Research Article

Random Dynamics of the Stochastic Boussinesq Equations Driven by Lévy Noises

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1. Introduction

Dynamical systems driven by non-Gaussian processes, such as Lévy processes, have attracted a lot of attention recently. Ordinary differential equations driven by Lévy processes have been summarized in [1]. Peszat and Zabczyk [2] have presented a basic framework for partial differential equations driven by Lévy processes.

The Navier–Stokes fluid equations are often coupled with other equations, especially, with the scalar transport equations for fluid density, salinity, or temperature. These coupled equations model a variety of phenomena arising in environmental, geophysical, and climate systems. The related Boussinesq fluid equations [3–5] under Gaussian fluctuations have been recently studied, for example, existence and uniqueness of solutions [6], stochastic flow, dynamical impact under random dynamical boundary conditions [7, 8], and large deviation principles [9, 10], among others.

Motivated by a recent work on a simple stochastic partial differential equation with Lévy noise [11], we study the stochastic Boussinesq equations driven by some special Lévy noises, and we consider the random dynamics of this stochastic system. Specifically, for a given bounded $C^1$-smooth domain $D \subset \mathbb{R}^2$ with sufficient smooth boundary, we consider the following stochastic Boussinesq equations driven by subordinator Lévy noise:

$$
\begin{align*}
\frac{du}{dt} &= \left( \frac{1}{\text{Re}} \Delta u - \nabla p - u \cdot \nabla u - \frac{1}{\text{Fr}^2} \theta e_2 \right) + dY_1(t), \quad \text{on } D \times \mathbb{R}_+,
\frac{d\theta}{dt} &= \left( \frac{1}{\text{RePr}} \Delta \theta - u \cdot \nabla \theta \right) + dY_2(t), \quad \text{on } D \times \mathbb{R}_+, 
\text{div } u &= 0, \quad \text{on } D \times \mathbb{R}_+,
\end{align*}
$$

(1)

where $u = u(x,t) = (u^1, u^2) \in \mathbb{R}^2$ is the velocity vector, $\theta = \theta(t, x) \in \mathbb{R}$ is salinity, $p(t, x) \in \mathbb{R}$ is the pressure term, $x = (\xi, \eta) \in D \subset \mathbb{R}^2$, $\Delta$ denotes the Laplacian operator, and $\nabla$ denotes the gradient operator. Moreover, Fr is the Froude number, Re is the Reynolds number, Pr is the Prandtl number, and $e_2 \in \mathbb{R}^2$ is a unit vector in the upward vertical direction. The initial data $u_0, \theta_0$ are given. Both $Y_1(t)$ and $Y_2(t)$ are subordinator Lévy processes on Hilbert spaces $H_1$ and $H_2$, which will be specified later. The present paper is
devoted to the existence, uniqueness, regularity, and the cocycle property of solution for stochastic Boussinesq equations (1).

This paper is organized as follows. In Section 2, we first present some properties of the subordinator Lévy process \( Y(t) \), then review some fundamental properties of the stochastic integral with respect to Lévy process \( Y(t) \). Section 3 is devoted to the existence, uniqueness, regularity, and the cocycle property of the stochastic Boussinesq equations. Finally, some discussions on the global weak solution of stochastic Boussinesq equations driven by general Lévy noise are also presented in Section 4.

2. Preliminaries

In this section, we introduce some operators and fraction spaces and then present some properties of the subordinator Lévy process \( Y(t) \) and the stochastic integral with respect to Lévy process \( Y(t) \).

In order to reformulate the stochastic Boussinesq equations (1) as an abstract stochastic evolution, we introduce the following function spaces.

Denote \( L^2(D) \) to be the space of functions defined on \( D \), which are \( L^2 \)-integrable with respect to the Lebesgue measure \( dx = dx_1 dx_2 \), endowed with the usual scalar product and norm, that is, for \( u, v \in L^2(D) \),

\[
(u, v) = \int_D u(x) v(x) \, dx, \quad |u| = \{(u, u)\}^{1/2}.
\]

For \( m \in \mathbb{Z}^+ \cup \{0\} \) and \( q \in (1, \infty) \), define

\[
H^{m,q}(D) = \left\{ u \in L^q(D) : D^\alpha u \in L^q(D), \, \alpha \in \mathbb{N}^2, \, 0 \leq |\alpha| \leq m \right\}
\]

as the usual Sobolev space with scalar product

\[
(u, v)_m = \sum_{0 \leq |\alpha| \leq m} \langle D^\alpha u, D^\alpha v \rangle_{L^q(D)}
\]

and the induced norm

\[
|u|_m = \|u\|_{H^m(D)} = \left( \sum_{0 \leq |\alpha| \leq m} |D^\alpha u|_{L^q(D)}^q \right)^{1/2},
\]

where \( D^\alpha u \) is the \( \alpha \)-th order weak derivative of \( u \).

For \( s \in \mathbb{R} \), let \( H^{s,q}(D) \) be defined by the complex interpolation method [12] as follows.

\[
H^{\beta,q}(D) = \left[ H^{k,q}(D), H^{m,q}(D) \right]_{\beta},
\]

where \( k, m \in \mathbb{N} \), \( \theta \in (0, 1) \), and \( k < m \) are chosen to satisfy

\[
\beta = (1 - \theta) k + \theta m.
\]

The closure of \( C_0^\infty(D) \) in the Banach space \( H^{s,q}(D) \), \( s \geq 0 \), \( q \in (1, \infty) \), will be denoted by \( H^{s,q}_0(D) \).

The following product spaces are needed:

\[
\mathcal{V} = \left\{ u = (u^1, u^2) \in (C_c^\infty(D))^2 \times C_c^\infty(D), \, \nabla \cdot u = 0 \right\},
\]

\[
L^q(D) = (L^q(D))^2 \times L^q(D),
\]

\[
\mathcal{V}^{s,q} = (H^{s,q}(D))^2 \times H^{s,q}(D),
\]

\[
\forall \beta (\theta) = \left\{ u = (u^1, u^2) \in (H^{s,q}(D))^2 \times H^{s,q}(D), \, \nabla \cdot u = 0 \right\},
\]

\[
\mathcal{V}^{s,q}_0 = \left\{ u = (u^1, u^2) \in H^{s,q}_0(D), \, \nabla \cdot u = 0 \right\}.
\]

Let \( H^{s,q}(D) \) denote the closure of \( \mathcal{V} \) with respect to the \( H^{s,q} \)-norm, \( V^{s,q}(D) \) denote the closure of \( \mathcal{V} \) with respect to the \( V^{s,q} \)-norm, and \( V' \) be the dual space of \( V^{s,q}(D) \). In particular, we denote by \( H^{1,2} \) and \( V^{1,2} \) \( H \) and \( V \), respectively. Denote

\[
A_1 u = \Delta u(t), \quad A_2 \theta = \Delta \theta(t),
\]

\[
B_1 (u_1, u_2) = (u_1 \cdot \nabla) u_2, \quad B_2 (u_1, \theta_2) = (u_1 \cdot \nabla) \theta_2,
\]

\[
U_0 = \left( \begin{array}{c} u_0 \\ \theta_0 \end{array} \right) \in H, \quad U(t) = \left( \begin{array}{c} u(t) \\ \theta(t) \end{array} \right) \in V,
\]

\[
R(U) = \left( \begin{array}{cc} - \frac{1}{Fr^2} \theta e_2 \\ 0 \end{array} \right),
\]

\[
Y(t) = \left( \begin{array}{c} Y_1(t) \\ Y_2(t) \end{array} \right) \in H = H_1 \times H_2,
\]

where \( \nu = 1 / \text{Re} \) and \( k = 1 / \text{Re Pr} \).

Now, we define the following two operators:

\[
A : V \rightarrow V' : V \ni u = (u, \theta) \mapsto AU = \left( \begin{array}{c} \nu A_1 u_1 \\ k A_2 \theta \end{array} \right),
\]

\[
B : V \times V \rightarrow V' : V \times V \ni (U_1, U_2) \mapsto B(U_1, U_2) = \left( \begin{array}{c} B_1 (u_1, u_2) \\ B_2 (u_1, \theta_2) \end{array} \right).
\]

Then, the stochastic Boussinesq system (1) can be rewritten as the following abstract stochastic evolution equation:

\[
dU(t) + [AU(t) + B(U(t), U(t)) + R(U(t))] \, dt = dY(t),
\]

\[
U(0) = U_0.
\]

In order to apply the technique of the reproducing Kernel Hilbert space, it is better to introduce the definition \( \gamma \)-radonifying.

**Definition 1** (see [13]). Let \( K \) and \( X \) be Banach spaces, a bounded linear operator \( L : K \rightarrow X \) is called \( \gamma \)-radonifying.
Abstract and Applied Analysis

if and only if \( L(y_K) \) is \( \sigma \)-additive, where \( y_K \) is the canonical cylindrical \( \sigma \)-additive set-valued function (also called a Gaussian distribution) on \( K \).

The following is our standing assumption:

**Assumption 1.** Space \( E \subset H \cap L^4 \) is a Hilbert space such that for some \( \delta \in (0,1/2) \),

\[
A^{-\delta} : E \rightarrow H \cap L^4 \text{ is } y \text{-radonifying.} \tag{12}
\]

**Remark 2.** Under the above assumption, we have the facts that \( E \subset H \) and the Banach space \( U \) is taken as \( H \cap L^4 \) (see [11,14,15] for more details and related results). In fact, space \( E \) is the reproducing kernel Hilbert space of noise \( W(t) \) on \( H \cap L^4 \).

It is well known that subordinators form the subclass of increasing Lévy processes, which can be thought of as a general random dynamical system, which are taken from [7].

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a Banach space, and let \( E \) be a complete separable metric space and \( \mathbb{P} \) be a probability space. Let \( H \subset L^4 \) be a Banach space, and let \( \mathcal{F} \subset \mathcal{P} \) be a probability space. Let \( \gamma \) be the Lévy process with the intensity measure:

\[
\gamma(t) = \nu(\Gamma \cap B(U, (0,1))), \quad \Gamma \in \mathcal{B}(U), \tag{16}
\]

\[
B(U, (0,1)) \text{ denotes the unit ball in } U,
\]

and \( N_2 \) be the Lévy process with the intensity measure \( \nu_2 = \nu - \nu_1 \). Then \( N_2 \) can be defined as a compound Poisson process with the intensity measure \( \nu_2 \), and \( N_1, N_2 \) can be defined by the Poisson random measure \( \pi \) which is defined as follows:

\[
\pi([0,1] \times \Gamma) = \sum_{k \geq 1} \Delta Y(s), \quad \Gamma \in \mathcal{B}(U), \tag{17}
\]

where \( \Delta Y(s) = Y(s^+) - Y(s^-), s \geq 0 \). Here, the symbol \( \Delta \) denotes the increment of \( Y \).

We assume that the process \( Y \) is right-continuous with left-hand side limits. Thus

\[
\Delta Y(s) = Y(s^-) - Y(s^+), \quad s \geq 0. \tag{18}
\]

Notice that as \( \pi \) is a time homogeneous Poisson random measure, \( Y \) can be expressed as

\[
Y(t) = \sum_{k \geq 1} \Delta Y(s) = \int_0^t \int_U u\pi(du,ds), \quad t \geq 0. \tag{19}
\]

Hence,

\[
N_1(t) = \sum_{k \geq 1} 1_{\{|\Delta Y(s)| < 1\}} \Delta Y(s) = \int_0^t \int_{|u| < 1} u\pi(du,ds),
\]

\[
N_2(t) = \sum_{k \geq 1} 1_{\{|\Delta Y(s)| \geq 1\}} \Delta Y(s) = \int_0^t \int_{|u| \geq 1} u\pi(du,ds). \tag{20}
\]

Assume that the operator \( \Psi(t), t \in [0,T] \), is a strongly measurable function taking values in the space of all bounded linear operator from \( U \) to \( E \). Let \( 0 < \tau_1 < \tau_2 < \tau_3 < \cdots \rightarrow \infty \) be the jump times for \( N_2 \) and \( \Delta N_2(\tau_k) = \Delta Y(\tau_k) = Y(\tau_k) - Y(\tau_{k-1}), k = 1,2, \ldots \). Then, the stochastic integral of \( \Psi(t) \) with respect to jump process \( N_2(t), t \geq 0 \), can be defined as

\[
\int_0^t \Psi(s)dN_2(s) = \sum_{k \geq 1} \Psi(\tau_k) \Delta N_2(\tau_k). \tag{21}
\]

Since the operator \( \Psi \) is taking values in \( E \), it follows from the decomposition of \( Y \) that the sum of sequences is finite. Hence the stochastic integral of the operator \( \Psi \) with respect to \( N_2 \) is taking values in \( E \). Moreover, the stochastic integral of the operator \( \Psi(t), t \in [0,T] \), with respect to Lévy process \( Y \) can be defined by

\[
\int_0^t \Psi(s)dy(s) = \int_0^t \Psi(s)dN_1(s) + \int_0^t \Psi(s)dN_2(s) \tag{22}
\]

and takes values in \( E \) as well (see [11] for more details).

Next, we recall some basic definitions and properties for general random dynamical systems, which are taken from [7]. Let \( (H,d) \) be a complete separable metric space and \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space.
Definition 5. A random dynamical system (RDS) with time \( T \) on \( (H, d) \) over \( \{\theta_t\} \) on \( (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}}) \) is a \((\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})\) measurable map:
\[
\Pi : T \times H \times \Omega \rightarrow H \times \Omega, \quad \Pi (t, s, \omega) = (S(t, \omega)x, \theta_t^\ast \omega)
\]
such that

(i) \( S(0, \omega) = Id \) (identity on \( H \)) for any \( \omega \in \Omega \),

(ii) (Cocycle property) \( S(t + s, \omega) = S(t, \theta_t^\ast \omega) \circ S(s, \omega) \) for all \( s, t \in T \) and \( \omega \in \Omega \).

An RDS is said to be continuous or differentiable if for all \( t \in T \), and an arbitrary outside outside \( \mathbb{P} \)-nullset \( B \subset \Omega, \omega \in B \) the map \( S(t, \omega) : H \rightarrow H \) is continuous or differentiable, respectively.

Assume that the bounded linear operator \( A \) generates a \( C_0 \)-semigroup \( S = (e^{tA})_{t \geq 0} \) on a Hilbert space \( E \) and \( Y \) defined on a filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) is a subordinator Lévy process taking values in a Hilbert space \( U \).

Consider the following stochastic Langevin equation:
\[
dX(t) = AX(t)dt + dY(t), \quad t \geq t_0,
\]
\[
X(t_0) = x \in E.
\]

Definition 6. Let \( x \in E \) be a square integrable \( \mathcal{F}_{t_0} \)-measurable random variable in \( E \). A predictable process \( X : [t_0, \infty) \times \Omega \rightarrow E \) is called a mild solution of the Langevin equation (24) with initial data \((t_0, x)\) if it is an adapted \( E \)-valued stochastic process and satisfies
\[
X(t) = S(t - t_0)x + \int_{t_0}^t S(t - s)dY(s), \quad t \geq t_0.
\]

It is well known that the Ornstein-Uhlenbeck process \( X(t), t \geq 0 \), has some important integrability. Here we need the Banach space to be of type \( p \), for \( p \in (1, 2] \). First we recall the definition briefly (see [14] for more details).

Definition 7 (see [14]). For \( p \in (1, 2] \), the Banach space \( E \) is called as type \( p \) if and only if there exists a constant \( K_p(E) > 0 \) such that for any finite sequence of symmetric independent identically distribution random variables \( \varepsilon_1, \ldots, \varepsilon_n : \Omega \rightarrow [-1, 1], n \in \mathbb{N} \), and any finite sequence \( x_1, \ldots, x_n \) from \( E \), satisfying
\[
\left| \sum_{i=1}^n \varepsilon_i x_i \right|^p \leq K_p(E) \sum_{i=1}^n |x_i|^p.
\]

Moreover, if there exists a constant \( L_p(E) > 0 \) such that for every \( E \)-valued martingale \( \{M_n\}_{n=0}^N, N \in \mathbb{N} \), satisfying
\[
\sup_n |M_n|^p \leq L_p(E) \sum_{n=0}^N \sum_{i=1}^n |X_i|^p,
\]

the separable Banach space \( E \) is called a separable martingale type \( p \)-Banach space.

Lemma 8 (see [11, Corollary 8.1, Proposition 8.4]). Assume that \( p \in (1, 2] \), \( Z \) is a subordinator Lévy process from the class Sub(\( p \), \( E \) is a separable type \( p \)-Banach, \( U \) is a separable Hilbert space \( U, E \subset U \), and \( W = (W(t))_{t \geq 0} \) is a \( U \)-valued Wiener process.

Define the \( U \)-valued Lévy process as
\[
Y(t) = W(Z(t)), \quad t \geq 0,
\]

and define the process as
\[
X(t) = \int_0^t e^{(t-s)A}dY(s).
\]

Then, with probability 1, for all \( T > 0 \),
\[
\int_0^T |X(t)|_p^p dt < \infty,
\]
\[
\int_0^T |X(t)|_4^4 dt < \infty.
\]

We have the following existence and regularity results, which have been studied in [2, 11].

Theorem 9. Assume that \( E = U, S = S(t), t \geq 0 \) is \( C_0 \) semigroup generated by the bounded linear operator \( A \) in the space \( E \). Then, if one of the following conditions is satisfied:

(i) \( p \in (0, 1] \) or

(ii) \( p \in (1, 2] \) and the Banach space \( E \) is of separable martingale type \( p \)-Banach space,

the Langevin equation (24) admits one mild solution \( X(t) \in E, t \geq 0 \). Moreover, if \( p \in (1, 2] \), \( S = S(t), t \geq 0 \), is a \( C_0 \)-group in the separable martingale type \( p \)-Banach space \( E \), then the mild solution \( X \) of the Langevin equation is a càdlàg (right-continuous with left-hand side limits) process.

Proof. As \( S = S(t), t \geq 0 \), is a \( C_0 \)-group in the separable martingale type \( p \)-Banach space \( E \), the Hilbert space \( H \) is the reproducing kernel Hilbert space of \( W(1) \), and the embedding operator \( i : H \rightarrow E \) satisfies the \( \gamma \)-radonifying property. The proof of Theorem 9 is just a simple application of Theorems 4.1 and 4.4 in [11].

3. Cocycle Property of the Stochastic Boussinesq Equations

In this section, we will show the existence, uniqueness, regularity, and the cocycle property of the stochastic Boussinesq equations (11).

It is well known that both \( A_1 \) and \( A_2 \) are positive definite, self-adjoint operators, and denote \( D(A_1) \) and \( D(A_2) \) to be the domains of \( A_1 \) and \( A_2 \), respectively. Hence, the domain of the operator \( A \) can be represented as \( D(A) = D(A_1) \times D(A_2) \).
It follows from Lemma 2.2 in [7] that there exists positive numbers \( \mu_1, \mu_2 \), such that
\[
(A_1 u, u) \geq \mu_1 \|u\|_{L^2}^2, \quad (A_2 (u, \theta), (u, \theta)) \geq \mu_2 \|u, \theta\|_{L^2}^2.
\]
(31)

Let \( \lambda = \min(\mu_1, \mu_2) \). Then
\[
(\mathbf{A} u, u) \geq \lambda \|u\|_{L^2}^2.
\]
(32)

For any arbitrary \( U, V, W \in \mathbb{V} \), we can define the following trilinear form \( b : U \times V \times W \rightarrow \mathbb{R} \) by
\[
b(u, v, w) = \langle B(u, v), w \rangle,
\]
b\((U, V, W) = b_1(u, v, w) + b_2(u, \tilde{v}, \tilde{w})
\]
\[
b_1(u, v, w) = \int_D \sigma_j(u) \frac{\partial \phi_j}{\partial x_i} v_j(\omega) \, dx,
\]
\[
b_2(u, \tilde{v}, \tilde{w}) = \int_D \sigma_j(u) \frac{\partial \phi_j}{\partial x_i} \tilde{w}_j(\omega) \, dx.
\]
(33)

We have the following results.

**Lemma 10** (see [7, Lemma 2.3]). If \( U, V, W \in \mathbb{V} \), then
\[
b(U, V, W) = -b(U, W, V),
\]
\[
(b(V, U), U) = b(V, U, U) = 0.
\]
(34)

**Lemma 11** (see [7, Lemma 2.4]). There exists a constant \( c_0 > 0 \) such that if \( u \in V_1, \theta, \eta \in V_2, \phi = (u, \theta), \) then
\[
\sup |b(u, v, w)| \leq c_0 \|u\|_{L^2} \|v\|_{L^2} \|w\|_2, \quad u \in V, \quad v \in D(A), \quad w \in H,
\]
\[
\sup |b(u, v, w)| \leq c_0 \|u\|_{L^2} \|v\|_{L^2} \|w\|_{L^2} \|w\|_{L^2} \|w\|_{L^2}, \quad u \in V, \quad v \in V, \quad w \in D(A), \quad w \in H,
\]
\[
\sup |b(u, v, w)| \leq c_0 \|u\|_{L^2} \|v\|_{L^2} \|v\|_2, \quad u \in V, \quad v \in V,
\]
\[
\sup |b(u, \theta, w)| \leq c_0 \|u\|_{L^2} \|\theta\|_{L^2} \|w\|_{H^\gamma}, \quad u \in V, \quad \theta \in V, \quad w \in W,
\]
\[
\sup |b(u, \theta, w)| \leq c_0 \|u\|_{L^2} \|\theta\|_{L^2} \|w\|_{H^\gamma}, \quad u \in V, \quad \theta \in V, \quad w \in W,
\]
\[
\sup |b(u, \theta, w)| \leq c_0 \|u\|_{L^2} \|\theta\|_{L^2} \|w\|_{H^\gamma}, \quad u \in V, \quad \theta \in V, \quad w \in W.
\]
(35)

**Definition 12.** An \( H \)-valued \((\mathcal{F})_{t \geq 0} \) adapted and \( \mathbb{H}^{2,2}(D) \)-valued càdlàg process \( u(t) \) (\( t \geq 0 \)) is considered as a solution to (11), if for each \( T > 0 \),
\[
\sup_{0 \leq t \leq T} |U(t)|_H^2 + \int_0^T \|U(t)\|^4_{L^4(D)} \, dt \leq \infty, \quad \text{a.s.,}
\]
(36)

and for any \( \psi \in V \cap \mathbb{H}^{2,2}(D) \), and for any \( t > 0 \), \( \mathbb{P} \)-a.s.,
\[
(U(t), \psi) - (U_0, \psi) - \int_0^t (U(s), \Delta \psi) \, ds
\]
\[
+ \int_0^t (B(U, U), \psi(s)) \, ds + \int_0^t (\mathcal{R}(U), \psi(s)) \, ds = (\psi, Y(t)).
\]
(37)

Denote
\[
\mathbb{H}^{1,2}(0, T) = \{ \text{the space of all functions } v \in L^2(0, T; V) \cap \mathbb{H}^{2,2}(D) \text{ satisfying } v' \in L^2(0, T; V') \}.
\]
(38)

**Lemma 13.** Assume that \( z \in L^4(0, T; L^4(D)) \), \( g \in L^2(0, T, V') \), and \( v_0 \in H \). Then there exists a unique \( v \in \mathbb{H}^{1,2}(0, T) \) such that
\[
\frac{dv}{dt} + Av + B(v, z) + B(z, v) + B(v, v) = g, \quad t \geq 0,
\]
(39)

Moreover,
\[
\sup_{t \in [0, T]} |v(t)|_2^2 \leq K_2^2 L^2, \quad \int_0^T |v(t)|_2^2 \, dt \leq M_2^2,
\]
\[
\int_0^T |v'(t)|_2^2 \, dt \leq M_2^2, \quad \int_0^T |v(t)|^4_{L^4(D)} \, dt \leq 2T^{1/2} K_3^2 L^2 M,
\]
(40)

and the mapping \( L^2(0, T, V') \times H \ni (g_0, v_0) \mapsto v \in \mathbb{H}^{1,2}(0, T) \) is analytic.

**Proof.** It can be shown by the same approach as the one in Proposition 8.7 in [11].

**Lemma 14** (see [2, Proposition 10.1]). Let \( u : [0, T] \rightarrow B \) be a continuous function whose left derivative
\[
\frac{d}{dt} u(t) = \lim_{\epsilon \to 0, \epsilon > 0} \frac{u(t + \epsilon) - u(t)}{\epsilon}
\]
exists at \( t_0 \in [0, T] \). Then the function \( v(t) = |u(t)|_B \), \( t \in [0, T] \), is left differentiable at \( t_0 \) and
\[
\frac{d}{dt} v(t) = \min \left\{ |x'|, \frac{d}{dt} u(t) \right\}, \quad x' \in \partial |u(t)|_B.
\]
(41)

In order to apply the Yosida approximation for the solution of (11), we need to introduce some definitions of dissipative mapping (operator) (see [17] for details).

**Definition 15.** Let \( (B, | \cdot |_B) \) be a separable Banach space, \( B^* \) be the dual space of \( B \). The subdifferential \( \partial |x|_B \) of norm \( | \cdot |_B \) at \( x \in B \) is defined by the formula
\[
\partial |x|_B := \{ x^* \in B^* : |x + y|_B - |x|_B \geq (x^*, y), \forall y \in B \}.
\]
(42)
A mapping $F : D(F) \subset B \to B$ is said to be dissipative, if for any $x, y \in D(F)$, there exists $z^* \in \partial |x - y|_B$ such that
\[
\langle z^*, F(x) - F(y) \rangle \leq 0.
\] (44)

A dissipative mapping $F : D(F) \subset B \to B$ is called an $m$-dissipative mapping or maximal dissipative if the image of $I - \lambda F$ is equal to the whole space $B$ for some $\lambda > 0$ (and then for any $\lambda > 0$), that is,
\[
\text{range} (I - \lambda A) = B, \quad \text{for some} \ \lambda > 0.
\] (45)

Assume that $F$ is an $m$-dissipative mapping. Then its resolvent $I_\alpha$ and respectively the Yosida approximations $F_\alpha$, $\alpha > 0$, are defined by
\[
I_\alpha x = (I - \alpha F)^{-1} x \in \text{dom} \ F,
\]
\[
F_\alpha x = \frac{1}{\alpha} (I_\alpha x - x), \quad \forall x \in \text{dom} \ I_\alpha = \text{range} (I - \alpha F).
\] (46)

**Lemma 16** (see [2, Proposition 10.2]). Let $F : D(F) \to B$ be an $m$-dissipative mapping on $B$. Then

1. for all $\alpha > 0$ and $x, y \in B$, $|I_\alpha (x) - I_\alpha (y)|_B \leq |x - y|_B$;
2. the mapping $F_\alpha$, $\alpha > 0$, are dissipative and Lipschitz continuous:
\[
|F_\alpha (x) - F_\alpha (y)|_B \leq \frac{2}{\alpha} |x - y|_B, \quad \forall x, y \in B.
\] (47)

Moreover, $|F_\alpha (x)|_B \leq |F(x)|_B$, for all $x \in D(F)$; and
3. $\lim_{\alpha \to 0} F_\alpha (x) = x$, for all $x \in D(F)$.

The following theorem is one of the main results of this paper, which will be proved by applying the well-known Yosida approach.

**Theorem 17.** For every $u_0 \in H$, under Assumption 1, the stochastic Boussinesq system (11) admits a unique càdlàg mild solution $u(t)$.

**Proof.** Denote $Z_\alpha (\omega)$ to be the stationary solution of Langevin equation (24). Let $V = U - Z_\alpha$. Then (11) is converted into the following evolution equation with random coefficients:
\[
dV = [AV + B(V + Z_\alpha) + R(V + Z_\alpha)] dt, \quad t \geq 0,
\]
\[
V(0) = U_0,
\] (48)

where $(A, D(A))$ generates an analytic $C_0$-semigroup $S$ (see Section 2.2 in [2]). It follows from the proof of Theorem 10.1 in [2] that, for $\alpha > 0$, $\beta > 0$, and sufficiently small $\eta$, the mappings $A + \eta$ and $B(\cdot, \cdot) + R(\cdot) + \eta$ are $m$-dissipative. Hence, the Yosida approximations of the $m$-dissipative mappings $A + \eta$ and $B(\cdot, \cdot) + R(\cdot) + \eta$ can be respectively denoted by
\[
(A + \eta)_\beta = \frac{1}{\beta} \left( (I - \beta (A + \eta))^{-1} - I \right),
\]
\[
((B + R) + \eta)_\alpha = \frac{1}{\alpha} \left( (I - \alpha ((B + R) + \eta))^{-1} - I \right).
\] (49)

Now consider the following random approximate equation:
\[
\frac{d}{dt} Y_{\alpha,\beta}(t) = (A + \eta)_\beta Y_{\alpha,\beta} + (B + R + \eta)_\alpha \left(Y_{\alpha,\beta} + Z_\alpha (t-)\right) - 2\eta Y_{\alpha,\beta} - \eta Z_\alpha (t-),
\]
\[
Y_{\alpha,\beta} (0) = U_0.
\] (50)

It is easy to verify that $(A + \eta)_\beta, D((A + \eta)_\beta)$ generates an analytic $C_0$-semigroup $S_\beta$. Notice that the Yosida approximate operators are Lipschitz. Therefore the random approximation equation (50) has a unique continuous solution $Y_{\alpha,\beta}$.

Next we will show that
\[
\lim_{\alpha \to 0} \left[ \lim_{\beta \to 0} Y_{\alpha,\beta}(t) \right] = Y(t), \quad t \geq 0,
\] (51)
in $H$, and this limit is actually the mild solution of stochastic Boussinesq equation (48).

For the sake of simplification, we just present the estimations when $\eta = 0$, and the remaining estimates can be obtained by the similar arguments for $\eta \neq 0$.

Let $Y_{\alpha,\beta}$ be the solution of the integral equation:
\[
Y_{\alpha,\beta}(t) = S(t) U_0 + \int_0^t S(t-s) \times \left( B (Y_{\alpha,\beta}(s)+Z_\alpha (s-)), Y_{\alpha,\beta}(s)+Z_\alpha (s-) \right)
+ R (Y_{\alpha,\beta}(s)+Z_\alpha (s-)) \right)_\alpha ds.
\] (52)

Notice that the operator $(B(\cdot, \cdot) + R(\cdot))_\alpha$ is Lipschitz continuous and $Z_\alpha$ is càdlàg. Hence, there exists a solution of random approximate equation (50), which is continuous in $H$.

For $\alpha > 0$ and $\beta > 0$, direct computation implies
\[
Y_{\alpha,\beta} - Y_{\alpha,\beta} = S(t) U_0 - S_\beta (t)
+ \int_0^t \left[ S(t-s) - S_\beta (t-s) \right]
\times \left[ B (Y_{\alpha,\beta}(s)+Z_\alpha (s-)), Y_{\alpha,\beta}(s)+Z_\alpha (s-) \right)
+ R (Y_{\alpha,\beta}(s)+Z_\alpha (s-)) \right)_\alpha ds
\]
\[
+ \int_0^t \left[ S_\beta (t-s) \right]
\times \left[ B (Y_{\alpha,\beta}(s)+Z_\alpha (s-)), Y_{\alpha,\beta}(s)+Z_\alpha (s-) \right)
+ R (Y_{\alpha,\beta}(s)+Z_\alpha (s-)) \right)_\alpha ds.
\] (53)
Since both $A$ and $B + R$ are $m$-dissipative. Therefore, there exists constant $M, \omega,$ and $C_{\alpha}$ such that for all $t \geq 0, V, W \in H$,

$$
\|S_\beta(t)\|_{\mathcal{L}(H,H)} \leq Me^{\omega t},
$$

$$
\|B(V) + R(V)\|_\alpha - \|B(U) + R(U)\|_\alpha \leq C_\alpha|V - U|_H. \tag{54}
$$

Then

$$
\begin{align*}
\left| Y_\alpha(t) - Y_{\alpha,\beta}(t) \right| \\
\leq \left| S(t)U_0 - S_\beta(t)U_0 \right| + MC_\alpha \int_0^t e^{\omega(t-s)} \left| Y_\alpha(s) - Y_{\alpha,\beta}(s) \right| ds \\
+ \int_0^t \left[ \left\| S_\beta(t-s) - S(t-s) \right\| \right] \times \left[ B \left( Y_{\alpha,\beta}(s) + Z_\alpha(s) \right), Y_{\alpha,\beta}(s) + Z_\alpha(s) \right] \\
+ R \left( Y_{\alpha,\beta}(s) + Z_\alpha(s) \right) \right\|_\alpha \right] ds.
\end{align*}
\tag{55}
$$

By the Hille-Yosida theorem, it follows that

$$
S_\beta(t)U_0 \to S(t)U_0, \quad \text{as} \ \beta \to 0 \tag{56}
$$

uniformly in $t$ on compact subsets $U_0$ of $H$. Hence, it follows that

$$
\left| Y_\alpha(t) - Y_{\alpha,\beta}(t) \right| \leq MC_\alpha \int_0^t \left| Y_\alpha(s) - Y_{\alpha,\beta}(s) \right| ds \tag{57}
$$

uniformly on bounded intervals as $\beta \to 0$.

By Gronwall inequality, we have

$$
\lim_{\beta \to 0, t \leq T} \sup \left| Y_\alpha(t) - Y_{\alpha,\beta}(t) \right| = 0, \ \forall T < \infty. \tag{58}
$$

By Lemma 14,

$$
\begin{align*}
\frac{d}{dt} \left| Y_{\alpha,\beta}(t) \right| \\
= \min \left\{ \left< x^*, \frac{d}{dt} Y_{\alpha,\beta}(t) \right> : x^* \in \partial \left| Y_{\alpha,\beta}(t) \right| \right\} \\
= \min \left\{ \left< x^*, A_{\beta} Y_{\alpha,\beta}(t) \right> + \left[ B \left( Y_{\alpha,\beta}(s) + Z_\alpha(s) \right), Y_{\alpha,\beta}(s) + Z_\alpha(s) \right] \\
+ R \left( Y_{\alpha,\beta}(s) + Z_\alpha(s) \right) \right\} : x^* \in \partial \left| Y_{\alpha,\beta}(t) \right| \right\},
\end{align*}
\tag{59}
$$

Recalling that both $A_{\beta}$ and $[B(\cdot, \cdot) + R(\cdot)]$ are $m$-dissipative and $A_{\beta}$ is linear, we obtain

$$
\begin{align*}
\frac{d}{dt} \left| Y_{\alpha,\beta}(t) \right| \leq \left| B \left( Y_{\alpha,\beta}(t) + Z_\alpha(t) \right), Y_{\alpha,\beta}(s) + Z_\alpha(t) \right) \\
+ R \left( Y_{\alpha,\beta}(t) + Z_\alpha(t) \right) \right\|_\alpha \right] \leq \left| B \left( Y_{\alpha,\beta}(t) + Z_\alpha(t) \right), Y_{\alpha,\beta}(s) + Z_\alpha(t) \right) \\
+ R \left( Y_{\alpha,\beta}(t) + Z_\alpha(t) \right) \right\|_\alpha,
\end{align*}
\tag{60}
$$

that is,

$$
\begin{align*}
\left| Y_{\alpha,\beta}(t) \right| & \leq \left| U_0 \right| + \int_0^t \left| B \left( Y_{\alpha,\beta}(s) + Z_\alpha(s) \right), Y_{\alpha,\beta}(s) + Z_\alpha(s) \right) \\
+ R \left( Y_{\alpha,\beta}(s) + Z_\alpha(s) \right) \right\|_\alpha \right] ds, \quad t \geq 0.
\end{align*}
\tag{61}
$$

It follows from the estimate (58) that, for any $\alpha > 0$, and $t \in [0, T],

$$
\begin{align*}
\left| Y_\alpha(t) \right| \leq \left| U_0 \right| + \int_0^t \left| B \left( Y_{\alpha,\beta}(s) + Z_\alpha(s) \right), Y_{\alpha,\beta}(s) + Z_\alpha(s) \right) \\
+ R \left( Y_{\alpha,\beta}(s) + Z_\alpha(s) \right) \right\|_\alpha \right] ds.
\end{align*}
\tag{62}
$$

Similarly, by Lemma 16, for $t \in [0, T],

$$
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left| Y_{\alpha,\beta}(t) - Y_{\gamma,\beta}(t) \right|^2 \\
= \left< \frac{d}{dt} \left( Y_{\alpha,\beta}(t) - Y_{\gamma,\beta}(t) \right), Y_{\alpha,\beta}(t) - Y_{\gamma,\beta}(t) \right> \\
= \left< \left( A_{\beta} Y_{\alpha,\beta}(t) - A_{\beta} Y_{\gamma,\beta}(t) \right) + \left[ (B + R)_{\alpha} \right] \\
\times \left( \left( Y_{\alpha,\beta}(t) + Z_\alpha(\omega(t)) \right) \right) - \left[ (B + R)_{\gamma} \right] \\
\times \left( \left( Y_{\alpha,\beta}(t) + Z_\alpha(\omega(t)) \right) \right), Y_{\alpha,\beta}(t) - Y_{\gamma,\beta}(t) \right> \\
\leq \left( (B + R)_{\alpha} \right) \left( Y_{\alpha,\beta}(t) + Z_\alpha(\omega(t)) \right) - \left[ (B + R)_{\gamma} \right] \\
\times \left( \left( Y_{\alpha,\beta}(t) + Z_\alpha(\omega(t)) \right) \right), Y_{\alpha,\beta}(t) - Y_{\gamma,\beta}(t) \right> \\
\leq (\gamma + \alpha) \left[ (B + R)_{\alpha} \left( Y_{\alpha,\beta}(t) + Z_\alpha(\omega(t)) \right) \right] \\
+ \left[ (B + R)_{\gamma} \left( Y_{\alpha,\beta}(t) + Z_\alpha(\omega(t)) \right) \right]^2 \\
\leq (\gamma + \alpha) \left[ (B + R) \left( Y_{\alpha,\beta}(t) + Z_\alpha(\omega(t)) \right) \right] \\
+ \left[ (B + R) \left( Y_{\alpha,\beta}(t) + Z_\alpha(\omega(t)) \right) \right]^2.
\end{align*}
\tag{63}
$$

By the dissipation of the operators $A, B$, and $R$ and estimates (63), there exists a constant $C > 0$ such that

$$
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left| Y_{\alpha,\beta}(t) - Y_{\gamma,\beta}(t) \right|^2 \leq C (\alpha + \gamma), \quad t \in [0, T].
\end{align*}
\tag{64}
$$
Thus, by the reflexivity of $H$ function $Y(t)$ weakly converges in $L^2$. Let $h \in L^2$, then

$$\langle Y(t), h \rangle_{L^2} = \langle S(t)U_0, h \rangle_{L^2} + \int_0^t \langle (B+R)Y(s) + Z_A(s), S^\ast(t-s)h \rangle_{L^2} ds.$$  

Moreover

$$J_\alpha (Y(s) + Z_A(s)) \rightarrow Y(s) + Z_A(s), \text{ as } \alpha \rightarrow 0.$$  

Notice that $(B+R)(J_\alpha (Y(s) + Z_A(s))) \rightarrow (B+R)(Y(s) + Z_A(s))$ weakly converges in $L^2$. So, letting $\alpha \rightarrow 0$, we obtain

$$\langle Y(t), h \rangle_{L^2} = \langle S(t)U_0, h \rangle_{L^2} + \int_0^t \langle (S(t-s)(B+R)Y(s) + Z_A(s)), h \rangle_{L^2} ds.$$  

It follows from the arbitrariness of $h$ that

$$Y(t) = S(t)U_0 + \int_0^t S(t-s)(B+R)(Y(s) + Z_A(s)) ds, \quad t \in [0, T].$$  

Thus, $Y(t)$ is a mild solution of random Boussinesq equation (50).

**Theorem 18.** For any $U_0 \in H$, the map $\Phi : T \times \Omega \times H \rightarrow H$ defined by the solution of stochastic Boussinesq equation (11) as $U(t) = \Phi(t, \theta(t))U_0$ has the cocycle property; that is, the solution of stochastic Boussinesq equation (11) generates a random dynamical system $(\Omega, F, P, (\theta_t)_{t \geq 0}, \Phi)$.

**Proof.** From Theorem 17, stochastic Boussinesq equation (11) admits a unique solution $V(t, Z(\omega)(t), x)$. Define the map

$$\Phi : \mathbb{R}^+ \times \Omega \times H \rightarrow H,$$

$$\Phi(t, \omega) x = V(t \cdot Z(\omega)(t)) (x - Z(\omega)(0)) + Z(\omega)(t + s).$$

(i) By the similar argument of Theorem 17, every solution $Y_\alpha(t)$ of the Yosida approximation equation (50) is measurable. Notice that $Y_\alpha(t) \rightarrow Y(t)$ uniformly as $\alpha \rightarrow 0$. Hence, the limit function $Y(t)$ is also measurable. Thus, the mapping $\Phi$ is measurable.

(ii) Obviously, $\Phi(0, \omega) = I$.

(iii) It suffices to verify that the cocycle property holds for the mapping $\Phi$, that is,

$$\Phi(t + s, \omega) x = V(t + s, Z_A(\omega)(t + s)) (x - Z_A(\omega)(0)) + Z_A(\omega)(t + s).$$

In fact, recalling that $Z_A(\omega)(s) = Z_A(\theta(t))(0)$, it follows that

$$\Phi(t, \theta(t)) [\Phi(s, \omega) x]$$

$$= V(t, Z_A(\theta(t))(0)) (\Phi(s, \omega)x - Z_A(\theta(t))(0)) + Z_A(\theta(t))(0)$$

$$\times V(s, Z_A(\omega)(s))(x - Z_A(\omega)(0)) + Z(\omega)(s) - Z(\theta(t))(0)) + Z(\omega)(t)$$

$$= V(t, Z_A(\omega)(t)) V(s, Z_A(\omega)(s))(x - Z_A(\omega)(0)) + Z_A(\omega)(t)$$

$$\times (x - Z_A(\omega)(0)) + Z_A(\omega)(t)$$

$$= V_1(t).$$

Moreover,

$$V(t + s, Z_A(\omega)(t + s))(x - Z_A(\omega)(0))$$

$$= V(t, Z_A(\omega)(t)) V(s, Z_A(\omega)(s))(x - Z_A(\omega)(0))$$

$$= V_2(t).$$

Since

$$V(0, Z_A(\omega)(0))(x - Z_A(\omega)(0)) = x - Z_A(\theta(t))(0),$$

$$V(0, Z_A(\omega)(0))(x - Z_A(\omega)(0)) = x - Z_A(\theta(t))(0).$$

(77)
Thus,
\[
V_1(0) = V(s, Z_A(\omega)(s))(x - Z_A(\omega)(0)) \\
= V(0, Z_A(\theta_\omega)(0)) V(s, Z_A(\omega)(s)) \\
\times (x - Z_A(\omega)(0)) = V_2(0),
\]
\[
\frac{dV_1(t)}{dt} + AV_1(t) \\
\quad + B(V_1(t) + Z_A(\omega)(t + s), V_1(t) + Z_A(\omega)(t + s)) \\
= -R(V_1(t) + Z_A(\theta_{t+s}\omega)),
\]
\[
\frac{dV_2(t)}{dt} + AV_2(t) \\
\quad + B(V_2(t) + Z_A(\theta_\omega)(t), V_2(t) + Z_A(\theta_\omega)(t)) \\
= -R(V_2(t) + Z_A(\theta_\omega)(t)).
\]

Therefore, we obtain
\[
\frac{dV_1(t)}{dt} + AV_1(t) \\
\quad + B(V_1(t) + Z_A(\omega)(t + s), V_1(t) + Z_A(\omega)(t + s)) \\
= -R(V_1(t) + Z_A(\theta_{t+s}\omega)),
\]
\[
\frac{dV_2(t)}{dt} + AV_2(t) \\
\quad + B(V_2(t) + Z_A(\theta_\omega)(t), V_2(t) + Z_A(\theta_\omega)(t)) \\
= -R(V_2(t) + Z_A(\theta_\omega)(t)).
\]

The uniqueness of the solution implies that almost surely \( V_1(t) = V_2(t) \) holds, that is,
\[
\Phi(t, \theta_\omega) \left[ \Phi(s, \omega)x \right] = \Phi(t + s, \theta_{t+s}(\omega))x.
\]

Thus, the cocycle property for the mapping \( \Phi \) holds.

By the definition of random dynamical systems [18], the solution mapping of the stochastic Boussinesq equation (11) generates a random dynamical system \( \Phi \). Thus, the proof of Theorem 18 is complete.

4. Discussion

In Section 3, we have studied the long-time behavior of stochastic Boussinesq equations (1) driven by subordinator Lévy noise and have shown the cocycle property of random dynamical systems generated by the mild solution of stochastic Boussinesq equation (1). To prove the existence of random attractor, it suffices to show the existence of random absorbing set and the compactness of random dynamical system \( \Phi \), we refer the similar argument to [13].

Here, we are also interested in the stochastic Boussinesq equations driven by Poisson noise and Wiener noise, and we are trying to show the existence of random dynamical systems. To the end, we consider the following stochastic Boussinesq equations driven by Lévy noises followed as
\[
\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u - v\Delta u + \nabla p &= \theta e_2 + b_1 dt + dW^1(t) + \int_X f(x) N^1(dt, dx), \\
\frac{\partial \theta}{\partial t} + (u \cdot \nabla) \theta - k \Delta \theta &= u_2 + b_2 dt + dW^2(t) + \int_X g(x) N_2(dt, dx),
\end{align*}
\]
where \( W^1(\cdot) \) and \( W^2(\cdot) \) are \( H \)-valued Brownian motion, \( b_1 \) and \( b_2 \) are constants vector in \( H \), \( f \) and \( g \) are measurable mappings from some measurable space \( X \) to \( H \), and \( N^1 \) and \( N_2 \) are compensated Poisson measure on \([0, \infty) \times X\) with intensity measure \( \nu_1 \) and \( \nu_2 \), respectively, where \( \nu_1 \) and \( \nu_2 \) are \( \sigma \)-finite measure on \( \mathbb{B}(X), f(x), \) and \( g(x) \) satisfying
\[
\begin{align*}
\int_U |f(x)|^2 e^{\alpha|f(x)|^2} \nu(dx) &< \infty, \\
\int_U |g(x)|^2 e^{\beta|g(x)|^2} \nu(dx) &< \infty, \quad \forall \alpha > 0, \forall \beta > 0.
\end{align*}
\]

Let \( D([0, T], H) \) be the space of all càdlàg paths from \([0, T]\) to \( H \) endowed with the uniform convergence topology. Since there are finite jumps when the character measure \( \lambda(dx) < \infty \), we can rearrange the jump time of \( N(dt, dx) \) as \( \sigma_1(\omega) < \sigma_2(\omega) < \cdots \). Since there is no jump on the interval \([\sigma_1(\omega), \sigma_2(\omega)]\), just as the approach in [19], we can apply Banach fixed point theorem to prove that there exists a unique solution \( \phi(t) \) in \( L^2([0, \sigma_1(\omega)); V) \cap D([0, \sigma_1(\omega)); H) \). Define
\[
\phi^{(1)}(t) = \begin{cases} 
\phi(t), & 0 \leq t < \sigma_1(\omega), \\
\phi(\sigma_1^-) + f(\phi(\sigma_1^-), P_{\sigma_1}), & t = \sigma_1(\omega).
\end{cases}
\]

On \([\sigma_1(\omega), \sigma_2(\omega)]\), define
\[
\bar{\phi}_0 = \phi^{(1)}(\sigma_1) 1_{(\sigma_1, \infty)} , \\
\bar{\sigma}_2 = (\sigma_2 - \sigma_1) 1_{(\sigma_1, \infty)} + \infty 1_{(\sigma_1, \infty)} ,
\]
\[
\bar{\mathcal{F}}_t = \mathcal{F}_{\sigma_1+t} , \\
P_t = (\theta_{\sigma_1}, P) 1_{(\sigma_1, \infty)}.
\]

Similar to the argument in [11], since \( P_t \) is stationary Poisson point process on \( \mathbb{R}^+ \times \mathbb{Z} \) with intensity measure \( \lambda(dx)dt \),
then $\tilde{P}_t$ is also a stable Poisson point process on $R^t \times Z$ with intensity measure $\lambda(dx)dt$. Define
\[
\phi^{(2)}(t) = \begin{cases} 
\phi^{(1)}(t), & 0 \leq t < \sigma_1(\omega), \\
\phi^{(2)}(t - \sigma_1), & \sigma_1(\omega) < t < \sigma_2(\omega), \\
\phi^{(2)}((\sigma_2 - \sigma_1) -) + f(\tilde{\phi}^{(2)}((\sigma_2 - \sigma_1) -), P_{\sigma_2}), & t = \sigma_2(\omega),
\end{cases}
\]
\[
\phi^{(n)}(t) = \begin{cases} 
\phi^{(n-1)}(t), & t < \sigma_{n-1}(\omega), \\
\tilde{\phi}^{(n)}(t - \sigma_{n-1}), & \sigma_{n-1}(\omega) < t < \sigma_n(\omega), \\
\tilde{\phi}^{(n)}((\sigma_n - \sigma_{n-1}) -) + f(\tilde{\phi}^{(n)}((\sigma_n - \sigma_{n-1}) -), P_{\sigma_n}), & t = \sigma_n.
\end{cases}
\]

Hence, $\phi^{(n)}(t)$ is càdlàg on $[0, T]$ such that $B(\phi^{(n)}, \tilde{\phi}^{(n)}) \in H$ and $A_p(\phi^{(n)}) \in H$, $P$ a.s. for all $t \geq 0$, and
\[
P \left( \int_0^t \left[ ||\phi(s)|| + ||B(\phi(s) + z_A(s), \phi(s) + z_A(s))|| \\
+ 2\mu_0 ||R(\phi(s) + z_A(s))|| ds < \infty \right] \right) = 1, \quad \forall t > 0.
\]

Therefore, $\phi^{(n)}(t)$ is a unique global weak solution of (81). We can verify the existence of random dynamical systems generated by the global weak solution of (81).

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