Research Article

Asymptotic Behavior of Solutions to Fast Diffusive Non-Newtonian Filtration Equations Coupled by Nonlinear Boundary Sources

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This paper is concerning the asymptotic behavior of solutions to the fast diffusive non-Newtonian filtration equations coupled by the nonlinear boundary sources. We are interested in the critical global existence curve and the critical Fujita curve, which are used to describe the large-time behavior of solutions. It is shown that the above two critical curves are both the same for the multidimensional problem we considered.

1. Introduction

In this paper, we study the non-Newtonian filtration equations coupled by the nonlinear boundary sources

\[ \frac{\partial u}{\partial t} = \text{div} \left( |\nabla u|^{p-2} \nabla u \right), \quad \frac{\partial v}{\partial t} = \text{div} \left( |\nabla v|^{q-2} \nabla v \right), \]

\((x, t) \in \left( \mathbb{R}^N \setminus B_1(0) \right) \times (0, T), \)

\[ |\nabla u|^{p-2} \nabla u \cdot \vec{v} = u^\alpha (x, t), \quad |\nabla v|^{q-2} \nabla v \cdot \vec{y} = u^\beta (x, t), \]

\((x, t) \in \partial B_1(0) \times (0, T) , \)

\[ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \mathbb{R}^N \setminus B_1(0), \]

(1)

where \( 1 < p, q < 2, \alpha, \beta \geq 0, N \geq 2, B_1(0) \) is the unit ball in \( \mathbb{R}^N \) with boundary \( \partial B_1(0) \), \( \vec{v} \) is the inward normal vector on \( \partial B_1(0) \), and \( u_0(x) \) and \( v_0(x) \) are nonnegative, suitably smooth, and bounded functions satisfying the appropriate compatibility conditions.

The system (1)–(3) can be used to describe the models in population dynamics, chemical reactions, heat propagation, and so on. It is well known that the classical solutions do not exist because the equations in (1) are degenerate in \( \{x(t); \nabla u(x, t) = 0\} \), while the local existence and the comparison principle of the weak solutions can be obtained; see [1, 2]. In this paper, we investigate the asymptotic behavior of solutions to the system (1)–(3), including blowup in a finite time and global existence in time.

Since the beginning work on critical exponent done by Fujita in [3], there are a lot of Fujita type results established for various equations; see the survey papers [4, 5] and the references therein and also the papers [6–9]. We recall some results on the fast diffusion case. In [10], the authors obtained the critical exponents for the single one-dimensional fast diffusive equation on \((0, +\infty) \times (0, +\infty)\), that is,

\[ \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( \left| \frac{\partial u}{\partial x} \right|^{p-2} \frac{\partial u}{\partial x} \right), \quad x > 0, \quad t > 0, \]

\[ \left| \frac{\partial u}{\partial x} \right|^{p-2} \frac{\partial u}{\partial x} (0, t) = u^\alpha (0, t), \quad t > 0, \]

\[ u(x, 0) = u_0(x), \quad x > 0. \]

(2)

(3)

(4)

They showed that the global critical exponent is \( \alpha_0 = 2(p - 1)/p \) and the critical Fujita exponent is \( \alpha_c = 2(p - 1) \).
After that, the corresponding results on the coupled $p$-laplace equations can be found in the paper [11], in which the authors consider multiple equations coupled by boundary sources. For the system,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( |\nabla u|^{p-2} \frac{\partial u}{\partial x} \right),$$

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x} \left( |\nabla v|^{q-2} \frac{\partial v}{\partial x} \right), \quad x > 0, \ t > 0,$$

$$\left| \frac{\partial u}{\partial x} \right|^{p-2} \frac{\partial u}{\partial x}(0, t) = v^\alpha(0, t),$$

$$\left| \frac{\partial v}{\partial x} \right|^{q-2} \frac{\partial v}{\partial x}(0, t) = u^\beta(0, t), \quad t > 0,$$

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad x > 0,$$  

the results in [11] are that the critical global existence curve is $\alpha \beta = 4(\frac{p-1}{p-q})$ and when $\alpha \beta > 4(\frac{p-1}{p-q})$, the critical Fujita curve is on the strictly right of the critical global existence curve.

As for the multidimensional problem, the single equation case of (1)–(3) was discussed in [12]; that is,

$$\frac{\partial u}{\partial t} = \text{div} \left( |\nabla u|^{p-2} \nabla u \right), \quad (x, t) \in (\mathbb{R}_+ N \setminus B_1(0)) \times (0, T),$$

$$|\nabla u|^{p-2} \nabla u \cdot \nabla \tilde{v} = v^\alpha(x, t), \quad (x, t) \in \partial B_1(0) \times (0, T),$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}_+ N \setminus B_1(0)$$

with $1 < p < 2$, $\alpha \geq 0$, $N \geq 2$. It was shown that the critical global exponent and the critical Fujita exponent both are $\alpha = \frac{p-1}{p-2}$.

Motivated by the papers mentioned above, the aim of this paper is to study the asymptotic behavior of solutions to the system (1)–(3). We show that the phenomenon that the two critical exponents for multi-dimensional equation coincide also occurs in the coupled equations.

Furthermore, by virtue of the radial symmetry of the exterior domain of the unit ball, we note that the above result can be extended to the following more general problems:

$$\frac{\partial}{\partial t} \left( |x|^\lambda u \right) = \text{div} \left( |x|^\lambda |\nabla u|^{p-2} \nabla u \right),$$

$$\frac{\partial}{\partial t} \left( |x|^\lambda v \right) = \text{div} \left( |x|^\lambda |\nabla v|^{q-2} \nabla v \right),$$

$$\left| \frac{\partial u}{\partial r} \right|^{p-2} \frac{\partial u}{\partial r}(1, t) = v^\alpha(1, t),$$

$$\left| \frac{\partial v}{\partial r} \right|^{q-2} \frac{\partial v}{\partial r}(1, t) = u^\beta(1, t), \quad t > 0.$$
where \(1 < p, q < 2, \tilde{\alpha}, \tilde{\beta}, \tilde{\lambda} = \lambda_1 + N - 1 > p - 1, \tilde{\lambda}_2 > q - 1, \alpha, \beta \geq 0, u_0, v_0 \) are nonnegative, suitably smooth, and bounded functions. It is not difficult to check that the solution \((u, v)(r, t)\) of the system (12)–(14) is also the solution of the system (7)–(9) if \(u_0(x)\) and \(v_0(x)\) are radially symmetrical. We now study the asymptotic behavior of solutions to the system (12)–(14).

**Proposition 3.** If \(\alpha \beta \leq 4(p-1)(q-1)/pq\), then all nonnegative solutions of the system (12)–(14) exist globally in time.

**Proof of Proposition 3.** We prove this proposition by constructing a kind of global upper solutions. Let

\[
\bar{u}(r, t) = e^{\beta t} \left( K + e^{-M\eta_1} \right), \quad \eta_1 = (r - 1) e^{\beta t}, \quad r > 1, t > 0,
\]

\[
\bar{v}(r, t) = e^{\beta t} \left( K + e^{-M\eta_2} \right), \quad \eta_2 = (r - 1) e^{\beta t}, \quad r > 1, t > 0,
\]

where \(M_1 = (K + 1)^{\alpha/(p-1)}, M_2 = (K + 1)^{\beta/(q-1)}, f_1 = L_1(2-p)/p, f_2 = L_2(2-q)/q, \) and \(K = 1\) are large constants satisfying

\[
K > \max \left\{ \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{2 - p}{pe}, \frac{2 - q}{qe} \right\},
\]

\[
L_1 > \max \left\{ \frac{p(p - 1) e^{M_1}}{p - 2 + peK}, \frac{\alpha pq}{2(p - 1)(q - 2 + peK)} \right\}.
\]

Obviously, we have \(\bar{u}(r, 0) \geq u_0(r)\) and \(\bar{v}(r, 0) \geq v_0(r)\) for \(r > 1\), and a direct computation yields

\[
\frac{\partial}{\partial r} \left( \frac{\partial^2 u}{\partial r^2} \right) = (p - 1) M_1^{p-1} e^{((p-1)M_1-1)e^{\beta t}} e^{-(p-1)(p-1)M_1 e^{\beta t}} < 0,
\]

\[
\frac{\partial}{\partial r} \left( \frac{\partial^2 v}{\partial r^2} \right) = (p - 1) M_2^{p-1} e^{((p-1)M_2-1)e^{\beta t}} e^{-(p-1)(p-1)M_2 e^{\beta t}} < 0.
\]

Noticing that \(- ye^{-y} \geq -e^{-1}\) for \(y > 0\), we have

\[
\frac{\partial}{\partial t} \bar{u} = L_1 e^{\beta t} \left( K + e^{-M_1(r-1)e^{\beta t}} \right) + f_1 e^{\beta t} \left( -M_1 (r - 1) e^{\beta t} e^{-M_1(r-1)e^{\beta t}} \right)
\]

\[
\geq L_1 e^{\beta t} \left( K + e^{-M_1(r-1)e^{\beta t}} \right) - f_1 e^{\beta t} e^{-1}
\]

\[
\geq \left( L_1 - f_1/K \right) e^{\beta t}.
\]

Due to the choice of \(L_1\) and \((p - 1)L_1 + pf_1 = L_1, \tilde{\lambda}_1 > 0\), then we get

\[
\frac{\partial}{\partial t} \bar{u} \geq \frac{\partial}{\partial r} \left( \frac{\partial^2 u}{\partial r^2} \right) + f_1 \frac{\partial}{\partial r} \frac{\partial^2 u}{\partial r^2} \geq 0.
\]

On the other hand,

\[
\frac{\partial}{\partial t} \left( \frac{\partial^2 v}{\partial r^2} \right) \geq (p + 1)^{\beta} e^{(p-1)M_2 e^{\beta t}} e^{-(p-1)(p-1)M_2 e^{\beta t}} < 0.
\]

Similarly, we have

\[
\frac{\partial}{\partial t} \left( \frac{\partial^2 v}{\partial r^2} \right) \geq (p - 1) M_2^{p-1} e^{((p-1)M_2-1)e^{\beta t}} e^{-(p-1)(p-1)M_2 e^{\beta t}} < 0.
\]

Since that \(\alpha \beta \leq 4(p-1)(q-1)/(pq)\), then

\[
\bar{u}(1, t) = e^{\beta t} (K + 1)^{\beta} \leq (K + 1)^{\beta} e^{4(p-1)(q-1)/pq} < \bar{u}(1, t).
\]

This indicates that

\[
\frac{\partial}{\partial t} \left( \frac{\partial^2 v}{\partial r^2} \right) \geq \frac{\partial}{\partial t} \left( \frac{\partial^2 u}{\partial r^2} \right) \geq 0.
\]

Noticing the global existence in time of \((\bar{u}, \bar{v})\), we get that the solution of the problem (12)–(14) exists globally by the comparison principle. The proof is complete.

**Proposition 4.** If \(\alpha \beta > 4(p-1)(q-1)/(pq)\), then the nonnegative nontrivial solutions of the system (12)–(14) blow up in finite time for large initial data.

**Proof of Proposition 4.** The proposition is proved by constructing a kind of lower blow-up solutions. Set

\[
\bar{u}(r, t) = (T - t)^{-k_1} f_1(\bar{\xi}), \quad \bar{\xi} = (r - 1) (T - t)^{-l_1}, \quad r > 1, t > 0,
\]

\[
\bar{v}(r, t) = (T - t)^{-k_2} f_2(\bar{\eta}), \quad \bar{\eta} = (r - 1) (T - t)^{-l_2}, \quad r > 1, t > 0,
\]

where \(T > 0\), and

\[
k_1 = \frac{(q - 1)(2(p - 1) + \alpha p)}{\alpha \beta - 4(p - 1)(q - 1)}, \quad l_1 = \frac{1 + (2 - p) k_1}{p},
\]

\[
k_2 = \frac{(p - 1)(2(q - 1) + \beta q)}{\alpha \beta - 4(p - 1)(q - 1)}, \quad l_2 = \frac{1 + (2 - q) k_2}{q}.
\]
Due to $1 < p, q < 2$, $\alpha\beta > 4(p - 1)(q - 1)/(pq)$, we see that $k_1, l_1, k_2, l_2 > 0$ and

$$
k_1 + 1 = (p - 1)k_1 + pl_1, \\
k_2 + 1 = (q - 1)k_2 + ql_2,
$$

where $0 < A_1, A_2 < 1, B_1, B_2 > 0$ are the constants to be determined. It is clear that the above $f_1, f_2$ satisfy (32). Now, we verify that $f_1, f_2$ satisfy (33)–(36) in the distribution sense. We claim that (33) is valid for $0 < \xi < A_1/4$. In fact, we only need to verify that

$$
\frac{1}{2}\left(\left|f_1'(\xi)\right|^{p'-2}f_1''(\xi)\right)^{r} \geq -\tilde{\lambda}_1 T^l [f_1'(\xi)]^{p'-2} f_1''(\xi),
$$

that is,

$$
\frac{p - 1}{2} \left|f_1''(\xi)\right|^{p-2} f_1''(\xi) \geq -\tilde{\lambda}_1 T^l f_1'(\xi),
$$

In the first place, we compute each term in (40) as follows:

$$
\frac{p - 1}{2} f_1''(\xi) = \frac{p - 1}{2} \left[\frac{2pB_1^2}{(p - 2)^2} (A_1 + B_1\xi)^{(4-p)/(p-2)},
$$

$$
-\tilde{\lambda}_1 T^l f_1'(\xi) = -\tilde{\lambda}_1 T^l \left(\frac{pB_1}{p - 2} (A_1 + B_1\xi)^{(2p-2)},
$$

$$
\frac{p - 1}{2} \left|f_1''(\xi)\right|^{p-2} f_1''(\xi) \geq k_1 f_1'(\xi),
$$

Then, the inequalities in (40) are valid provided that, for $0 < \xi < A_1/4$,

$$
\frac{p - 1}{2} \left(\frac{B_1}{p - 2}\right)^{(4-p)/(p-2)} \geq k_1,
$$

So, we choose $B_1 = k_1^{1/(2-p)}[(p-1)p^{p-1}]^{1/(2-p)}$, and $A_1$ is small enough that

$$
A_1 \leq \frac{B_1 (p - 1)}{\tilde{\lambda}_1 T^l (2 - p)}. \tag{43}
$$

Similarly, choose $B_2$, $A_2$, satisfying that

$$
B_2 = \left(\frac{k_2 (2 - q)^{q}}{(q - 1) d^{q-1}}\right)^{(1/q)}, \quad A_2 \leq \frac{B_2 (q - 1)}{\tilde{\lambda}_2 T^l (2 - q)} \tag{44}
$$

to get the validity of (35).

Note that

$$
\left|f_1'(\xi)\right|^{p'-2} f_1''(\xi),
$$

has an easier solution to the problem (12)–(14) with $u_0(r) \geq u(r, 0), v_0(r) \geq v(r, 0)$ if the following inequalities hold:

$$
\left(\left|f_1'(\xi)\right|^{p'-2} f_1''(\xi)\right)^{r} \geq -\tilde{\lambda}_1 T^l [f_1'(\xi)]^{p'-2} f_1''(\xi),
$$

$$
\left(\left|f_2'(\eta)\right|^{p'-2} f_2''(\eta)\right)^{r} \geq -\tilde{\lambda}_2 T^l [f_2'(\eta)]^{p'-2} f_2''(\eta),
$$

Note that

$$
\left(\left|f_1'(\xi)\right|^{p'-2} f_1''(\xi)\right)^{r} \geq -\tilde{\lambda}_1 T^l [f_1'(\xi)]^{p'-2} f_1''(\xi),
$$

So if we assume that $f_1, f_2$ satisfy

$$
f_1(\xi), f_2(\eta) \geq 0, \quad f_1'(\xi), f_2'(\eta) \leq 0,
$$

then (27)–(30) hold provided that

$$
\left(\left|f_1'(\xi)\right|^{p'-2} f_1''(\xi)\right)^{r} \geq -\tilde{\lambda}_1 T^l [f_1'(\xi)]^{p'-2} f_1''(\xi),
$$

$$
\left(\left|f_2'(\eta)\right|^{p'-2} f_2''(\eta)\right)^{r} \geq -\tilde{\lambda}_2 T^l [f_2'(\eta)]^{p'-2} f_2''(\eta),
$$

Take

$$
f_1(\xi) = \left((A_1 + B_1\xi)^{(4-p-2)} - (2A_1)^{(4-p-2)}\right)^{r}, \quad \xi \geq 0,
$$

$$
f_2(\eta) = \left((A_2 + B_2\eta)^{(4-p-2)} - (2A_2)^{(4-p-2)}\right)^{r}, \quad \eta \geq 0,
$$

where $0 < A_1, A_2 < 1, B_1, B_2 > 0$ are the constants to be determined. It is clear that the above $f_1, f_2$ satisfy (32). Now, we verify that $f_1, f_2$ satisfy (33)–(36) in the distribution sense. We claim that (33) is valid for $0 < \xi < A_1/4$. In fact, we only need to verify that

$$
\frac{1}{2}\left(\left|f_1'(\xi)\right|^{p'-2} f_1''(\xi)\right)^{r} \geq -\tilde{\lambda}_1 T^l [f_1'(\xi)]^{p'-2} f_1''(\xi),
$$

that is,

$$
\frac{p - 1}{2} \left|f_1''(\xi)\right|^{p-2} f_1''(\xi) \geq k_1 f_1'(\xi),
$$

In the first place, we compute each term in (40) as follows:

$$
\frac{p - 1}{2} f_1''(\xi) = \frac{p - 1}{2} \left[\frac{2pB_1^2}{(p - 2)^2} (A_1 + B_1\xi)^{(4-p)/(p-2)},
$$

$$
-\tilde{\lambda}_1 T^l f_1'(\xi) = -\tilde{\lambda}_1 T^l \left(\frac{pB_1}{p - 2} (A_1 + B_1\xi)^{(2p-2)},
$$

$$
\frac{p - 1}{2} \left|f_1''(\xi)\right|^{p-2} f_1''(\xi) \geq k_1 f_1'(\xi),
$$

Then, the inequalities in (40) are valid provided that, for $0 < \xi < A_1/4$,

$$
\frac{p - 1}{2} \left(\frac{B_1}{p - 2}\right)^{(4-p)/(p-2)} \geq k_1,
$$

So, we choose $B_1 = k_1^{1/(2-p)}[(p-1)p^{p-1}]^{1/(2-p)}$, and $A_1$ is small enough that

$$
A_1 \leq \frac{B_1 (p - 1)}{\tilde{\lambda}_1 T^l (2 - p)}. \tag{43}
$$

Similarly, choose $B_2$, $A_2$, satisfying that

$$
B_2 = \left(\frac{k_2 (2 - q)^{q}}{(q - 1) d^{q-1}}\right)^{(1/q)}, \quad A_2 \leq \frac{B_2 (q - 1)}{\tilde{\lambda}_2 T^l (2 - q)} \tag{44}
$$

to get the validity of (35).
Next, we verify that $f_1$ and $f_2$ given by (37) and (38) also satisfy the boundary conditions (34) and (36):

$$-|f_1'(0)|^{p-2}f_1'(0) = \left(\frac{pB_1}{2-p}A_1^{2/(p-2)}\right)^{p-1}$$

$$= \left(\frac{pB_1}{2-p}\right)^{p-1}A_1^{(p-1)/(p-2)},$$

(45)

$$f_2'(0) = \left(A_2^{q/(q-2)} - 2A_2^{q/(q-2)}\right)\sigma$$

$$= \left(1 - 2^{q/(q-2)}\right)\gamma A_2^{q/(q-2)}.$$

From (34), we need

$$\left(\frac{pB_1}{2-p}\right)^{p-1}A_1^{(p-1)/(p-2)} \leq \left(1 - 2^{q/(q-2)}\right)^{a}A_2^{a/(q-2)}.$$

Set $A_2 = \sigma^\alpha$, where $\sigma$ is to be determined. Then rewriting the last inequality, we get

$$A_1^{2(p-1)/(p-2) - q\alpha/(q-2)} \leq \left(\frac{2 - p}{pB_1}\right)^{p-1},$$

(47)

which can be obtained by choosing $A_1 > 0$ small enough, if

$$\frac{2(p-1)}{p-2} - \frac{q\alpha}{q-2} > 0.$$

(48)

In a similar discussion on the boundary condition (36), for enough small $A_1$, the following inequality is needed:

$$\frac{2\sigma(q-1)}{q-2} - \frac{p\beta}{p-2} > 0.$$

Note that $1 < p$ and $q < 2$; therefore, (48)-(49) are equal to

$$\frac{2(p-1)}{q-2} < \frac{\sigma p}{q-2} < \frac{p\beta}{2(q-1)}.$$

(50)

Recall the assumption that $\alpha\beta > 4(p-1)(q-1)/pq$, so there exists a constant $\sigma > 0$, such that (50) holds. Furthermore, we can choose $A_1$ as small as needed.

Therefore, the solution $(u, v)$ of the problem (12)–(14) blows up in a finite time if $(u_0(r), v_0(r))$ is large enough such that

$$u_0(r) \geq \bar{u}(r, 0), \quad v_0(r) \geq \bar{v}(r, 0), \quad r > 1.$$

The proof is completed. \qed

**Proposition 5.** If $\alpha\beta \neq (p-1)(q-1)$, then every nonnegative nontrivial solution of the system (12)–(14) with small initial data exists globally.

**Proof of Proposition 5.** We seek the steady-state solution of the system (12)–(14):

$$u(r, t) = \bar{u}(r), \quad v(r, t) = \bar{v}(r), \quad r > 1, \quad t > 0.$$

(52)

A direct calculation shows that $\bar{u}(r), \bar{v}(r)$ should satisfy

$$\left(r^\lambda |\bar{u}'|^p \bar{u}'\right)' = 0, \quad \left(r^\lambda |\bar{v}'|^q \bar{v}'\right)' = 0, \quad r > 1,$$

$$-|\bar{u}'(1)|^{p-2} \bar{u}'(1) = \bar{v}(1), \quad -|\bar{v}'(1)|^{q-2} \bar{v}'(1) = \bar{u}(1),$$

(53)

which implies that

$$\bar{u}' = -A_2^{\alpha/(p-1)} r^{-\lambda_1/(p-1)}, \quad \bar{v}' = -A_1^{\beta/(q-1)} r^{-\lambda_2/(q-1)}, \quad r > 1.$$

(54)

with $A_1 = \bar{u}(1), A_2 = \bar{v}(1)$. Integrating the above equalities yields

$$\bar{u}(r) = \left(A_1 - \frac{A_2^{\alpha/(p-1)}}{\lambda_1/(p-1) - 1}\right)$$

$$+ \frac{A_2^{\alpha/(p-1)}}{\lambda_1/(p-1) - 1} r^{1 - \lambda_1/(p-1)},$$

(55)

$$\bar{v}(r) = \left(A_2 - \frac{A_1^{\beta/(q-1)}}{\lambda_2/(q-1) - 1}\right)$$

$$+ \frac{A_1^{\beta/(q-1)}}{\lambda_2/(q-1) - 1} r^{1 - \lambda_2/(q-1)}.$$

\qed

In particular, we let

$$A_1 - \frac{A_2^{\alpha/(p-1)}}{\lambda_1/(p-1) - 1} = 0, \quad A_2 - \frac{A_1^{\beta/(q-1)}}{\lambda_2/(q-1) - 1} = 0.$$  

(56)

Define $C_1 = \lambda_1/(p-1) - 1, C_2 = \lambda_2/(q-1) - 1$, and then $C_1 > 0, C_2 > 0$. Noticing that $\alpha\beta \neq (p-1)(q-1)$, we get

$$A_2 = \left(C_1^{\beta/(q-1)} C_2^{\alpha/(p-1)}\right) (p-1)(q-1)/(\alpha\beta).$$

(57)

$$A_1 = C_1 A_2^{\alpha/(p-1)}.$$

Therefore, the bounded positive functions:

$$\bar{u}(r) = \frac{A_2^{\alpha/(p-1)}}{C_1} r^{-C_1}, \quad \bar{v}(r) = \frac{A_1^{\beta/(q-1)}}{C_2} r^{-C_2}$$

are just a couple of steady-state solutions of the problem (12)–(14) with the initial data $\bar{u}(r), \bar{v}(r)$. By the comparison principle, for any initial data $u_0(r), v_0(r)$ which is small enough to satisfy

$$u_0(r) \leq \bar{u}(r), \quad v_0(r) \leq \bar{v}(r), \quad r > 1,$$

(59)

the solutions of the problem (12)–(14) exist globally in time.
Remark 6. Due to the fact that \((p-1)(q-1) < 4(p-1)(q-1)/(pq)\) for \(1 < p, q < 2\), it is seen from Propositions 3 and 5 that all nonnegative solutions to the system (12)–(14) with enough small initial data exist globally in time.

Now, we prove the main result for the system (7)–(9), that is, Theorem 2.

Proof of Theorem 2. Noticing that the functions \(u_0(x), v_0(x)\) are bounded, we can choose two bounded, radially symmetric functions denoted by \(u_1(x), v_1(x)\) satisfying that \(u_1(|x|) \geq u_0(x), v_1(|x|) \geq v_0(x)\), respectively. By using Proposition 3 and the comparison principle, we can obtain the global existence of solutions to the system (7)–(9).

For the initial data \((u_0, v_0)\) is large enough such that \(u_0(x) \geq u(|x|, 0), v_0(x) \geq v(|x|, 0)\), here \(u(|x|, 0), v(|x|, 0)\) are defined in the proof of Proposition 4 if \(\alpha \beta > 4(p-1)(q-1)/(pq)\), then the solutions of the system (7)–(9) with such \((u_0, v_0)\) blow up in a finite time by the comparison principle and Proposition 4.

On the other hand, using the comparison principle again and combining with Proposition 5, we see that the solution \((u, v)\) of (7)–(9) exists globally if

\[
\begin{align*}
u_0(x) &\leq \frac{(p-1)A_2^{\alpha/(p-1)}}{\lambda_1 + N - p} |x|^{-(\lambda_1+p-1)/(p-1)}, \\
u_0(x) &\leq \frac{(q-1)A_1^{\beta/(q-1)}}{\lambda_2 + N - q} |x|^{-(\lambda_2+q-1)/(q-1)},
\end{align*}
\]

where

\[
A_2 = \left( \frac{\lambda_1 + N - p}{p - 1} \right)^{\beta/(q-1)} \times \left( \frac{\lambda_2 + N - q}{q - 1} \right)^{(p-1)/(q-1) - \alpha \beta/(p-1)},
\]

\[
A_1 = \left( \frac{\lambda_1 + N - p}{p - 1} \right)^{\alpha/(p-1)}.
\]

The proof is complete. \(\square\)

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References


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