Research Article

Complex Oscillation of Higher-Order Linear Differential Equations with Coefficients Being Lacunary Series of Finite Iterated Order

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1. Definitions and Notations

In this paper, we assume that readers are familiar with the fundamental results and standard notations of the Nevanlinna theory of meromorphic functions (see [1–3]). In order to describe the growth of order of entire functions or meromorphic functions more precisely, we first introduce some notations about finite iterated order. Let us define inductively, for \( r \in (0, +\infty) \),
\[
\exp_1 r = e^r \quad \text{and} \quad \exp_{i+1} r = \exp(\exp_i r), \quad i \in \mathbb{N}.
\]
For a sufficiently large \( r \), we define \( \log_0 r = \log r \) and \( \log_{i+1} r = \log(\log_i r) \), \( i \in \mathbb{N} \). We also denote \( \exp_0 r = r = \log_0 r \) and \( \exp_{i,1} r = \log_i r \). Moreover, we denote the logarithmic measure of a set \( E \subset (0, +\infty) \) by \( m_E = \int_E \frac{dt}{t} \), and the upper logarithmic density of \( E \subset (0, +\infty) \) is defined by
\[
\log \text{dens} E = \lim_{r \to \infty} \frac{m_r(E \cap [1, r])}{\log r}.
\]

Throughout this paper, we use \( p \) to denote a positive integer. In the following, we recall some definitions of entire functions or meromorphic functions of finite iterated order (see [4–10]).

Definition 1. The \( p \)-iterated order of a meromorphic function \( f(z) \) is defined by
\[
\sigma_p (f) = \lim_{r \to \infty} \frac{\log_p T(r, f)}{\log r}.
\]

Remark 2. If \( f(z) \) is an entire function, then the \( p \)-iterated order of \( f(z) \) is defined by
\[
\sigma_p (f) = \lim_{r \to \infty} \frac{\log_p T(r, f)}{\log r} = \lim_{r \to \infty} \frac{\log_{p+1} M(r, f)}{\log r}.
\]

If \( p = 1 \), the classical growth of order of \( f(z) \) is defined by
\[
\sigma (f) = \lim_{r \to \infty} \frac{\log T(r, f)}{\log r} = \lim_{r \to \infty} \frac{\log M(r, f)}{\log r}.
\]

If \( p = 2 \), the hyperorder of \( f(z) \) is defined by
\[
\sigma_2 (f) = \lim_{r \to \infty} \frac{\log_2 T(r, f)}{\log r} = \lim_{r \to \infty} \frac{\log_2 M(r, f)}{\log r}.
\]
Definition 3. If \( f(z) \) is an entire function with \( 0 < \sigma_p(f) = \sigma < \infty \), then the \( p \)-iterated type of \( f(z) \) is defined by
\[
\tau_p(f) = \lim_{r \to \infty} \frac{\log M(r,f)}{r^\sigma}.
\]

Definition 4. The \( p \)-iterated lower order of an entire function \( f(z) \) is defined by
\[
\mu_p(f) = \lim_{r \to \infty} \frac{\log T(r,f)}{\log r} = \lim_{r \to \infty} \frac{\log M_{p+1}(r,f)}{\log r}.
\]

Definition 5. The finiteness degree of the iterated order of a meromorphic function \( f(z) \) is defined by
\[
i(f) = \begin{cases} 
0 & \text{for } f(z) \text{ rational}, \\
\min \{ p \in \mathbb{N} : \sigma_p(f) < \infty \} & \text{for } f(z) \text{ transcendental for which some } p \in \mathbb{N} \text{ with } \sigma_p(f) < \infty \text{ exists}, \\
\infty & \text{or } f(z) \text{ with } \sigma_p(f) = \infty \forall p \in \mathbb{N}.
\end{cases}
\]

Definition 6. The \( p \)-iterated exponent of convergence of \( a \)-point of a meromorphic function \( f(z) \) is defined by
\[
\lambda_p(f,a) = \lim_{r \to \infty} \frac{\log P_n(r,a)}{\log r} = \lim_{r \to \infty} \frac{\log N(r,a)}{\log r}.
\]

The \( p \)-iterated lower exponent of convergence of \( a \)-point of a meromorphic function \( f(z) \) is defined by
\[
\lambda_p(f) = \lim_{r \to \infty} \frac{\log M_{p+1}(r)\log r}{r^\sigma} = \lim_{r \to \infty} \frac{\log \bar{N}(r,1/f)}{\log r},
\]

2. Introductions and Main Results

In the past ten years, many authors have investigated the complex oscillation properties of the higher-order linear differential equations
\[
\begin{align*}
&f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = 0, \\
&f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = F(z)
\end{align*}
\]
with \( A_j(z) \) \( (j = 0, \ldots, k-1) \), \( F(z) \) being entire functions or meromorphic functions of fast growing (e.g., see [4–12]), and obtained the following results.

Theorem A (see [8]). Let \( A_0(z), \ldots, A_{k-1}(z) \) be entire functions, if \( i(A_j) \leq p \) \( (j = 0, \ldots, k-1) \), then (12) satisfies (9) and \( \sigma_{p+1}(f) = \max \{ \sigma_{p}(A_j), j = 0, \ldots, k-1 \} \) hold for all solutions of (12).

Theorem B (see [8]). Let \( A_0(z), \ldots, A_{k-1}(z) \) be entire functions and let \( i(A_0) = p \). Assume that \( \max \{ \sigma_{p}(A_j) \} \leq p+1 \) and \( \sigma_{p+1}(f) = \sigma_{p}(A_0) \) hold for all nontrivial solutions of (12).

Theorem C (see [4, 12]). Let \( A_0(z), \ldots, A_{k-1}(z) \) be entire functions and let \( i(A_0) = p \). Assume that \( \max \{ \sigma_{p}(A_j) \} \leq p+1 \) and \( \max \{ \tau_{p}(A_j) \} = \sigma_{p}(A_0) \). Then, every solution (12) satisfies (9) and (12) holds for all nontrivial solutions of (12).

Theorem D (see [10]). Let \( A_0(z), \ldots, A_{k-1}(z) \) be entire functions of finite iterated order satisfying \( i(A_0) = p \), \( \sigma_{p}(A_0) = \sigma \), and \( \lim_{r \to \infty} \{ \sum_{j=1}^{k-1} m(r,A_j)/m(r,A_0) \} < 1 \). Then, every nontrivial solution \( f(z) \) of (12) satisfies \( \sigma_{p+1}(f) = \sigma_{p}(A_0) = \sigma \).

Theorem E (see [10]). Let \( A_0(z), \ldots, A_{k-1}(z) \) be entire functions of finite iterated order satisfying \( \max \{ \sigma_{p}(A_j) \}, j \neq 0 \) \( < \mu_p(A_0) = \sigma_{p}(A_0) \) and \( \lim_{r \to \infty} \{ \sum_{j=1}^{k-1} m(r,A_j)/m(r,A_0) \} < 1 \). Then, every nontrivial solution \( f(z) \) of (12) satisfies \( \sigma_{p+1}(f) = \mu_p(A_0) = \sigma_{p}(A_0) \).

Theorem F (see [5]). Let \( A_j(z) \) \( (j = 0, \ldots, k-1) \) be entire functions of finite iterated order such that there exists one transcendental \( A_d \) \( (0 \leq d \leq k-1) \) satisfying \( \sigma_{p}(A_d) \leq \sigma_{p}(A_j) \leq \sigma_{p}(A_0) \) for all \( j \neq s \), then (12) has at least one solution \( f(z) \) that satisfies \( i(f) = p+1 \) and \( \sigma_{p+1}(f) = \sigma_{p}(A_d) \).

Remark 7. Theorems B–E are investigating the growth of solutions of (12) when the coefficients are of finite iterated order and \( A_d(z) \) grows faster than other coefficients in (12).

What can we have if there exists one middle coefficient \( A_d(z) \) \( (1 \leq d \leq k-1) \) such that \( A_d(z) \) grows faster than other coefficients in (12) or (13)? Many authors have investigated this question when \( A_d(z) \) is of finite order and obtained many results (e.g., see [13–15]). Here, our question is that under what conditions can we obtain similar results with Theorems B–C if \( A_d(z) \) \( (1 \leq d \leq k-1) \) is of finite iterated order and grows faster than other coefficients in (12) or (13).

In 2009, Tu and Liu make use of the proposition of lacunary power series to investigate the above question in the case \( p = 1 \) and obtain the following result.

Theorem G (see [15]). Let \( A_j(z) \) \( (j = 0, \ldots, k-1) \), \( F(z) \) be entire functions satisfying \( \max \{ \sigma_{p}(A_j), \sigma(A,F) \} < \sigma(A_d) \leq \sigma_{p}(A_d) \) \( (1 \leq d \leq k-1) \). Suppose that \( A_d(z) = \sum_{n=0}^{\infty} c_n z^{d_n} \) is an entire function of regular growth such that the sequence of exponents \( \{ \lambda_n \} \) satisfies Fabry gap condition
\[
\frac{\lambda_n}{n} \to \infty \quad (n \to \infty),
\]
then one has

(i) if \( F(z) \equiv 0 \), then every transcendental solution \( f(z) \) of (13) satisfies \( \sigma_2(f) = \sigma(A_d) \);

(ii) if \( F(z) \not\equiv 0 \), then every transcendental solution \( f(z) \) of (13) satisfies \( \lambda_2(f) = \lambda_2(f) = \sigma(A_d) \).

In this paper, we continue our research in this area and obtain the following results.

**Theorem 8.** Let \( A_j(z) (j = 0, \ldots, k - 1) \), \( F(z) \) be entire functions of finite iterated order and satisfying \( 0 < \max \{ \sigma_p(A_j), j \neq d \} < \sigma(A_d) \) and \( \tau(F) < \tau_p(A_d) \), then every transcendental solution \( f(z) \) of (13) satisfies \( \lambda_{p+1}(f) = \lambda_{p+1}(f) = \sigma_p(A_d) \); furthermore if \( F(z) \not\equiv 0 \), then every transcendental solution \( f(z) \) of (13) satisfies \( \lambda_{p+1}(f) = \lambda_{p+1}(f) = \sigma_p(A_d) \).

**Remark 12.** Theorem 10 implies that all the solutions of (13) are of regular growth if \( A_d \) is of regular growth under some conditions; Theorem 11 is an improvement of the Theorem in [14, page 2694] and Theorems 1–2 in [16, page 624]. In fact, by Lemma 15, the gap condition (15) in Theorem 8 implies that \( T(r, A_d) \sim \log M(r, A_d) \) as \( r \to \infty \) outside a set of \( r \) of finite logarithmic measure; therefore, Theorem 11 is a generalization of Theorem 8 in a sense, but the condition on \( A_d \) in Theorem 8 is more stringent than that in Theorem 11.

In addition, Theorems 8–11 may have polynomial solutions of degree \( < d \) if \( d > 1 \).

### 3. Preliminary Lemmas

**Lemma 13** (see [17]). Let \( f(z) \) be a transcendental meromorphic function, and let \( a > 1 \) be a given constant, for any given \( \varepsilon > 0 \),

(i) there exist a constant \( B > 0 \) and a set \( E_1 \subset (0, +\infty) \) having finite logarithmic measure such that for all \( z \) satisfying \( |z| = r \not\in E_1 \), one has

\[
\left| \frac{f^{(i)}(z)}{f(z)} \right| \leq B \left[ \frac{T(\log r)^\alpha}{r} \log T(\alpha r, f) \right]^{j-i} (0 \leq i < j).
\]

(ii) There exists a set \( H_1 \subset [0, 2\pi) \) that has linear measure zero a constant \( B > 0 \) that depends only on \( \alpha \), for any \( \theta \in [0, 2\pi) \setminus H_1 \), there exists a constant \( R_0 = R_0(\theta) > 1 \) such that for all \( z \) satisfying \( \arg z = \theta \) and \( |z| = r > R_0 \), one has

\[
\left| \frac{f^{(i)}(z)}{f^{(j)}(z)} \right| \leq B[T(\alpha r, f) \log T(\alpha r, f)]^{j-i} (0 \leq i < j).
\]
Lemma 15 (see [18]). Let $f(z) = \sum_{\lambda_n \gg 0} c_{\lambda_n} z^{\lambda_n}$ be an entire function and the sequence of exponents $\{\lambda_n\}$ satisfies the gap condition (15). Then for any given $\epsilon > 0$,

$$\log L(r, f) > (1 - \epsilon) \log M(r, f)$$  \hspace{1cm} (18)

holds outside a set of finite logarithmic measure, where $M(r, f) = \sup_{|z|=r} |f(z)|$, $L(r, f) = \inf_{|z|=r} |f(z)|$.

Lemma 16 (see [4]). Let $f(z)$ be an entire function of finite iterated order satisfying $0 < \sigma_p(f) = \sigma < \infty$ and $\tau_p(f) = \tau > 0$, then for any given $\beta < \tau$, there exists a set $E_2 \subset (0, +\infty)$ having infinite logarithmic measure such that for all $r \in E_2$, one has

$$M(r, f) = \exp_p \{\beta r^\sigma\}.$$  \hspace{1cm} (19)

Lemma 17. Let $f(z) = \sum_{\lambda_n \gg 0} c_{\lambda_n} z^{\lambda_n}$ be an entire function of finite iterated order satisfying $0 < \sigma_p(f) = \sigma < \infty$ and $\tau_p(f) = \tau > 0$ such that the sequence of exponents $\{\lambda_n\}$ satisfies the gap condition (15). Then, for any given $\beta > \tau$, there exists a set $E_3 \subset (0, +\infty)$ having infinite logarithmic measure such that for all $|z| = r \in E_3$, one has

$$|f(z)| > M(r, f)^{(1 - \epsilon)} \exp_p \{\beta r^\sigma\},$$  \hspace{1cm} (22)

where $E_3 = E_2 \setminus E_1$ is a set having infinite logarithmic measure.

Lemma 18 (see [13]). Let $f(z)$ be a transcendental entire function. Then, there is a set $E_1 \subset (0, +\infty)$ having finite logarithmic measure such that for all $|z| = r \notin E_1$ and $|f(z)| = M(r, f)$, one has

$$\left| \frac{f(z)}{f^{(j)}(z)} \right| \leq 2 r^j, \ (j \in \mathbb{N}).$$  \hspace{1cm} (23)

Lemma 19 (see [7, 9, 10]). Let $A_0(z), \ldots, A_{k-1}(z)$, $F(z) \neq 0$ be meromorphic functions, and let $f(z)$ be a meromorphic solution of (13) satisfying one of the following conditions:

(i) max{|$i(A_j)(j = 0, \ldots, k - 1)$| < $i(f)$} = $p + 1$ ($p, q \in \mathbb{N}$);

(ii) $b = \max\{|\sigma_{p+1}(F), \sigma_{p+1}(A_j)(j = 0, \ldots, k - 1)| < \sigma_{p+1}(f)$,

then $\lambda_{p+1}(f) = \sigma_{p+1}(f)$.

Lemma 20 (see [8]). Let $A_0(z), \ldots, A_{k-1}(z)$, $F(z)$ be entire functions of finite iterated order, if $i(A_j) \leq p, i(F) \leq p$ ($j = 0, \ldots, k - 1$). Then $\sigma_{p+1}(f) \leq \max\{|\sigma_{p+1}(A_j), \sigma_{p+1}(F), j = 0, \ldots, k - 1\}$ holds for all solutions of (13).

Lemma 21 (see [2]). Let $g : (0, +\infty) \to R, h : (0, +\infty) \to R$ be monotonically increasing functions such that

(i) $g(r) \leq h(r)$ outside of an exceptional set of finite linear measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.

(ii) $g(r) \leq h(r)$ outside of an exceptional set of finite logarithmic measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(r^\alpha)$ for all $r > r_0$.

Lemma 22 (see [19]). Let $f(z)$ be an entire function of finite iterated order satisfying $\mu_p(f) = \mu < \infty$. Then, for any given $\epsilon > 0$, there exists a set $E_3 \subset (0, +\infty)$ having finite logarithmic measure such that for all $|z| \in E_3$ one has

$$M(r, f) < \exp_p \{r^{\mu + \epsilon}\}.$$  \hspace{1cm} (24)

Lemma 23 (see [2, 20]). Let $f(z)$ be a transcendental entire function, let $0 < \eta_1 < 1/4$ and $z, 0$ a point such that $|z| = r$ and that $|f(z)| = M(r, f)\gamma(r)^{-1} h_1 \eta_1$ holds. Then, there exists a set $E_1 \subset (0, +\infty)$ of finite logarithmic measure such that

$$\frac{|f^{(j)}(z)|}{f(z)} = \frac{|f_j(r)|}{z^r} \left(1 + o(1)\right)$$  \hspace{1cm} (25)

holds for all $j \in \mathbb{N}$ and all $r \notin E_1$, where $\gamma_j(r)$ is the central index of $f(z)$.

Lemma 24 (see [7, 9]). Let $f(z)$ be an entire function of finite iterated order satisfying $\sigma_p(f) = \sigma, \mu_p(f) = \mu, p, q \in \mathbb{N}$. Then, one has

$$\lim_{r \to \infty} \frac{\log \gamma_j(r)}{\log r} = \sigma,$$

$$\lim_{r \to \infty} \frac{\log \gamma_j(r)}{\log r} = \mu.$$  \hspace{1cm} (26)

Lemma 25. Let $A_0(z), \ldots, A_{k-1}(z), F(z)$ be entire functions of finite iterated order satisfying max$|\sigma_p(A_j), j \neq d, \sigma_d(F)| \leq \mu_p(A_d) < \infty$. Then, every solution $f(z)$ of (13) satisfies $\mu_{p+1}(f) \leq \mu_p(A_d)$.

Proof. By (13), we have

$$\left| \frac{f^{(k)}(z)}{f(z)} \right| \leq \left| A_{k-1} \frac{f^{(k-1)}(z)}{f(z)} \right| + \cdots + \left| A_1 \frac{f'}{f} \right| + |A_0| + \left| \frac{F(z)}{f(z)} \right|.$$  \hspace{1cm} (27)

By Lemma 23, there exists a set $E_1$ having finite logarithmic measure such that for all $|z| = r \notin E_1$ and $|f(z)| = M(r, f)$, we have

$$\frac{|f^{(j)}(z)|}{f(z)} = \left(\frac{\gamma_j(r)}{z^r}\right) \left(1 + o(1)\right), \ (j \in \mathbb{N}).$$  \hspace{1cm} (28)
By Lemma 22, for any given $\epsilon > 0$, there exists a set $E_4$ having infinite logarithmic measure such that for all $|z| = r \in E_4$ and $|f(z)| = M(r, f) > 1$, we have

$$\left| A_{d} (z) \right| \leq \exp_{p} \left\{ \rho^{p}_{\mu_{p} (A_{d})} (z) \right\},$$

$$\left| A_{j} (z) \right| \leq \exp_{p} \left\{ \rho^{p}_{\mu_{p} (A_{j})} (z) \right\},$$

$$\left| F (z) \right| \leq |F (z)| < \exp_{p} \left\{ \rho^{p}_{\mu_{p} (A_{j})} (z) \right\}. \quad (29)$$

Hence from (27)–(29), for any given $\epsilon > 0$ and for all $z$ satisfying $|z| = r \in E_4 \setminus E_1$ and $|f(z)| = M(r, f)$, we have

$$\left( \frac{\nu_{j} (r)}{r} \right)^{k} \left( 1 + o (1) \right) \leq (k + 1) \left( \frac{\nu_{j} (r)}{r} \right)^{k-1} \left( 1 + o (1) \right) \exp_{p} \left\{ \rho^{p}_{\mu_{p} (A_{j})} (z) \right\}. \quad (30)$$

By (30) and Lemma 24, we have $\mu_{p+1} (f) \leq \mu_{p} (A_{d})$. Therefore, we complete the proof of Lemma 25. \hfill \Box

**Lemma 26.** Let $A_{0}, A_{1}, \ldots, A_{k-1}$, $F \neq 0$ be meromorphic functions of finite iterated order; if $f$ is a meromorphic solution of the (13) and satisfies $b = \max \{ \sigma_{p+1} (F), \sigma_{p+1} (A_{j}), j = 0, \ldots, k-1 \} < \mu_{p+1} (f)$, then $\lambda_{p+1} (f) = \lambda_{p+1} (f) = \mu_{p+1} (f)$. \hfill \Box

**Proof.** By (13), we have

$$\frac{1}{f} = \frac{1}{F} \left( \frac{f^{(k)}}{f} + A_{k-1} \frac{f^{(k-1)}}{f} + \cdots + A_{0} \right). \quad (31)$$

By (31), we get

$$N \left( r, \frac{1}{f} \right) \leq kN \left( r, \frac{1}{f} \right) + N \left( r, \frac{1}{F} \right) + \sum_{j=0}^{k-1} N \left( r, A_{j} \right). \quad (32)$$

By the lemma of logarithmic derivative and (31), we have

$$m \left( r, \frac{1}{f} \right) \leq m \left( r, \frac{1}{F} \right) + \sum_{j=0}^{k-1} m \left( r, A_{j} \right) + o \left\{ \log \left( rT \left( r, f \right) \right) \right\} \quad (r \notin E_1). \quad (33)$$

By (32) and (33), we have

$$T \left( r, f \right) = T \left( r, \frac{1}{f} \right) + o \left( 1 \right) \leq kN \left( r, \frac{1}{f} \right) + T \left( r, \frac{1}{F} \right) + \sum_{j=0}^{k-1} T \left( r, A_{j} \right) + o \left\{ \log \left( rT \left( r, f \right) \right) \right\} \quad (34)$$

$$= kN \left( r, \frac{1}{f} \right) + T \left( r, F \right) + \sum_{j=0}^{k-1} T \left( r, A_{j} \right) + o \left\{ \log \left( rT \left( r, f \right) \right) \right\} \quad (r \notin E_1). \quad (35)$$

Since $\max \{ \sigma_{p+1} (F), \sigma_{p+1} (A_{j}), j = 0, \ldots, k-1 \} < \mu_{p+1} (f)$, for sufficiently large $r$, we have

$$T \left( r, F \right) = o \left\{ T \left( r, f \right) \right\}, \quad (36)$$

$$T \left( r, A_{j} \right) = o \left\{ T \left( r, f \right) \right\}, \quad j = 0, \ldots, k-1. \quad (37)$$

$$\log \left( rT \left( r, f \right) \right) = o \left\{ T \left( r, f \right) \right\}. \quad (38)$$

By (34)–(35), we have

$$(1 - o (1)) T \left( r, f \right) \leq kN \left( r, \frac{1}{f} \right), \quad (r \notin E_1). \quad (39)$$

By Lemma 21 (i) and (36), we have $\lambda_{p+1} (f) = \lambda_{p+1} (f) = \mu_{p+1} (f)$. \hfill \Box

**Lemma 27.** Let $f(z)$ be a transcendental entire function, for each sufficiently large $|z| = r$, and let $z_{r} = re^{i \theta}$ be a point satisfying $|f(z_{r})| = M(r, f)$. Then, there exists a constant $\delta_{r} (\geq 0)$ such that for all $z$ satisfying $|z| = r \notin E_1$ and $\arg z = \theta \in [\theta_{r} - \delta_{r}, \theta_{r} + \delta_{r}]$, one has

$$f^{(j)} (z_{r}) = \left( \frac{\nu_{j} (r)}{z_{r}} \right)^{j} \left( 1 + o (1) \right) \quad (j \in \mathbb{N}). \quad (40)$$

**Proof.** If $z_{r} = re^{i \theta}$ is a point satisfying $|f(z_{r})| = M(r, f)$, since $|f(z)|$ is continuous in $|z| = r$, then there exists a constant $\delta_{r} (\geq 0)$ such that for all $z$ satisfying $|z| = r$ (large enough) and $\arg z = \theta \in [\theta_{r} - \delta_{r}, \theta_{r} + \delta_{r}]$, we have

$$|f(z) - f(z_{r})| < \epsilon;$$

that is,$|f(z)| = \left| f\left( z_{r} \right) \right| > \frac{1}{2} \left| f\left( z_{r} \right) \right|$$

$$= \frac{1}{2} M(r, f) > M(r, f) \nu_{j} (r)^{-1/4 + \eta_{j}}.$$ 

By Lemma 23, we have

$$f^{(j)} (z_{r}) = \left( \frac{\nu_{j} (r)}{z_{r}} \right)^{j} \left( 1 + o (1) \right), \quad (j \in \mathbb{N}). \quad (41)$$

holds for all $z$ satisfying $|z| = r \notin E_1$ and $\arg z = \theta \in [\theta_{r} - \delta_{r}, \theta_{r} + \delta_{r}]$. \hfill \Box

**Lemma 28.** Let $f(z)$ be a transcendental entire function, for each sufficiently large $|z| = r$, and let $z_{r} = re^{i \theta}$ be a point satisfying $|f(z_{r})| = M(r, f)$. Then, there exists a constant $\delta_{r} (\geq 0)$ such that for all $z$ satisfying $|z| = r \notin E_1$ and $\arg z = \theta \in [\theta_{r} - \delta_{r}, \theta_{r} + \delta_{r}]$, one has

$$\left| f (z) \right| \leq 2r^{j} \quad (j \in \mathbb{N}). \quad (42)$$

**Proof.** If $z_{r} = re^{i \theta}$ is a point satisfying $|f(z_{r})| = M(r, f)$, then by Lemma 27 there exists a constant $\delta_{r} (\geq 0)$ such that for all
Let \( f(z) \) be an entire function of order \( 0 < \sigma_p(f) = \sigma < \infty \). Then for any given \( \varepsilon > 0 \), there exists a set \( E_\varepsilon \subset (0, +\infty) \) with positive upper logarithmic density such that for all \( |z| = r \in E_\varepsilon \), one has

\[
M(r, f) > \exp_p \{ r^{\sigma-\varepsilon} \}.  
\]

Proof. Since \( m(r, f) \sim \log M(r, f) \) as \( r \to \infty \) outside a set \( r \) of finite logarithmic measure and for all \( r \in E_\varepsilon \) with positive upper logarithmic density and a set \( H_2 \subset [0, 2\pi) \) having linear measure zero such that for all \( z \) satisfying \( \arg z = \theta \in [0, 2\pi) \) \( \setminus H_2 \), we have

\[
|f(re^{i\theta})| > M(r, f)^{1-\varepsilon} \quad (r \notin E_\varepsilon). 
\]

By Lemma 29, for any given \( \varepsilon > 0 \), there exists a set \( E_\varepsilon \subset (0, +\infty) \) with positive upper logarithmic density, we have

\[
M(r, f) > \exp_p \{ r^{\sigma-\varepsilon} \}. 
\]
Hence from (51)–(54), for all $z$ satisfying $|z| = r \in E_3 \setminus E_1$ and $|f(z)| = M(r, f)$, we have
\[
\exp \left\{ \beta_1 r^{\sigma_r(A_d)} \right\} 
\leq 2B(k + 1)r^d \exp \left\{ \alpha_1 r^{\sigma_r(A_d)} \right\} [T(2r, f)]^{2k}.
\] (55)

By (55), we have
\[
\sigma_{p+1}(f) = \lim_{r \to \infty} \frac{\log_{p+1} T(r, f)}{\log r} \geq \sigma_p(A_d). \tag{56}
\]

On the other hand, by Lemma 20, we have $\sigma_{p+1}(f) \leq \sigma_p(A_d)$. Therefore, every transcendental solution $f(z)$ of (13) satisfies $\sigma_{p+1}(f) = \sigma_p(A_d)$. Furthermore if $F(z) \not\equiv 0$, then by Lemma 19, we have that every transcendental solution $f(z)$ of (13) satisfies $\sigma_{p+1}(f) = \sigma(A_d)$. (ii) Assume that $f$ is a solution of (13). By the elementary theory of differential equations, all the solutions of (13) are entire functions and have the form
\[
f = f^* + C_1 f_1 + C_2 f_2 + \cdots + C_k f_k,
\] (57)
where $C_1, \ldots, C_k$ are complex constants, $\{f_1, \ldots, f_k\}$ is a solution base of (12), and $f^*$ is a solution of (13) and has the form
\[
f^* = D_1 f_1 + D_2 f_2 + \cdots + D_k f_k,
\] (58)
where $D_1, \ldots, D_k$ are certain entire functions satisfying
\[
D_j' = F \cdot G_j(f_1, \ldots, f_k) \cdot W(f_1, \ldots, f_k)^{-1} \quad (j = 1, \ldots, k),
\] (59)
where $G_j(f_1, \ldots, f_k)$ are differential polynomials in $f_1, \ldots, f_k$ and their derivative with constant coefficients, and $W(f_1, \ldots, f_k)$ is the Wronskian of $f_1, \ldots, f_k$. By Theorem A, we have $\sigma_{p+1}(f_j) \leq \sigma_p(A_d) (j = 1, 2, \ldots, k)$; then by (57)–(59), we get
\[
\sigma_{p+1}(f) \leq \max \{\sigma_{p+1}(f_i), \sigma_{p+1}(F), j = 1, \ldots, k\} \leq \sigma_p(A_d).
\] (60)

Since $\sigma_p(F) > \sigma_p(A_d)$, it is easy to see that $\sigma_p(f) \geq \sigma_p(F)$ by (13).

(iii) Suppose that $f$ is a solution of (13), it is easy to see that $\sigma_{p+1}(f) \geq \sigma_{p+1}(F)$ by (13). On the other hand, since $\sigma_{p+1}(F) > \sigma_{p+1}(A_d)$ and by (57)–(59), we have
\[
\sigma_{p+1}(f) \leq \max \{\sigma_{p+1}(f_i), \sigma_{p+1}(F), j = 1, \ldots, k\} \leq \sigma_{p+1}(F).
\] (61)

Therefore, all solutions of (13) satisfy $\sigma_{p+1}(f) = \sigma_{p+1}(F)$.

By the same proof in Theorem 4.2 in [8, page 401], we can obtain that all solutions of (13) satisfying $\lambda_{p+1}(f) = \lambda_{p+1}(F)$ with at most one exceptional solution $f_0$ satisfying $\lambda_{p+1}(f_0) < \sigma_{p+1}(F)$.

Proof of Theorem 10. Suppose that $f(z)$ is a transcendental solution of (13), by the same proof in Theorem 8, we have $\sigma_{p+1}(f) = \sigma_p(A_d) = \sigma$. Thus, it remains to show that $\mu_{p+1}(f) = \mu_p(A_d) = \sigma$. We choose $\alpha_2, \beta_2$ to satisfy
\[
\max \{\sigma_p(A_j), j \neq d, \sigma_p(F)\} < \alpha_2 < \beta_2 < \sigma. \tag{62}
\]

Since the sequence of exponents $\{\lambda_j\}$ of $A_d$ satisfies (15) and $\mu_p(A_d) = \sigma$, then by Lemma 15, there exists a set $E_1$ having finite logarithmic measure such that for all sufficiently large $r \not\in E_1$, we have
\[
|A_d(z)| \geq \exp \{r^{\beta_1}\}, \tag{63}
\]
\[
|A_j(z)| \leq \exp \{r^{\alpha_j}\}, \quad j \neq d.
\]

Hence from (51), (52), (54), and (63), for all $z$ satisfying $|z| = r \not\in E_1$ and $|f(z)| = M(r, f)$, we have
\[
\exp \{r^{\beta_1}\} \leq 2B(k + 1)r^d \exp \{r^{\alpha_j}\} [T(2r, f)]^{2k}. \tag{64}
\]

Since $\beta_2$ is arbitrarily close to $\sigma$, by (64) and Lemma 21 (ii), we have
\[
\mu_{p+1}(f) = \lim_{r \to \infty} \frac{\log_{p+1} T(r, f)}{\log r} \geq \sigma. \tag{65}
\]

On the other hand, by Lemma 26, we have $\mu_{p+1}(f) \leq \mu_p(A_d) = \sigma$; therefore, every transcendental solution of (13) satisfies $\mu_{p+1}(f) = \sigma$.

Proof of Theorem 11. (i) By Lemma 20, we know that every solution of (13) satisfies $\sigma_{p+1}(f) \leq \sigma_p(A_d)$. In the following, we show that every transcendental solution $f(z)$ of (13) satisfies $\sigma_{p+1}(f) \geq \sigma_p(A_d)$. Suppose that $f(z)$ is a transcendental solution of (13). For each sufficiently large circle $|z| = r$, we take a point $z_r = r e^{i \theta}$, satisfying $|f(z_r)| = M(r, F)$. By Lemma 28, there exist a constant $\delta_r > 0$ and a set $E_1$ such that for all $z$ satisfying $|z| = r \not\in E_1$ and arg $z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r]$, we have
\[
\left| \frac{f(z)}{f^{(j)}(z)} \right| \leq 2r^d. \tag{66}
\]

By Lemma 13 (ii), there exist a set $H_1 \subset [0, 2\pi)$ having linear measure zero and a constant $B > 0$ such that for sufficiently large $r$ and for all $z$ satisfying arg $z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H_1$, we have
\[
\left| \frac{f^{(j)}(z)}{f^{(i)}(z)} \right| \leq B[T(2r, f)]^{2k} \quad (0 \leq i < j \leq k). \tag{67}
\]

Setting $\max \{\sigma_p(A_j), \sigma_p(F), j \neq d\} = b \leq \sigma_p(A_d)$, for all $z$ satisfying $|z| = r \not\in E_1$, we have
\[
\left| \frac{F(z)}{f(z)} \right| \leq |F(z)| \leq \exp \{r^{\alpha_j}\}. \tag{68}
\]
Since $T(r, A_d) \sim \log M(r, A_d)$ as $r \to \infty (r \notin E_1)$, by Lemma 30, for any given $\varepsilon > 0$, there exists a set $E_\varepsilon \subset (0, \infty)$ with $\log \text{dents} E_\varepsilon > 0$ and a set $H_2 \subset [0, 2\pi)$ with linear measure zero such that for all $z$ satisfying $|z| = r \in E_\varepsilon$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus H_2$, we have

$$|A_d(z)| > \exp_p \left\{ \rho^{(A_d)} - \varepsilon \right\}. \quad (69)$$

Substituting (66)–(69) into (51), for all $z$ satisfying $|z| = r \in E_\varepsilon \setminus E_1$ and $\arg z = \theta \in [\theta_r - \delta_r, \theta_r + \delta_r] \setminus (H_1 \cup H_2)$, we have

$$\exp_p \left\{ \rho^{(A_d)} - \varepsilon \right\} \leq (k + 1) B[T(2r, f)]^{2k} \cdot 2^d \cdot \exp_p \left\{ \rho^{(A_d)} - \varepsilon \right\} \cdot \rho^{(A_d)}(f). \quad (70)$$

From (70), we have $\sigma_{p+1}(f) \geq \sigma_p(A_d)$. Therefore, every transcendental solution $f(z)$ of (13) satisfies $\sigma_{p+1}(f) = \sigma_p(A_d)$. Furthermore, if $F(z) \neq 0$, then every transcendental solution $f(z)$ of (13) satisfies $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f) = \sigma_p(A_d)$. 

(ii)–(iv) By the same proof in Theorems 8 and 10, we can obtain the conclusions (ii)–(iv).

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