Relaxation Problems Involving Second-Order Differential Inclusions

Adel Mahmoud Gomaa

1 Taibah University, Faculty of Applied Science, Department of Applied Mathematics, Al-Madinah, Saudi Arabia
2 Mathematics Department, Faculty of Science, Helwan University, Cairo, Egypt

Correspondence should be addressed to Adel Mahmoud Gomaa; gomaa5@hotmail.com

Received 12 November 2012; Accepted 12 March 2013

1. Introduction

Second-order differential inclusions of three boundary conditions were studied by many authors [1–6], using Hartman-type functions. Such a function was first introduced by [7] for two boundary conditions. Moreover, in [8] we consider second-order differential inclusions with four boundary conditions,

\[ \ddot{u}(t) \in F(t, u(t), \dot{u}(t)), \quad \text{a.e. on } [0, T], \]
\[ u(0) = x_0, \quad u(\eta) = u(\theta) = u(T), \]

where \( 0 < \eta < \theta < T \) and \( F \) is a multifunction from \([0, T] \times \mathbb{R}^n \times \mathbb{R}^n \) to the nonempty compact convex subsets of \( \mathbb{R}^n \), while in [9] we study four-point boundary value problems for differential inclusions and differential equations with and without multivalued moving constraints.

In the present paper, we study relaxation results for the second-order differential inclusions, with four boundary conditions,

\[ \ddot{u}(t) \in F(t, u(t), \dot{u}(t)), \quad \text{a.e. on } [0, 1], \]
\[ u(0) = 0, \quad u(\eta) = u(\theta) = u(1), \]

and, with \( m \geq 3 \) boundary conditions,

\[ \ddot{u}(t) \in F(t, u(t), \dot{u}(t)), \quad \text{a.e. on } [0, 1], \]
\[ \dot{u}(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i), \]

where \( 0 < \eta < \theta < 1 \) and \( F \) is a multifunction from \([0, 1] \times \mathbb{R}^n \) to the nonempty compact subsets of \( \mathbb{R}^n \).

In conjunction with Problem (P) and Problem (Q) we also consider the following problems:

\[ \ddot{u}(t) \in \text{ext } F(t, u(t), \dot{u}(t)), \quad \text{a.e. on } [0, 1], \]
\[ \dot{u}(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i), \]

where \( 0 < \eta < \theta < 1 \) and \( F \) is a multifunction from \([0, 1] \times \mathbb{R}^n \) to the nonempty compact subsets of \( \mathbb{R}^n \).
2. Notations and Preliminaries

Throughout this paper we let \( I = [0,1] \) and \( 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1 \). We will use the following definitions, notations, and summarize some results.

(i) A multifunction \( F \) from a metric space \((X, d)\) to the set \( P_f(Y)\) of all closed subsets of another metric space \( Y \) is lower semicontinuous (l. s. c.) at \( x_0 \in X \) if for every open subset \( V \) of \( Y \) with \( F(x_0) \cap V \neq \emptyset \) there exists an open subset \( U \) of \( X \) such that \( x_0 \in U \) and \( F(x) \cap V \neq \emptyset \) for all \( x \in U \). \( F \) is l. s. c. if it is l. s. c. at each \( x \in X \).

(ii) \( F \) is upper semicontinuous (u. s. c.) at \( x_0 \in X \) if for every open subset \( V \) in \( Y \) and containing \( F(x_0) \) there exists an open subset \( U \) in \( X \) such that \( x_0 \in U \) and \( F(x) \subseteq V \), for all \( x \in U \). \( F \) is u. s. c. if it is u. s. c. at each \( x \in X \).

(iii) A multifunction \( F \) from \( I \) into the set \( P_f(X) \) of all closed subsets of \( X \) is measurable if for all \( x \in X \) the function \( t \to d(x, F(t)) = \inf \{ \| x - y \| : y \in F(t) \} \) is measurable \([10–13]\).

(iv) Let \((\Omega, \Sigma)\) be a measurable space and \( X \) a separable Banach space. We say that \( F : \Omega \to P_f(X) \) is graph measurable if

\[
gr(F) = \{(x, \xi) \in \Omega \times X : \xi \in F(x)\} \in \Sigma \times \mathcal{B}(X),
\]

where \( \mathcal{B}(X) \) is the Borel \( \sigma \)-field of \( X \). For further details we refer to \([14–16]\).

(v) \( F \) is continuous if it is lower and upper semicontinuous.

(vi) For each \( A, B \in P_f(X) \), the Hausdorff metric is defined by

\[
d_H(A, B) = \max \left( \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right).
\]

It is known that the space \((P_f(X), d_H)\) is a generalized metric space, if the sets are not bounded (see, for instance, \([14, 15]\)).

(vii) A multifunction \( F \) is Hausdorff continuous \((d_H\text{-continuous})\) if it is continuous from \( X \) into the metric space \((P_f(Y), d_H)\).

(viii) If \( F \) has compact values in \( Y \), then \( F \) is \( d_H\text{-continuous} \) if and only if it is continuous \([14, 17]\).

(ix) We denote by \( P_{cke}(\mathbb{R}^n) \) the nonempty compact convex subsets of \( \mathbb{R}^n \).

(x) The Banach spaces \( C(I, \mathbb{R}^n), C^1(I, \mathbb{R}^n) \), and \( C^2(I, \mathbb{R}^n) \) endowed with the norms

\[
\|u\|_{C^1} = \max_{t \in I} |u(t)|, \quad \|u\|_{C^2} = \max \{ \|u\|_{C^1}, \| \dot{u} \|_{C^1} \},
\]

\[
\|u\|_{C^2} = \max \{ \|u\|_{C^0}, \| \ddot{u} \|_{C^0}, \| \dot{u} \|_{C^1} \},
\]

respectively.

(xi) \( L^1_w(I, \mathbb{R}^n) \) denotes the space \( L^1(I, \mathbb{R}^n) \) equipped with the weak norm \( \| \cdot \|_w \) which is defined by

\[
\|u\|_w = \sup \left\{ \int_a^b h(t) \, dt : 0 \leq a \leq b \leq 1 \right\}.
\]

(xii) \( W^{2,1}(I, \mathbb{R}^n) \) is the Sobolev space of functions \( u : I \to \mathbb{R}^n \) and \( \dot{u}, \ddot{u} \) are both absolutely continuous functions so \( \dot{u}(t) \in L^1(I, \mathbb{R}^n) \) and it is equipped with the norm \( \| u \|_{W^{2,1}(I, \mathbb{R}^n)} = \| uu \|_{L^1(I, \mathbb{R}^n)} + \| \ddot{u} \|_{L^1(I, \mathbb{R}^n)} \).

(xiii) Let \( R : I \to \mathbb{R}^n \) be a multifunction and \( \delta_R = \{ h \in L^1(I, \mathbb{R}^n) : h(t) \in R(t) \} \).

(xiv) By a solution of \((P)\) (resp., of \((P_a)\)) we mean a function \( u \in W^{2,1}(I, \mathbb{R}^n) \) such that \( \dot{u}(t) = h(t) \) a.e. on \( I \) with \( h \in \delta_{F(\cdot, \cdot; \eta)} \) (resp., \( h \in \delta_{F(\cdot, \cdot; \eta)}^{ext} \)) and \( u(0) = 0 \), \( u(1) = u(0) \).

(xv) By a solution of \((Q)\) (resp., of \((Q_a)\)) we mean a function \( u \in W^{2,1}(I, \mathbb{R}^n) \) such that \( \dot{u}(t) = h(t) \) a.e. on \( I \) with \( h \in \delta_{F(\cdot, \cdot; \eta)} \) (resp., \( h \in \delta_{F(\cdot, \cdot; \eta)}^{ext} \)) and \( u(0) = 0 \), \( u(1) = \sum_{i=1}^n a_i \xi_i \).

(xvi) In the sequel by \( \Delta_P \) (resp., \( \Delta_{P_a} \)) we denote the solution set of Problem \((P)\) (resp., of Problem \((P_a)\)). Moreover, by \( \Delta_Q \) (resp., \( \Delta_{Q_a} \)) we denote the solution set of Problem \((Q)\) (resp., of Problem \((Q_a)\)).

Definition 1. Let \( E \) be a Banach space and let \( Y \) be a metric space. A multifunction \( G : I \times Y \to P_{cke}(E) \) has the Scorza-Dragoni property (the SD-property) if for every \( \epsilon > 0 \) there exists a closed set \( A \subset I \) such that the Lebesgue measure \( \mu(I \setminus A) \) is less than \( \epsilon \) and \( GL_{\epsilon} \) is continuous. The multifunction \( G \) is called integrally bounded on compacta in \( Y \) if, for any compact subset \( Q \subset Y \), we can find an integrable function \( \mu_Q : I \to \mathbb{R}^+ \) such that \( \| y \| : y \in G(t, z) \leq \mu_Q(t) \), for almost every \( z \in Q \).

Theorem 2 (see \([18]\)). Let \( Y \) be a complete metric space, \( E \) a separable Banach space, \( E_\sigma \) the Banach space \( E \) endowed with the weak topology, \( M : I \times Y \to P_{cke}(E_\sigma) \), and \( K \) a compact subset of \( C(I, Y) \). Furthermore, let \( R : K \to 2^{L^1(I,E)} \) be a multifunction defined by

\[
R(y) = \left\{ g \in L^1(I,E) : g(t) \in M(t,y(t)) \text{ a.e. on } I \right\}.
\]

If \( M \) has the SD-property and is integrably bounded on compacta in \( Y \), then the set

\[
A_K = \left\{ f \in C(K,L^1_w(I,E)) : f(y) \in R(y) \forall y \in K \right\}
\]

is nonempty complete subset of the space \( C(K,L^1_w(I,E)) \). Moreover, \( A_K = A_{ext}^{CK} \) where \( L^1_w(I,E) \) is the space of equivalence classes of Bochner-integrable functions \( v : I \to E \) with the norm \( \| v \|_w = \sup_{t \in I} \int_0^1 v(s) \, ds \) and

\[
A_{ext}^{CK} = \left\{ f \in C(K,L^1_w(I,E)) : f(y) \in ext R(y) \forall y \in K \right\}.
\]

Lemma 3 (see \([19]\)). For \( p \) such that \( 1 < p < \infty \) let \( u_n, u \in \mathbb{R}^n \leq L^p(I, \mathbb{R}^n), \sup_{n \in \mathbb{N}} \| u_n \|_p < \infty \) and \( u_n \rightharpoonup u \) with respect to the weak norm \( \| \cdot \|_w \). Then \( u_n \rightharpoonup u \) weakly in \( L^p(I, \mathbb{R}^n) \).

Next we state a preliminary lemma, for \( 0 < \eta < \theta < 1 \), which is useful in the study of four boundary problems for the differential equations and the differential inclusions, and
moreover we summarize some properties of a Hartman-type function.

Lemma 4 (see [8]). Let $G : l \times l \rightarrow \mathbb{R}$ be the function defined as follows:

$$G(t, \tau) = \begin{cases} \frac{-\tau}{\theta - \eta} & \text{if } 0 \leq \tau < t, \\ \frac{t(t - \theta) + \eta (r - t - 1)}{\theta - \eta} & \text{if } \eta < \tau \leq \theta, \\ \frac{1 - \tau}{1 - \theta} & \text{if } \theta < \tau \leq 1, \end{cases}$$

when $\eta < t < \theta$,

$$G(t, \tau) = \begin{cases} \frac{-\tau}{\theta - \eta} & \text{if } 0 \leq \tau \leq \eta, \\ \frac{\tau (t - \theta + 1) + \eta (r - t - 1)}{\theta - \eta} & \text{if } \eta < \tau \leq t, \\ \frac{t (t - \theta) + \eta (r - t - 1)}{\theta - \eta} & \text{if } t < \tau \leq \theta, \\ \frac{1 - \tau}{1 - \theta} & \text{if } \theta < \tau \leq 1, \end{cases}$$

lastly if $\theta < t \leq 0$,

$$G(t, \tau) = \begin{cases} \frac{-\tau}{\theta - \eta} & \text{if } 0 \leq \tau \leq \eta, \\ \frac{\eta (r - t - 1) + \tau (t - \theta + 1)}{\theta - \eta} & \text{if } \eta < \tau \leq t, \\ \frac{1 - \tau}{1 - \theta} & \text{if } t < \tau \leq \theta, \\ \frac{1 - \tau}{1 - \theta} & \text{if } \theta < \tau \leq 1. \end{cases}$$

Then the following hold.

(i) If $u \in W^{2,1} (1, \mathbb{R}^n)$ with $u(0) = x_0, u(1) = u(\theta) = u(\eta)$, then

$$u(t) = x_0 + \int_0^1 G(t, \tau) \dot{u}(\tau) d\tau, \quad \forall t \in I;$$

(ii) If $w \in L^1 (1, \mathbb{R}^n)$, then for all $t \in I$,

$$\int_0^1 G(t, \tau) w(\tau) d\tau = \int_0^t (t - \tau) w(\tau) d\tau - \int_0^\eta \frac{\eta (r - \tau) (t + 1)}{\theta - \eta} w(\tau) d\tau$$

$$+ \int_0^\theta \frac{\theta (t - \tau) + (r - \eta)}{\theta - \eta} w(\tau) d\tau + \int_\theta^1 \frac{1 - \tau}{1 - \theta} w(\tau) d\tau;$$

(iii) $\sup_{t \in I} |G(t, \tau)| \leq 2$, $\sup_{t \in I} |\partial G(t, \tau) / \partial t| \leq 1.$

Let $c_1, c_2, a \in L^p(I, \mathbb{R}^n)$, $1 < p < \infty$, and let $L$ be a linear operator from $C(I, \mathbb{R}) \times C(I, \mathbb{R})$ to $C(I, \mathbb{R}) \times C(I, \mathbb{R})$ defined by $L(f, g) = (f, g)$ such that, for all $t \in I$,

$$f(t) = \int_0^T |G(t, \tau)| (c_1 (r) f(\tau) + c_2 (r) g(\tau)) d\tau,$$

$$g(t) = \int_0^T \frac{\partial G(t, \tau)}{\partial t} (c_1 (r) f(\tau) + c_2 (r) g(\tau)) d\tau.$$

If $c_1 = c_2 = 0$, then clearly $L = 0$. We note that if $\mathcal{K} = \{h_1, h_2 \in C(I, \mathbb{R}) \times C(I, \mathbb{R}) : h_1(t), h_2(t) \geq 0, \forall t \in I\}$, then $L(\mathcal{K}) \subseteq \mathcal{K}$. Moreover, the spectral radius $r(L) = \lim \|L^n\|^{1/n}$ is an eigenvalue of $L$ with an eigenvector in $\mathcal{K}$ [20].

3. Relaxation Theorems

In this section, both Theorems 5 and 7 improve [19, Theorem 4.1] with [21, Theorem 6]. Indeed in [19] Papageorgiou considered $(P)$ and $(P_\alpha)$ with the two boundary conditions $u(0) = u(1) = 0$ and in [21] Ibrahim and Gomaa study the same problems with three boundary conditions $u(0) = x_0, u(\eta) = u(1)$.

Theorem 5. Let $F : I \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow P_{c,\alpha}(\mathbb{R}^n)$ be a multifunction such that

(i) for each $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, the multifunction $F(\cdot, x, y)$ is measurable,

(ii) $d_H(F(t, x, y), F(t, x', y')) \leq \alpha_1 (t) \| x - x' \| + \alpha_2 (t) \| y - y' \|$ a.e. with $\alpha_1, \alpha_2 \in L^1(I, \mathbb{R}^+) \text{ and } \| \alpha_1 + \alpha_2 \| < 1/2$,

(iii) for each $(t, x, y) \in I \times \mathbb{R}^n \times \mathbb{R}^n$,

$$\| F(t, x, y) \| = \sup \| v \| : v \in F(t, x, y) \leq a(t) + c_1 (t) \| x \| + c_2 (t) \| y \|$$

with $a, c_1, c_2 \in L^p(I, \mathbb{R}^+) \text{ } 1 < p < \infty$,

(iv) the spectral radius, $r(L)$, is less than 1.

Then for each solution $u \in \Delta_{p, \alpha}$ there is a sequence $(u_m(\cdot))_{m \in \mathbb{N}} \subset \Delta_{p}$ converging to $u(\cdot)$ in $C(I, \mathbb{R}^n)$, $\| || \cdot || \|_C$.

Proof. From [9, Theorem 2.1], we obtain $\Delta_{p, \alpha} \neq \emptyset$. Moreover, we can say that $\| F(t, x, y) \| \leq \alpha_1 (t)$ a.e. on $I$ for some $\alpha_1 \in L^p(I, \mathbb{R}^+)$. Let $u \in \Delta_{p, \alpha}$. Then

$$\dot{u}(t) = h(t), \text{ a.e. on } I,$$

$$u(0) = 0, \text{ } u(\eta) = u(\theta) = u(1),$$

where $h(t) \in F(t, u(t), \dot{u}(t))$ a.e. on $I$. Assume that $f : L^1(I, \mathbb{R}^n) \rightarrow C(I, \mathbb{R}^n)$ is a function such that, for each $h \in L^1(I, \mathbb{R}^n)$, $f(h) \in W^{1,2}(I, \mathbb{R}^n)$ is the unique solution of the second-order differential equation

$$\ddot{u}(t) = h(t), \text{ a.e. on } I,$$

$$u(0) = 0, \text{ } u(\eta) = u(\theta) = u(1).$$
Let $\delta = \{u \in L^1(I, \mathbb{R}^n) : \|u(t)\| \leq a_1(t) \text{ a.e. on } I\}$. It is easy to see that $f(\delta)$ is convex. Let $(u_n)_{n \in \mathbb{N}}$ be a sequence in $f(\delta)$. Hence, $u_n \in \mathcal{W}^{1,1}(I, \mathbb{R}^n)$ with $u_n(0) = x_0$, $u_n(\eta) = u_n(\theta) = u_n(\tau)$ and $\epsilon$ for each $v \in f(\delta)$. Define a multifunction $R : f(\delta) \to 2^{L^1(I, \mathbb{R}^n)}$ by

$$R(u) = \left\{ g \in L^1(I, \mathbb{R}^n) : g(t) \in F(\tau, u(t), \dot{u}(t)) \text{ a.e. on } I \right\}.$$  

(22)

Assume that $Y = \mathbb{R}^n \times \mathbb{R}^n$ and set a multifunction $M : I \times Y \to 2^{\mathbb{R}^n}$ such that $M(t, (x, y)) = F(t, x, y)$. From Theorem 3.1 in [23], $M$ has SD-property. Let $(g_n)_{n \in \mathbb{N}}$ be a sequence in $R(u)$ for some $u \in f(\delta)$. So, for each $t \in I$,

$$\lim_{n \to \infty} g_n(t) = g(t) \in F(t, u(t), \dot{u}(t))$$  

(23)

because $F$ has closed values in $\mathbb{R}^n$. Therefore, $g \in \delta^1_{F,\gamma(t),\gamma(t)}(\delta)$ which implies that $R(\cdot)$ has compact values in $\mathbb{R}^n$. We can apply Theorem 2 to find a continuous function $h : f(\delta) \to L^1(I, \mathbb{R}^n)$ such that $\theta(u) \in \text{ext} R(u)$, for all $u \in f(\delta)$. We see that $\theta(u(t)) \in \text{ext}(M(t, u(t), \dot{u}(t)))$ [24], hence $\theta(u(t)) \in \text{ext}(F(t, u(t), \dot{u}(t)))$. Let $(\xi_n)_{n \in \mathbb{N}}$ be a sequence in $\text{ext}(\delta^1_{F,\gamma(t),\gamma(t)}(\delta))$ be chosen such that $\xi_n \to \xi \in \mathcal{E}$, where $\xi \in \mathcal{E}$ is the function which for each $u \in f(\delta)$, $\eta(u) = \theta(u(t))$. For each $u \in f(\delta)$, we have $\|\theta(u(t))\| \leq a_1$ and so $\theta(u(t)) \in \delta$. Then, $h$ is a function from $f(\delta)$ into $f(\delta)$ and also we see that $\eta$ is continuous [19]. Now let $\epsilon_n \to 0$, $S_n = S_n$ and $\xi_n = \xi_n$. Then, for each $n \in \mathbb{N}$, the function $\text{fo}_{\xi_n}$ is continuous with respect to the compact set $f(\delta)$ into itself. From Schauder’s fixed point theorem, $\text{fo}_{\xi_n}$ has a fixed point $u_n$, by passing to a subsequence if necessary, we may assume that $u_n \to \tilde{u}$ in $C^1(I, \mathbb{R}^n)$. Then, we obtain

$$\|u_n(t) - u(t)\|$}

$$\leq \int_0^1 \int_0^t (t - \tau) \left(\xi_n(\tau) - \eta(t)\right) d\tau$$

$$- \int_0^t \int_0^\eta (t - \tau) (t + 1) \frac{\xi_n(\tau) - \eta(t)}{\tau - \eta} d\tau$$

$$+ \int_0^t \int_0^\eta \xi_n(\tau) - \eta(t) d\tau ds$$

$$\leq \int_0^1 \int_0^t \left(\right) d\tau$$

$$+ \int_0^t (t - \tau) \left(\right) d\tau$$

$$+ \int_0^t \int_0^\eta (t - \eta) (t + 1) \frac{\xi_n(\tau) - \eta(t)}{\tau - \eta} d\tau$$

$$+ \int_0^t \int_0^\eta \left(\right) d\tau$$

$$\leq \int_0^1 \int_0^t (t - \tau) \left(\right) d\tau$$

$$+ \int_0^t (t - \tau) \left(\right) d\tau$$

$$+ \int_0^t \int_0^\eta (t - \eta) (t + 1) \frac{\xi_n(\tau) - \eta(t)}{\tau - \eta} d\tau$$

$$+ \int_0^t \int_0^\eta \left(\right) d\tau$$

$$\leq \int_0^1 \int_0^t (t - \tau) \left(\right) d\tau$$

$$+ \int_0^t (t - \tau) \left(\right) d\tau$$

$$+ \int_0^t \int_0^\eta (t - \eta) (t + 1) \frac{\xi_n(\tau) - \eta(t)}{\tau - \eta} d\tau$$

$$+ \int_0^t \int_0^\eta \left(\right) d\tau$$
Abstract and Applied Analysis

As 

Moreover, we have

But \( \xi_n - S_n \to 0 \) with respect to the norm \( \| \cdot \| \) from Lemma 3 we get \( \xi_n - S_n \to 0 \) weakly in \( L^1(I, \mathbb{R}^n) \). So we have

Moreover,

\[
\int_0^1 \int_0^t (t - \tau) \| h(\tau) - S_n(\tau) \| \, d\tau + \int_0^1 \frac{1 - \tau}{1 - \theta} \| h(\tau) - S_n(\tau) \| \, ds
\leq \int_0^1 \int_0^t (t - \tau) (e_n + \alpha_1(\tau) \| u(\tau) - u_n(\tau) \|)
\]

Since by assumption (ii), \( \alpha_1 + \alpha_2 < 1/2 \) we get \( u = \hat{u} \).

So \( u_n \to u \) in \( C^1(I, \mathbb{R}^n) \) and \( u \in \Delta_p \) where the closure is taken in \( C^1(I, \mathbb{R}^n) \) which means that \( \Delta_p \subseteq \Delta_P \). Therefore, the proof is complete if we show that \( \Delta_P \) is closed. Indeed if \( y_n \in \Delta_P \) and \( y_n \to y \) in \( C^1(I, \mathbb{R}^n) \), then \( y_n \to f(y_n) \) for \( y_n \in \delta_{f^1}^{\mu, \nu}(I, \mathbb{K}) \). From assumption (iii) and the Dunford-Pettis theorem, \( \{y_n\}_{n \in \mathbb{N}} \) is weakly sequentially compact in \( L^1(I, \mathbb{R}^n) \). So we can say that \( \{y_n\}_{n \in \mathbb{N}} \) in \( L^1(I, \mathbb{R}^n) \). By [25, Theorem 3.1], we get

\[
y(t) \in \overline{\text{conv}} \lim_{n \to \infty} y_n(t) \subseteq \overline{\text{conv}} \lim_{n \to \infty} F(t, v_n(t), \dot{v}_n(t)) = F(t, v(t), \dot{v}(t)) \quad \text{a.e. on } I.
\]

Moreover, \( f(y_n) \to f(y) \) in \( L^1(I, \mathbb{R}^n) \) for \( y \in L^1(I, \mathbb{R}^n) \) and \( y(t) \in F(I, v(t), \dot{v}(t)) \) a.e. on \( I \). Hence, \( v \in \Delta_P \); that is \( \Delta_P \) is closed in \( C^1(I, \mathbb{R}^n) \).

Now we consider the following assumptions:

\[(A_1) \beta \in (0, \pi/2), a_i > 0 \text{ and } \sum_{i=1}^{m-2} a_i < 1; \]

\[(A_2) \sum_{i=1}^{m-2} a_i \cos \beta \xi_i - \cos \beta > 0 \text{ and } K_m = 1/\sum_{i=1}^{m-2} a_i \cos \beta \xi_i - \cos \beta; \]

\[(A_3) C_0 = (\sin \beta/\beta)(1 + K_m) \text{ and } C_1 = \min\{K_m + 1, K_m \sin^2 \beta\}; \]
\((A_4)\) \( S = \{ u \in C^2(I, \mathbb{R}^n) : \dot{u}(0) = 0, \ u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i) \} \);

\((A_5)\) \( \mathcal{G} : I \times I \to \mathbb{R} \) is defined by
\[
\mathcal{G}(t, s) = \begin{cases} 
\frac{1}{\beta} \sin \beta (t - s) & \text{if } 0 \leq s \leq t \leq 1 \\
0 & \text{if } 0 \leq t \leq s \leq 1 \\
\sin \beta (1 - s) - \sum_{i=1}^{m-2} a_i \sin \beta (\xi_i - s), & \text{if } 0 \leq s \leq \xi_1 \\
\sin \beta (1 - s) - \sum_{i=2}^{m-2} a_i \sin \beta (\xi_i - s), & \text{if } \xi_1 < s \leq \xi_2 \\
\sin \beta (1 - s) - \sum_{i=3}^{m-2} a_i \sin \beta (\xi_i - s), & \text{if } \xi_2 < s \leq \xi_3 \\
\vdots \\
\sin \beta (1 - s) - \sum_{i=k}^{m-2} a_i \sin \beta (\xi_i - s), & \text{if } \xi_{k-1} < s \leq \xi_k \\
\vdots \\
\sin \beta (1 - s), & \text{if } \xi_{m-2} < s \leq 1.
\end{cases}
\]

\((A_5)\) \( G : I \times I \to \mathbb{R} \) is defined by
\[
G(t, s) = \begin{cases} 
\cos \beta (t - s) & \text{if } 0 \leq s \leq t \leq 1 \\
0 & \text{if } 0 \leq t \leq s \leq 1 \\
\sin \beta (1 - s) - \sum_{i=1}^{m-2} a_i \sin \beta (\xi_i - s), & \text{if } 0 \leq s \leq \xi_1 \\
\sin \beta (1 - s) - \sum_{i=2}^{m-2} a_i \sin \beta (\xi_i - s), & \text{if } \xi_1 < s \leq \xi_2 \\
\sin \beta (1 - s) - \sum_{i=3}^{m-2} a_i \sin \beta (\xi_i - s), & \text{if } \xi_2 < s \leq \xi_3 \\
\vdots \\
\sin \beta (1 - s) - \sum_{i=k}^{m-2} a_i \sin \beta (\xi_i - s), & \text{if } \xi_{k-1} < s \leq \xi_k \\
\vdots \\
\sin \beta (1 - s), & \text{if } \xi_{m-2} < s \leq 1.
\end{cases}
\]

Lemma 6 (see [26]). If the assumptions \((A_1)\)–\((A_5)\) hold, then

(i) \( 0 \leq \mathcal{G}(t, s) \leq C_0 \) for all \((t, s) \in I \times I \),

(ii) \( \sup_{t,s \in I} |\partial \mathcal{G}(t, s)/\partial t| \leq C_1 \),

(iii) for each \( x \in C^1(I, \mathbb{R}^n) \) there exists a unique function \( u_x \in S \) such that
\[
u_x(t) = \int_0^t \mathcal{G}(s, s)x(s)ds,
\]

(iv) \( (\int_0^1 |\mathcal{G}(t, t)|^k dt)^{1/k} \leq C_0 \) and \( (\int_0^1 |\partial \mathcal{G}(t, t)/\partial t|^k dt)^{1/k} \leq C_1 \).

Proof. (ii) Since
\[
\frac{\partial \mathcal{G}(t, s)}{\partial t} = \begin{cases} 
\cos \beta (t - s) & \text{if } 0 \leq s \leq t \leq 1 \\
0 & \text{if } 0 \leq t \leq s \leq 1 \\
\sin \beta (1 - s) - \sum_{i=1}^{m-2} a_i \sin \beta (\xi_i - s), & \text{if } 0 \leq s \leq \xi_1 \\
\sin \beta (1 - s) - \sum_{i=2}^{m-2} a_i \sin \beta (\xi_i - s), & \text{if } \xi_1 < s \leq \xi_2 \\
\sin \beta (1 - s) - \sum_{i=3}^{m-2} a_i \sin \beta (\xi_i - s), & \text{if } \xi_2 < s \leq \xi_3 \\
\vdots \\
\sin \beta (1 - s) - \sum_{i=k}^{m-2} a_i \sin \beta (\xi_i - s), & \text{if } \xi_{k-1} < s \leq \xi_k \\
\vdots \\
\sin \beta (1 - s), & \text{if } \xi_{m-2} < s \leq 1.
\end{cases}
\]

Theorem 7. Assume that the assumptions \((A_1)\) and \((A_2)\) hold. Let \( F \) be a multifunction from \( I \times \mathbb{R}^n \times \mathbb{R}^n \) to \( P_k(\mathbb{R}^n) \) satisfying the following conditions:

(a) for each \( (x, y) \in \mathbb{R} \times \mathbb{R} \), the multifunction \( F(\cdot, x, y) \) is measurable;

(b) for each \( t \in I \), the function \( (x, y) \to F(t, x, y) \) is continuous with respect to the Hausdorff metric \( d_H \);

(c) for each \( (x, y) \in I \times \mathbb{R}^n \times \mathbb{R}^n \)
\[
\|F(t, x, y)\| \leq \sup \\{ \|v\| : v \in F(t, x, y) \}
\]
\[
\leq a(t) + c_1(t)\|x\| + c_2(t)\|y\|
\]

(d) the spectral radius \( r(L) \) of \( L \) is less than one.

Then Problem \((Q_e)\) admits a solution in \( S \).
Proof. We can say that $\| F(t, x, y) \| \leq a_1(t)$ a.e. on $I$ for some $a_1 \in L^p(I, \mathbb{R}^n)$ [9]. Let $x \in C^1(I, \mathbb{R}^n)$ and let $u \in C^2(I, \mathbb{R}^n)$ be the unique solution of the problem

$$
\hat{u}(t) = x(t), \quad \text{a.e. on } I,
$$

$$
\hat{u}(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad (\ast)
$$

From Lemma 6, we have $u(t) = \int_0^1 \mathcal{G}(t, s) x(s) ds, \forall t \in I$. Thus, we define a function $f : C^1(I, \mathbb{R}^n) \rightarrow C^2(I, \mathbb{R}^n)$ such that $f(x)$ is the unique solution of $(\ast)$. Let

$$
\mathcal{Y} = \{ x \in C^1(I, \mathbb{R}^n) : \| x(t) \| \leq a_1(t) \text{ a.e. on } I \}.
$$

From the Dunford-Pettis theorem, $\mathcal{Y}$ is weakly compact and then $f(\mathcal{Y})$ is convex and compact subset of $C^2(I, \mathbb{R}^n)$. Let $\mathcal{Y} = \mathbb{R}^n \times \mathbb{R}^n$. If $\mathbb{H} = f(\mathcal{Y})$, $\mathcal{R} : \mathcal{H} \rightarrow 2^{L^1(I, \mathbb{R}^n)}$ and $\mathcal{M} : I \times \mathcal{Y} \rightarrow 2^{\mathbb{R}^n}$, where $\mathcal{R}(u) = \{ g \in L^1(I, \mathbb{R}^n) : g(t) \in F(t,u(t),\hat{u}(t)) \text{ a.e. on } I \}$ and $\mathcal{M}(x,y) = F(x,y)$, then $\mathcal{R}$ has SD-property [23]. It is easy to show that $\mathcal{R}$ is nonempty and convex subset of $L^1(I, \mathbb{R}^n)$. If $(\mathcal{R}(u))_n$ is a sequence in $\mathcal{R}(u)$ for some $u \in \mathcal{H}$, then $\lim_{n \rightarrow \infty} \mathcal{R}(u)_n = f(t) \in F(t,u(t),\hat{u}(t))$, where the values of $F$ are closed. Therefore, the values of $\mathcal{R}$ are weakly compact. According to Theorem 5 there exists a continuous function $r : \mathcal{H} \rightarrow L^1_w(I, \mathbb{R}^n)$ with $r(u) \in \text{ext}(\mathcal{R}(u))$, for all $u \in \mathcal{H}$. Thus, $r(u)(t) \in \text{ext}(\mathcal{M}(u(t),u(t),\hat{u}(t)))$ a.e. on $I$ [24] which implies $r(u)(t) \in \text{ext}(F(t,u(t),\hat{u}(t)))$ a.e. on $I$. If $u \in f(\mathcal{Y})$, then $\| r(u)(t) \| \leq a_1$ and so $r(u) \in f(\mathcal{Y})$. Put $\theta : f(\mathcal{Y}) \rightarrow W^{2,1}(I, \mathbb{R}^n)$, then $\theta(u) = f(r(u))$, thus $\theta$ is a continuous function from $f(\mathcal{Y})$ into $f(\mathcal{Y})$ [19]. From Schauder's fixed point theorem, there exists $x \in f(\mathcal{Y})$ such that $x = \theta(x) = f(r(x))$ which means that there is $x \in S \subset C^2(I, \mathbb{R}^n)$ such that $x(t) \in \text{ext}(F(t,x(t),\hat{x}(t)))$.

Theorem 8. In the setting of Theorem 7, if one replaces condition (b) by the following condition:

$$
(b') d_H(F(t,x,y), F(t,x',y')) \leq k_1 \| x - x' \| + k_2 \| y - y' \| \text{ a.e. with } k_1 \geq 0, k_2 \geq 0 \text{ and } k_1 + k_2 < 1/2C_0.
$$

Then $\Delta_{Q_v} \subset \Delta_{Q_v}$ is the closure taken in $C^2(I, \mathbb{R}^n)$.

Proof. From Theorem 7, we have $\Delta_{Q_v} \neq \emptyset$. Moreover, $\| F(t,x,y) \| \leq b_1(t)$ a.e. on $I$ for some $b_1 \in L^p(I, \mathbb{R}^n)$. Let $u \in \Delta_{Q_v}$. Then

$$
\hat{u}(t) = h(t), \quad \text{a.e. on } I,
$$

$$
\hat{u}(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad (\ast)
$$

where $h(t) \in F(t,u(t),\hat{u}(t))$ a.e. on $I$. Assume that $f' : C^1(I, \mathbb{R}^n) \rightarrow C^2(I, \mathbb{R}^n)$ is a function such that, for each $h \in C^1(I, \mathbb{R}^n)$, $f'(h) \in C^2(I, \mathbb{R}^n)$ is the unique solution of the second-order differential equation

$$
\hat{u}(t) = h(t), \quad \text{a.e. on } I,
$$

$$
\hat{u}(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i). \quad (Q_h)
$$

Let $S = \{ u \in C^1(I, \mathbb{R}^n) : \| u(t) \| \leq b_1(t) \text{ a.e. on } I \}$. So $f'(S)$ is convex. Let $(u_n)_{n \in \mathbb{N}}$ be a sequence in $f'(S)$. Hence, $u_n \in C^2(I, \mathbb{R}^n)$ with $u_n(0) = 0, u_n(1) = \sum_{i=1}^{m-2} a_i u_n(\xi_i)$. Then from Lemma 6,

$$
\lim_{n \rightarrow \infty} u_n(t) = \int_0^1 \mathcal{G}(t, \tau) \hat{u}(\tau) \, d\tau = u(t), \quad (36)
$$

hence, $f'(S)$ is a compact subset of $C^2(I, \mathbb{R}^n)$. Set

$$
\mathcal{Q}_v(t) = \{ x \in F(t, v(t), \hat{v}(t)) : \| h(t) - x \| < \varepsilon + d(h(t), F(t,v(t),\hat{v}(t))) \}, \quad (37)
$$

where $\varepsilon > 0$ and $v \in f'(S)$. Hence, for each $t \in I$, $\mathcal{Q}_v(t) \neq \emptyset$. Assume that $\mathcal{R}(I)$ and $\mathcal{R}(\mathbb{R}^n)$ are the Borel-$\sigma$-fields of $I$ and $\mathbb{R}^n$, respectively. From condition (i), the function $t \rightarrow F(t,v(t),\hat{v}(t))$ is measurable. Hence, $grF(,v(,\hat{v}()) \in \mathcal{B}(I) \times \mathcal{B}(\mathbb{R}^n)$ and $(t,x) \rightarrow ed(h(t), F(t,v(t),\hat{v}(t)))-\| h(t) - x \|$ is measurable in $t$ and continuous in $x$ that is jointly measurable. Thus, by Aumann's selection theorem, there exists a measurable selection $s_v$ of $\mathcal{Q}_v$ such that $s_v(t) \in \mathcal{Q}_v(t)$ for each $t \in I$. Now we define a multifunction $\bar{\mathcal{Q}}_v : f'(S) \rightarrow 2^{C^2(I,\mathbb{R}^n)}$ by the following:

$$
\mathcal{Q}_v(v) = \{ x \in \delta_{F(v(,\hat{v}() \cap \mathcal{Q}_v(t)) : \| h(t) - x \| < \varepsilon + d(h(t), F(t,v(t),\hat{v}(t))) \text{ a.e. on } I \}, \quad (38)
$$

with $\mathcal{Q}_v(v(t)) \neq \emptyset$ for each $v \in f'(S)$. From [22, Proposition 4], $\mathcal{Q}_v$ is l. s. c. and clearly has decomposable values. Applying [22, Theorem 3], we have a continuous selection $S_v$ of $\mathcal{Q}_v$. Therefore,

$$
\| h(t) - S_v(v(t)) \| \leq \varepsilon + d(h(t), F(t,v(t),\hat{v}(t))) \leq \varepsilon + k_1(t) \| u(t) - v(t) \|
$$

$$
+ k_2(t) \| u(t) - \hat{v}(t) \| \quad \text{a.e. on } I. \quad (39)
$$

From Theorem 2, we find a continuous function $\xi^v_1 : f'(S) \rightarrow L^1_w(I, \mathbb{R}^n)$ such that $\xi^v_1(v) \in \text{ext}\delta_{F(v(,\hat{v}() \cap \mathcal{Q}_v(t)) and $S_v(v) - \xi^v_1(v) \| < \varepsilon$ for each $v \in f'(S)$. Define a multifunction $\mathcal{R}' : f'(S) \rightarrow 2^{C^2(I,\mathbb{R}^n)}$ by

$$
\mathcal{R}'(u) = \{ g \in C^1(I, \mathbb{R}^n) : g(t) \in F(t,u(t),\hat{u}(t)) \text{ a.e. on } I \}, \quad (40)
$$

Abstract and Applied Analysis 7
As in Theorem 5, let $Y = \mathbb{R}^n \times \mathbb{R}^n$ and set a multifunction $M : I \times Y \to 2^{\mathbb{R}^n}$ such that $M(t, (x, y)) = F(t, x, y)$ from [23, Theorem 3.1], $M$ has $SD$-property. $R'$ has nonempty convex values. Let $(g_n)_{n \in \mathbb{N}}$ be a sequence in $R'(u)$ for some $u \in f^0(S)$. So, for each $t \in I$,

$$
\lim_{n \to \infty} g_n(t) = g(t) \in F(t, u(t), \dot{u}(t))
$$

(41)

because $F$ has closed values in $\mathbb{R}^n$. Therefore, $g \in \delta^1_{F(\cdot, (\cdot), (\cdot))}$ which implies $R'(\cdot)$ has compact values in $\mathbb{R}^n$. We can apply Theorem 2 to find a continuous function $\theta' : f^0(S) \to L^1_u(I, \mathbb{R}^n)$ such that $\theta'(u) \in \text{ext}(R'(u))$, for all $u \in f^0(S)$. We see that $\theta'(u(t)) \in \text{ext}(M(t, u(t), \dot{u}(t)))$ [24], hence $\theta'(u(t)) \in \text{ext}F(t, u(t), \dot{u}(t))$ a.e. on $I$. Assume that $\eta' : f^0(S) \to C^2(I, \mathbb{R}^n)$ is the function which for which each $u \in f^0(S)$, $\eta'(u) = g(\theta'(u))$. For each $u \in f^0(S)$, we have $\|\theta'(u(t))\| \leq b_1$ and so $\theta'(u) \in S$. Then, $\eta'$ is a function from $f^0(S)$ into $f^0(S)$ and also we see that $\eta'$ is continuous [19]. Now let $e_n \to 0$, $S_n = S_n^0$ and $\eta_n^0 = \eta_n^0$. Then, for each $n \in \mathbb{N}$, the function $f^0\eta_n^0$ is a continuous function from the compact set $f^0(S)$ into itself. From Schauder's fixed point theorem, $f\eta_n^0$ has a fixed point $u_n$, but ext $\delta^1_{F(\cdot, (\cdot), (\cdot))} = \delta^1_{\text{ext}F(\cdot, (\cdot), (\cdot))}$ [24] so $u_n \in \Delta_{p_n}$. Assume that $u_n \to \tilde{u}$ in $C^2(I, \mathbb{R}^n)$. From Lemma 6, we obtain

$$
\|u_n(t) - u(t)\| \leq \frac{1}{2} \left[ \int_0^1 |\mathcal{G}(t, \tau)| \|\xi_n'(\tau) - S_n(\tau)\| \, d\tau + \int_0^1 |\mathcal{G}(t, \tau)| \|S_n(\tau) - h(\tau)\| \, d\tau \right] ds.
$$

(42)

But $\xi_n' - S_n \to 0$ with respect to the norm $\|\cdot\|_w$ and from Lemma 3 we get $\xi_n' - S_n \to 0$ weakly in $C^1(I, \mathbb{R}^n)$. So we have

$$
\int_0^1 |\mathcal{G}(t, \tau)| \|\xi_n'(\tau) - S_n(\tau)\| \, d\tau \to 0.
$$

(43)

Moreover, as $n \to \infty$ we have

$$
\|u(t) - u(t)\| \leq \|u - \tilde{u}\|_{C^1(I, \mathbb{R}^n)} \int_0^1 |\mathcal{G}(t, \tau)| (k_1(\tau) + k_2(\tau)) \, d\tau
\leq \|u - \tilde{u}\|_{C^1(I, \mathbb{R}^n)} \|k_1(\tau) + k_2(\tau)\|_{C_0}.
$$

(44)

Since by assumption (ii), $k_1 + k_2 < 1/2C_0$, thus from Lemma 6, we get $u(t) = \tilde{u}$. So $u_n \to \tilde{u}$ in $C^2(I, \mathbb{R}^n)$ and $u \in \Delta_{Q_0}$ where the closure is taken in $C^2(I, \mathbb{R}^n)$ which means that $\Delta_0 \subseteq \Delta_{Q_0}$. If $v_n \to v$ in $C^2(I, \mathbb{R}^n)$, then $v_n = \tilde{f}(y_n)$ for $y_n \in \delta^1_{\text{ext}F(\cdot, (\cdot), (\cdot))}$. From assumption (iii) and the Dunford-Pettis theorem, $\{y_n\}_{n \in \mathbb{N}}$ is weakly sequentially compact in $C^2(I, \mathbb{R}^n)$. By [25, Theorem 3.1], we get

$$
y(t) \in \text{conv} \lim_{n \to \infty} \{y_n(t)\}_{n \in \mathbb{N}} \subseteq \text{conv} \lim_{n \to \infty} \tilde{F}(t, v_n(t), \dot{v}_n(t)) = F(t, v(t), \dot{v}(t)) \text{ a.e. on } I.
$$

(45)

Moreover, $\tilde{f}'(y_n) \to \tilde{f}'(y)$ in $C^2(I, \mathbb{R}^n)$ for $y \in C^2(I, \mathbb{R}^n)$ and $y(t) \in F(t, v(t), \dot{v}(t))$ a.e. on $I$. Hence, $v \in \Delta_{Q_0}$; that is, $\Delta_0$ is closed in $C^2(I, \mathbb{R}^n)$.

\section*{Acknowledgments}

The author is deeply indebted and thankful to the deanship of the scientific research and his helpful and distinct team of employees at Tabah University, Al-Madinah Al-Munawarah, Saudia Arabia. This research work was supported by a Grant no. 3029/1434.

\section*{References}


