leq \sin x \leq \sum_{k=1}^{3} \frac{(-1)^k x^{2k-1}}{(2k-1)!}. \] (57)
By (54) and the above inequality, we have
\[
\sum_{k=1}^{6} \frac{(-1)^{k-1}}{(2k-1)!} \int_{0}^{1} s^{10k-3} ds \\
\leq \int_{0}^{1} s^2 \sin s^5 ds \leq \sum_{k=1}^{3} \frac{(-1)^{k-1}}{(2k-1)!} \int_{0}^{1} s^{10k-3} ds,
\]
\[
\frac{61249255037}{131964940800} \pi abc \leq I \leq \frac{3509}{7560} \pi abc. \tag{58}
\]

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