Research Article

The Multisoliton Solutions for the \((2+1)\)-Dimensional Sawada-Kotera Equation

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Applying bilinear form and extended three-wave type of ansätz approach on the \((2+1)\)-dimensional Sawada-Kotera equation, we obtain new multisoliton solutions, including the double periodic-type three-wave solutions, the breather two-soliton solutions, the double breather soliton solutions, and the three-solitary solutions. These results show that the high-dimensional nonlinear evolution equation has rich dynamical behavior.

1. Introduction

As is well known that the exact solutions of nonlinear evolution equations play an important role in nonlinear science field, especially in nonlinear physical science since they can provide much physical information and more insight into the physical aspects of the problem and thus lead to further applications. The search for exact solutions of nonlinear partial differential equations has long been an interesting and hot topic in nonlinear mathematical physics. Consequently, many methods are available to look for exact solutions of nonlinear evolution equations, such as the inverse scattering method, the Lie group method, the mapping method, Exp-function method, and ansätz technique [1–4]. Very recently, Wang et al. [5] proposed a new technique called extended three-wave approach to seek multiwave solutions for integrable equations, and this method has been used to investigate several equations [6, 7]. In this paper, we consider the following Sawada-Kotera equation:

\[
u_t = \left(u_{xxx} + 5uu_{xx} + \frac{5}{3}u^3 + 5u_{xy}\right)_x - 5 \int (u_{yy}) dx + 5uu_y + 5u_x \int (u_y) dx.
\]

Equation (1) was derived by B. G. Konopelchenko and V. G. Dubrovsky, and was called the Sawada-Kotera (SK) equation; for example, see [8]. By means of the two-soliton method, the exact periodic soliton solutions, N-soliton solutions, and traveling wave solutions of the SK equation were found [8–10].

In this paper, we discuss further the \((2+1)\)-dimensional SK equation, by using bilinear form and extended three-wave type of ansätz approach, respectively [5, 11–15], and some new multisoliton solutions are obtained.

2. The Multisoliton Solutions

We assume

\[u = -2(\ln f)_{xx},\]

where \(f = f(x, y, t)\) is an unknown real function. Substituting (2) into (1), we can reduce (1) into the following equation [8]:

\[
(D^6_x + 5D_x(D^3_x - 5D^2_x + D_xD_y) f \cdot f = 0,
\]
where the Hirota bilinear operator $D$ is defined by $(n, m \geq 0)$

$$D^n D^m f(x, t) \cdot g(x, t) = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^n \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^m f(x, t) \cdot g(x', t')$$

Substituting (7) into (2) yields the three-soliton solution of SK equation as follows:

$$u_1 = -\left( 2 \left[ 2 \sqrt{\delta_3} a_1^2 \cosh \left( \xi_1 + \frac{1}{2} \ln (\delta_3) \right) \right. \right.$$  

$$\left. - \frac{\delta_1 a_2^2 a_3^2 \cosh (\eta_1)}{a_1^2 - a_3^2} \right)^{-1} \times \frac{2 \sqrt{\delta_3} \cosh \left( \xi_1 + \frac{1}{2} \ln (\delta_3) \right) - \delta_1 \cosh (\gamma_1)}{a_1^2 - a_3^2} \right)$$

Case I. If $a_2 = 0$, then

$$b_1 = \frac{1}{4} a_1 (4a_1^2 + 3a_3^2), \quad b_2 = \frac{3}{2} i a_1^2 a_3,$$

$$b_3 = -\frac{1}{4} a_3 (6a_1^2 + a_3^2), \quad \delta_2 = -\frac{\delta_3 a_1^2}{a_1^2 - a_3^2},$$

$$\delta_3 = \delta_3,$$

$$c_1 = \frac{9}{16} a_1 \left( 5a_1^4 + 40a_1^2 a_3^2 + 16a_3^4 \right),$$

$$c_2 = -\frac{45}{4} i a_1^2 a_3 \left( 2a_1^2 + a_3^2 \right),$$

$$c_3 = \frac{9}{16} a_3 \left( a_1^2 + 20a_1^2 a_3^2 + 40a_3^4 \right),$$

where $a_1, a_3, \delta_1$, and $\delta_3$ are free real constants. Substituting (6) into (5) and taking $\delta_3 > 0$, we have

$$f_1 = 2 \sqrt{\delta_3} \cosh \left( a_1 x + K_1 y + L_1 t + \frac{1}{2} \ln (\delta_3) \right)$$

$$- \delta_1 \cosh (M_1 y + N_1 t) - \frac{\delta_3 a_2^2}{a_1^2 - a_3^2} \times \cosh (a_3 x - H_1 y + J_1 t),$$

where $K_1 = (1/4) a_1 (4a_1^2 + 3a_3^2), L_1 = (9/16) a_1 (5a_1^4 + 40a_1^2 a_3^2 + 16a_3^4), M_1 = -(3/2) a_1^2 a_3, N_1 = (45/4) a_1^2 a_3 (a_1^2 + a_3^2), H_1 = (1/4) a_3 (6a_1^2 + a_3^2),$ and $J_1 = (9/16) a_3 (a_1^4 + 20a_1^2 a_3^2 + 40a_3^4).$ Substituting (9) into (2) yields the double breather solution of SK equation as follows:

$$u_2 = -\left( 2 \left[ 2 \sqrt{\delta_3} \cosh \left( \xi_2 + \frac{1}{2} \ln (\delta_3) \right) \right. \right.$$  

$$\left. + \frac{\delta_1 a_2^2 A_3^2 \cos (\eta_2)}{a_1^2 + A_3^2} \right)^{-1} \times \frac{2 \sqrt{\delta_3} \cosh \left( \xi_2 + \frac{1}{2} \ln (\delta_3) \right) + \delta_1 \cos (\gamma_2)}{a_1^2 + A_3^2} \right)$$
\[ + 2 \left[ \left( 2a_1 \sqrt{\delta_3} \sinh \left( \xi_2 + \frac{1}{2} \ln (\delta_3) \right) \right. \right. \\
\left. \left. + \frac{\delta_1 a_1^2 A_3 \sin (\eta_2)}{a_1^2 + A_3^2} \right) \times \left( 2 \sqrt{\delta_3} \cosh \left( \xi_2 + \frac{1}{2} \ln (\delta_3) \right) \right. \right. \\
\left. \left. \left. + \delta_1 \cos (\eta_2) - \frac{\delta_1 a_1^2 \cos (\eta_2)}{a_1^2 + A_3^2} \right) \right] \right]^2, \]

where \( \xi_2 = a_1 x + K_2 y + L_2 t, \eta_2 = A_3 x - H_2 y + J_2 t, \) and \( y_2 = M_2 y + N_2 t. \)

**Case 2.** If \( a_2 \neq 0, \) then
\[
\begin{align*}
 b_1 &= -a_2^3, \quad b_2 = a_2^3, \quad b_3 = -a_2^3, \\
 \delta_1 &= \delta_1, \quad \delta_2 = \delta_2, \quad \delta_3 = \delta_3, \\
 c_1 &= 9a_2^5, \quad c_2 = 9a_2^5, \quad c_3 = 9a_2^5,
\end{align*}
\]
where \( a_1, a_2, a_3, \delta_1, \delta_2, \) and \( \delta_3 \) are free real constants. Substituting (11) into (5) and taking \( \delta_3 > 0, \) we have
\[
f_3 = 2 \sqrt{\delta_3} \cosh \left( a_1 x - a_1^3 y + 9a_2^5 t + \frac{1}{2} \ln (\delta_3) \right) \\
+ \delta_1 \cos (a_2 x + a_3^2 y + 9a_2^5 t) \\
+ \delta_2 \cosh (a_3 x - a_3^2 y + 9a_2^5 t). \]

Substituting (12) into (2) yields the breather two-soliton solution of SK equation as follows:
\[
u_3 = -\left( 2 \left[ \left( 2 \sqrt{\delta_3} a_2^3 \cosh \left( \xi_3 + \frac{1}{2} \ln (\delta_3) \right) \right. \right. \\
\left. \left. - \delta_1 a_2^3 \cos (\eta_3) + \delta_2 a_2^3 \cosh (\eta_3) \right) \right] \right) \times \left( 2 \sqrt{\delta_3} \cosh \left( \xi_3 + \frac{1}{2} \ln (\delta_3) \right) \right) \\
+ \left[ \left( 2 \left( 2 \sqrt{\delta_3} a_2 \sinh \left( \xi_3 + \frac{1}{2} \ln (\delta_3) \right) \right. \right. \\
\left. \left. - \delta_1 a_2 \sin (\eta_3) + \delta_2 a_2 \sinh (\eta_3) \right) \right] \right) \times \left( 2 \sqrt{\delta_3} \cosh \left( \xi_3 + \frac{1}{2} \ln (\delta_3) \right) \right. \\
\left. \left. + \delta_1 \cos (\eta_3) + \delta_2 \cosh (\eta_3) \right) \right] \right)^{-1}
\]
where \( \xi_3 = a_1 x - a_3^3 y + 9a_2^5 t, \eta_3 = a_2 x + a_2^3 y + 9a_2^5 t, \) and \( \eta_3 = a_3 x - a_3^3 y + 9a_2^5 t. \)

The expression \( (u_3) \) is the breather two-soliton solution of SK equation which is a periodic wave in \( x, y \) and meanwhile is a two-soliton in \( x, y \) (refer to Figure 1(b)).

**Case 3.** If \( a_3 = b_1 = 0, \) then
\[
a_1 = 2a_3, \quad b_2 = \sqrt{2a_3^5}, \quad b_3 = -\frac{3}{2} a_3^3, \\
c_1 = -\frac{169}{2} a_3^5, \\
c_2 = -20 \sqrt{2a_3^5}, \quad c_3 = -\frac{349}{4} a_3^5, \quad \delta_3 = \frac{5}{152} \delta_2^2 - \frac{7}{228} \delta_1^2,
\]
where \( a_3, \delta_1, \) and \( \delta_3 \) are free real constants. Substituting (14) into (5) and taking \( \delta_3 > 0, \) we have
\[
f_4 = 2 \sqrt{\delta_3} \cosh \left( 2 \left( -2a_3 x + \frac{169}{2} a_3^3 t - \frac{1}{2} \ln \left( \frac{5}{152} \delta_2^2 - \frac{7}{228} \delta_1^2 \right) \right) \right) \\
+ \delta_1 \cosh \left( -\sqrt{2a_3^3} y + 20 \sqrt{2a_3^5} t \right) \\
+ \delta_2 \cosh \left( -a_3 x + \frac{3}{2} a_3^3 y + \frac{349}{4} a_3^5 t \right).
\]

where \( (5/152)\delta_2^2 - (7/228)\delta_1^2 > 0. \) Substituting (15) into (2) yields the breather two-soliton solution of SK equation as follows:
\[
u_4 = -\left( 2 \left[ \left( 8 \sqrt{K_4} a_3^3 \cosh \left( \xi_4 - \frac{1}{2} \ln (K_4) \right) \right. \right. \\
\left. \left. + \delta_1 a_3^3 \cosh (\eta_4) \right) \right] \right) \times \left( 2 \sqrt{K_4} \cosh \left( \xi_4 - \frac{1}{2} \ln (K_4) \right) \right) \\
+ \delta_1 \cosh (\eta_4) + \delta_2 \cosh (\eta_4) \right)^{-1}
\]
where \( K_4 = (5/152)\delta_2^2 - (7/228)\delta_1^2, \xi_4 = -2a_3 x + (169/2)a_3^3 t, \eta_4 = -a_3 x + (3/2)a_3^3 y + (349/4)a_3^5 t, \) and \( \eta_4 = -\sqrt{2a_3^3} y + 20 \sqrt{2a_3^5} t. \)
The expression \((u_4)\) is the breather two-soliton solution of SK equation which is a periodic wave in \(y-t\) and meanwhile is a two-soliton in \(x, y\) and in \(x-t\), respectively (refer to Figure 1(c)).

Notice that \(u_3\) and \(u_4\) are also the breather two-soliton solutions, but their structure is different, because the two wave propagation directions are different in the \(u_3\) and \(u_4\), respectively (refer to Figures 1(b) and 1(c)).

If taking \(a_1 = iA_1, a_3 = iA_3\) in (12), then we have

\[
f_5 = 2 \cos \left( A_1 x + A_1^3 y + 9 A_1^5 t \right) \\
+ \delta_1 \cos \left( a_2 x + a_2^3 y + 9 a_2^5 t \right) \\
+ \delta_2 \cos \left( A_3 x + A_3^3 y + 9 A_3^5 t \right),
\]

(17) when \(\delta_3 = 1\). Substituting (17) into (2) gives the double-periodic three-wave solution of SK equation as follows:

\[
u_5 = \frac{2 \left[ 2 A_1^2 \cos (\xi_5) + \delta_1 a_2^3 \cos (\eta_5) + \delta_2 A_3^2 \cos (\gamma_5) \right]}{2 \cos (\xi_5) + \delta_1 \cos (\eta_5) + \delta_2 \cos (\gamma_5)} \\
+ 2 \left[ \frac{2 A_1 \sin (\xi_5) + \delta_1 a_2 \sin (\eta_5) + \delta_2 A_3 \sin (\gamma_5)}{2 \cos (\xi_5) + \delta_1 \cos (\eta_5) + \delta_2 \cos (\gamma_5)} \right]^2,
\]

(18)

where \(\xi_5 = A_1 x + A_1^3 y + 9 A_1^5 t, \eta_5 = a_2 x + a_2^3 y + 9 a_2^5 t,\) and \(\gamma_5 = A_3 x + A_3^3 y + 9 A_3^5 t,\)

3. Conclusion

By using bilinear form and extended three-wave type of ansatz approach, we discuss further the \((2 + 1)\)-dimensional
Sawada-Kotera equation and find some new multisoliton solutions. The result shows that the extended three-wave type of ansatz approach may provide us with a straightforward and effective mathematical tool for seeking multiwave solutions of high-dimensional nonlinear evolution equations.

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References


