Research Article


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By using a specific way of choosing the indexes, we introduce an up-to-date iterative algorithm for approximating common fixed points of a countable family of generalized quasi-$\phi$-asymptotically nonexpansive mappings and obtain a strong convergence theorem under some suitable conditions. As application, an iterative solution to a system of generalized mixed equilibrium problems is studied. The results extend those of other authors, in which the involved mappings consist of just finite families.

1. Introduction

Throughout this paper, we assume that $E$ is a real Banach space with its dual $E^*$, $C$ is a nonempty closed convex subset of $E$, and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$Jx = \{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \}, \quad \forall x \in E. \quad (1)$$

In the sequel, we use $F(T)$ to denote the set of fixed points of a mapping $T$.

Definition 1. (1) [1] A mapping $T : C \to C$ is said to be generalized quasi-$\phi$-asymptotically nonexpansive in the light of [1], if $F(T) \neq \emptyset$, and there exist nonnegative real sequences $\{v_n\}$ and $\{\mu_n\}$ with $v_n, \mu_n \to 0$ (as $n \to \infty$) such that

$$\phi(p, T^n x) \leq (1 + v_n) \phi(p, x) + \mu_n, \quad \forall n \geq 1, \quad x \in C, \quad p \in F(T), \quad (2)$$

where $\phi : E \times E \to \mathbb{R}$ denotes the Lyapunov functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \quad (3)$$

It is obvious from the definition of $\phi$ that

$$\left(\|x\| - \|y\| \right)^2 \leq \phi(x, y) \leq \left(\|x\| + \|y\| \right)^2. \quad (4)$$

(2) A mapping $T : C \to C$ is said to be uniformly $L$-Lipschitz continuous, if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L \|x - y\|, \quad \forall n \geq 1, \quad x, y \in C. \quad (5)$$

Example 2. Let $C$ be a unit ball in a real Hilbert space $l^2$, and let $T : C \to C$ be a mapping defined by

$$T(x_1, x_2, \ldots) = (0, x_1^2, a_2 x_2, a_3 x_3, \ldots), \quad (6)$$

where $\{a_i\}$ is a sequence in $(0, 1)$ satisfying $\prod_{i=2}^{\infty} a_i = 1/2$. It is shown by Goebel and Kirk [2] that

$$\phi(p, T^n y) \leq (1 + v_n) \phi(p, y) + \mu_n, \quad \forall n \geq 1, \quad y \in C, \quad p \in F(T), \quad (7)$$

where $\phi(x, y) = \|x - y\|^2$, $v_n = (2 \prod_{i=2}^{n} a_i)^2 - 1$, for all $n \geq 1$, and $\{\mu_n\}$ is a nonnegative real sequence with $\mu_n \to 0$ as $n \to \infty$. This shows that the mapping $T$ defined earlier is a generalized quasi-$\phi$-asymptotically nonexpansive mapping.
Let $\theta : C \times C \rightarrow \mathbb{R}$ be a bifunction, $\psi : C \rightarrow \mathbb{R}$ a real valued function, and $B : C \rightarrow E^*$ a nonlinear mapping. The so-called generalized mixed equilibrium problem (GMEP) is to find a $u \in C$ such that
\[ \theta(u, y) + \langle y - u, Bu \rangle + \psi(y) - \psi(u) \geq 0, \quad \forall y \in C, \] (8)
whose set of solutions is denoted by $\Omega(\theta, B, \psi)$.

The equilibrium problem is a unifying model for several problems arising in physics, engineering, science optimization, economics, transportation, network and structural analysis, Nash equilibrium problems in noncooperative games, and others. It has been shown that variational inequalities and mathematical programming problems can be viewed as a special realization of the abstract equilibrium problems. Many authors have proposed some useful methods to solve the equilibrium problem $\text{EP}$, generalized equilibrium problem $\text{GEP}$, mixed equilibrium problem $\text{MEP}$, and GMEP. Concerning the weak and strong convergence of iterative sequences to approximate a common element of the set of solutions for the GMEP, the set of solutions to variational inequality problems, the set of solutions to quasi-$\phi$-asymptotically nonexpansive mappings, and quasi-$\phi$-asymptotically nonexpansive mappings have been studied by many authors in the setting of Hilbert or Banach spaces (e.g., see [3–16] and the references therein).

Inspired and motivated by the study mentioned earlier, in this paper, by using a specific way of choosing the indexes, we propose an up-to-date iteration scheme for approximating common fixed points of a countable family of generalized quasi-$\phi$-asymptotically nonexpansive mappings and obtain a strong convergence theorem for solving a system of generalized mixed equilibrium problems. The results extend those of the authors, in which the involved mappings consist of just finite families.

2. Preliminaries

A Banach space $E$ is strictly convex if the following implication holds for $x, y \in E$:
\[ \|x\| = \|y\| = 1, \quad x \neq y \implies \| \frac{x + y}{2} \| < 1. \] (9)

It is also said to be uniformly convex if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that
\[ \|x\| = \|y\| = 1, \quad \|x - y\| \geq \varepsilon \implies \| \frac{x + y}{2} \| \leq 1 - \delta. \] (10)

It is known that if $E$ is uniformly convex Banach space, then $E$ is reflexive and strictly convex. A Banach space $E$ is said to be smooth if
\[ \lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \] (11)
exists for each $x, y \in S(E) := \{x \in E : \|x\| = 1\}$. In this case, the norm of $E$ is said to be Gâteaux differentiable. The space $E$ is said to have uniformly Gâteaux differentiable norm if for each $y \in S(E)$, the limit (11) is attained uniformly for $x \in S(E)$. The norm of $E$ is said to be Fréchet differentiable if for each $x \in S(E)$, the limit (11) is attained uniformly for $y \in S(E)$. The norm of $E$ is said to be uniformly Fréchet differentiable (and $E$ is said to be uniformly smooth) if the limit (11) is attained uniformly for $x, y \in S(E)$. Note that $E$ ($E^*$, resp.) is uniformly convex $\iff$ $E$ ($E^*$, resp.) is uniformly smooth.

Following Alber [17], the generalized projection $\Pi_C : E \rightarrow C$ is defined by
\[ \Pi_C(x) = \inf_{y \in C} \phi(y, x), \quad \forall x \in E. \] (12)

**Lemma 3** (see [17]). Let $E$ be a smooth, strictly convex, and reflexive Banach space, and let $C$ be a nonempty closed convex subset of $E$. Then, the following conclusions hold:

(i) If $E$ is uniformly smooth, then $J$ is uniformly continuous on each bounded subset of $E$.

(ii) If $E$ is reflexive and strictly convex, then $J^{-1}$ is norm-weak continuous.

(iii) If $E$ is a smooth, strictly convex, and reflexive Banach space, then the normalized duality mapping $J : E \rightarrow 2E^*$ is single valued, one-to-one, and onto.

(iv) A Banach space $E$ is uniformly smooth if and only if $E^*$ is uniformly convex.

(v) Each uniformly convex Banach space $E$ has the Kadec-Klee property; that is, for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $\|x_n\| \rightarrow \|x\|$, then $x_n \rightarrow x$, where $x_n \rightarrow x$ denotes that $\{x_n\}$ converges weakly to $x$.

**Lemma 5** (see [19]). Let $E$ be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, and let $C$ be a nonempty closed convex subset of $E$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in $C$ such that $x_n \rightarrow p$ and $\phi(x_n, y_n) \rightarrow 0$, where $\phi$ is the function defined by (3); then, $y_n \rightarrow p$.

**Lemma 6** (see [1]). Let $E$ and $C$ be the same as those in Lemma 5. Let $T : C \rightarrow C$ be a closed and generalized quasi-$\phi$-asymptotically nonexpansive mapping with nonnegative real sequences $\{\mu_n\}$ and $\{\mu_n\}$; then, the fixed point set $F(T)$ of $T$ is a closed and convex subset of $C$.

**Lemma 7** (see [20]). Let $E$ be a real uniformly convex Banach space, and let $B_r(0)$ be the closed ball of $E$ with center at the origin and radius $r > 0$. Then, there exists a continuous strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that
\[ \alpha \|x\|^2 + \beta \|y\|^2 \leq \alpha \|x + y\|^2 + \beta \|y - x\|^2 \] (13)
for all $x, y \in B_r(0)$, and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$. 

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3. Main Results

Theorem 8. Let $E$ be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, $C$ a nonempty closed convex subset of $E$, and $T_i : C \to C$, $i = 1, 2, \ldots$ a countable family of closed and generalized quasi-$\phi$-asymptotically nonexpansive mappings with nonnegative real sequences $\{\alpha_n\}^\infty_n_{=1}$ and $\{\mu_n\}$ satisfying $\alpha_n \to 0$ and $\mu_n \to 0$ (as $n \to \infty$ and for each $i \geq 1$), and each $T_i$ uniformly $L_i$-Lipschitz continuous. Let $\{x_n\}$ be a sequence in $[0, \epsilon]$ for some $\epsilon \in (0, 1)$, and let $\{\beta_n\}$ be a sequence in $[0, 1]$ satisfying $0 < \liminf_{n \to \infty} \beta_n (1 - \beta_n)$. Let $\{x_n\}$ be the sequence generated by

$$
x_1 \in C; \quad C_1 = C,
$$

$$
y_n = J^{-1} [\alpha_n Jx_n + (1 - \alpha_n) Jz_n],
$$

$$
z_n = J^{-1} [\beta_n Jx_n + (1 - \beta_n) JT_{\beta_n}^m x_n],
$$

$$
C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \phi(v, x_n) + \xi_n\},
$$

$$
x_{n+1} = \Pi_{C_{n+1}} x_n, \quad \forall n \geq 1,
$$

where $\xi_n := \nu_{x_n}^{(i)} \sup_{p \in F} \phi(p, x_n) + \nu_{m_n}^{(i)}$, $\Pi_{C_m}$ is the generalized projection of $E$ onto $C_{n+1}$, and $i_n$ and $m_n$ satisfy the positive integer equation: $n = i + (m - 1)/2$, $m \geq i$ ($m \geq i$, $n = 1, 2, \ldots$), that is, for each $n \geq 1$, there exist unique $i_n$ and $m_n$ such that

$$
i_1 = 1, \quad i_2 = 1, \quad i_3 = 2, \quad i_4 = 1, \quad i_5 = 2, \quad i_6 = 3, \quad i_7 = 1, \quad i_8 = 2, \ldots;
$$

$$
m_1 = 1, \quad m_2 = 2, \quad m_3 = 2, \quad m_4 = 3, \quad m_5 = 3, \quad m_6 = 3, \quad m_7 = 4, \quad m_8 = 4, \ldots.
$$

If $F := \bigcap_{i=1}^\infty F(T_i)$ is nonempty and bounded, then $\{x_n\}$ converges strongly to $\Pi_F x_1$.

Proof. We divide the proof into several steps.

(I) $F$ and $C_n$ (for all $n \geq 1$) both are closed and convex subsets in $C$.

In fact, it follows from Lemma 6 that each $F(T_i)$ is a closed and convex subset of $C$, so is $F$. In addition, with $C_1 (=C)$ being closed and convex, we may assume that $C_n$ is closed and convex for some $n \geq 2$. In view of the definition of $\phi$, we have that

$$
C_{n+1} = \{v \in C : \phi(v) \leq \alpha\} \cap C_n,
$$

where $\phi(v) = 2\langle v, Jx_n - Jy_n \rangle$, and $\alpha = \|x_n\|^2 - \|y_n\|^2 + \xi_n$. This shows that $C_{n+1}$ is closed and convex.

(II) $F$ is a subset of $\bigcap_{n=1}^\infty C_n$.

It is obvious that $F \subset C_1$. Suppose that $F \subset C_n$ for some $n \geq 2$. Since $E$ is uniformly smooth, $E^*$ is uniformly convex. Then, for any $p \in F \subset C_n$, we have that

$$
\phi(p, y_n) = \phi(p, J^{-1} [\alpha_n Jx_n + (1 - \alpha_n) Jz_n])
$$

$$
= \|p\|^2 - 2 \langle p, \alpha_n Jx_n + (1 - \alpha_n) Jz_n \rangle
$$

$$
+ \|\alpha_n Jx_n + (1 - \alpha_n) Jz_n\|^2
$$

$$
\leq \|p\|^2 - 2\alpha_n \langle p, Jx_n \rangle - 2 (1 - \alpha_n) \langle p, Jz_n \rangle
$$

$$
+ \alpha_n \|x_n\|^2 + (1 - \alpha_n) \|z_n\|^2
$$

$$
= \alpha_n \phi(p, x_n) + (1 - \alpha_n) \phi(p, z_n).
$$

(17)

Furthermore, it follows from Lemma 7 that for any $p \in F$, we have that

$$
\phi(p, z_n) = \phi(p, J^{-1} [\beta_n Jx_n + (1 - \beta_n) JT_{\beta_n}^m x_n])
$$

$$
= \|p\|^2 - 2 \langle p, \beta_n Jx_n + (1 - \beta_n) JT_{\beta_n}^m x_n \rangle
$$

$$
+ \|\beta_n Jx_n + (1 - \beta_n) JT_{\beta_n}^m x_n\|^2
$$

$$
\leq \|p\|^2 - 2\beta_n \langle p, Jx_n \rangle - 2 (1 - \beta_n)
$$

$$
\times \langle p, JT_{\beta_n}^m x_n \rangle + \beta_n \|x_n\|^2 + (1 - \beta_n) \|T_{\beta_n}^m x_n\|^2
$$

$$
- \beta_n (1 - \beta_n) g \left( \|Jx_n - JT_{\beta_n}^m x_n\| \right)
$$

$$
= \beta_n \phi(p, x_n) + (1 - \beta_n) \phi(p, T_{\beta_n}^m x_n)
$$

$$
- \beta_n (1 - \beta_n) g \left( \|Jx_n - JT_{\beta_n}^m x_n\| \right)
$$

$$
\leq \beta_n \phi(p, x_n) + (1 - \beta_n)
$$

$$
\times \left[ \phi(p, x_n) + \nu_{m_n}^{(i)} \sup_{p \in F} \phi(p, x_n) + \nu_{m_n}^{(i)} \right]
$$

$$
- \beta_n (1 - \beta_n) g \left( \|Jx_n - JT_{\beta_n}^m x_n\| \right)
$$

$$
\leq \phi(p, x_n) + \nu_{m_n}^{(i)} \sup_{p \in F} \phi(p, x_n) + \nu_{m_n}^{(i)}
$$

$$
- \beta_n (1 - \beta_n) g \left( \|Jx_n - JT_{\beta_n}^m x_n\| \right)
$$

$$
= \phi(p, x_n) + \xi_n - \beta_n (1 - \beta_n) g \left( \|Jx_n - JT_{\beta_n}^m x_n\| \right).
$$

(18)

Substituting (18) into (17) and simplifying it, we have that

$$
\phi(p, y_n) \leq \phi(p, x_n) + (1 - \alpha_n) \xi_n \leq \phi(p, x_n) + \xi_n.
$$

(19)

This implies that $p \in C_{n+1}$, and so $F \subset C_{n+1}$.

(III) $x_n \to x^* \in C$ as $n \to \infty$.

In fact, since $x_n = \Pi_{C_n} x_1$, from Lemma 3 (2), we have that $(x_n - y, Jx_1 - Jx_n) \geq 0$, for all $y \in C_n$. Again, since
\[ F \subset \bigcap_{m=1}^{\infty} C_n, \text{ we have that } (x_n - p, Jx_k - Jx_n) \geq 0, \text{ for all } p \in F. \text{ It follows from Lemma 3 (1) that for each } p \in F \text{ and for each } n \geq 1, \]
\[ \phi(x_n, x_1) = \phi(\Pi_{C_n} x_1, x_1) \leq \phi(p, x_1) - \phi(p, x_n) + \phi(p, x_1), \]
which implies that \( \{\phi(x_n, x_1)\} \) is bounded, so is \( \{x_n\} \). Since for all \( n \geq 1, x_n = \Pi_{C_n} x_1 \) and \( x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subseteq C_n \), we have \( \phi(x_n, x_1) \leq \phi(x_{n+1}, x_1) \). This implies that \( \{\phi(x_n, x_1)\} \) is nondecreasing; hence, the limit
\[ \lim_{n \to \infty} \phi(x_n, x_1) \text{ exists.} \]
(21)

Since \( E \) is reflexive, there exists a subsequence \( \{x_{n_i}\} \) of \( \{x_n\} \) such that \( x_{n_i} \rightharpoonup x^* \in C \) as \( i \to \infty \). Since \( C_n \) is closed and convex and \( C_{n+1} \subseteq C_n \), this implies that \( C_n \) is weakly closed and \( x^* \in C_n \) for each \( n \geq 1 \). In view of \( x_n = \Pi_{C_n} x_1 \), we have that
\[ \phi(x_{n_i}, x_1) \leq \phi(x^*, x_1), \quad \forall i \geq 1. \]
(22)

Since the norm \( \| \cdot \| \) is weakly lower semicontinuous, we have that
\[ \liminf_{i \to \infty} \phi(x_{n_i}, x_1) = \liminf_{i \to \infty} \left( \|x_{n_i}\|^2 - 2 \langle x_{n_i}, Jx_1 \rangle + \|x_1\|^2 \right) \]
\[ \geq \|x^*\|^2 - 2 \langle x^*, Jx_1 \rangle + \|x_1\|^2 \]
\[ = \phi(x^*, x_1), \]
(23)

and so
\[ \phi(x^*, x_1) \leq \liminf_{i \to \infty} \phi(x_{n_i}, x_1) \leq \limsup_{i \to \infty} \phi(x_{n_i}, x_1) \leq \phi(x^*, x_1). \]
(24)

This implies that \( \lim_{i \to \infty} \phi(x_{n_i}, x_1) = \phi(x^*, x_1) \), and so \( \|x_{n_i}\| \to \|x^*\| \) as \( i \to \infty \). Since \( x_{n_i} \rightharpoonup x^* \), by virtue of Kadec-Klee property of \( E \), we obtain that
\[ \lim_{i \to \infty} x_{n_i} = x^*. \]
(25)

Since \( \{\phi(x_{n_i}, x_1)\} \) is convergent, this, together with \( \lim_{i \to \infty} \phi(x_{n_i}, x_1) = \phi(x^*, x_1) \), shows that \( \lim_{n \to \infty} \phi(x_n, x_1) = \phi(x^*, x_1) \). If there exists some subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) such that \( x_{n_j} \to y \) as \( j \to \infty \), then, from Lemma 3 (1), we have that
\[ \phi(x^*, y) = \lim_{i,j \to \infty} \phi(x_{n_j}, x_{n_j}) = \lim_{i,j \to \infty} \phi(x_{n_j}, \Pi_{C_{n_j}} x_1) \]
\[ \leq \lim_{i,j \to \infty} \left( \phi(x_{n_j}, x_1) - \phi(\Pi_{C_{n_j}} x_1, x_1) \right) \]
\[ = \lim_{i,j \to \infty} \left( \phi(x_{n_j}, x_1) - \phi(x_{n_j}, x_1) \right) \]
\[ = \phi(x^*, x_1) - \phi(x^*, x_1) = 0; \]
(26)

that is, \( x^* = y \), and so
\[ \lim_{n \to \infty} x_n = x^*. \]
(27)

(IV) \( x^* \) is some member of \( F \).

Set \( \mathcal{K}_i = \{ k \geq 1 : k = i + (m - 1)m/2, m \geq i, m \in \mathbb{N} \} \) for each \( i \geq 1 \). Note that \( K_i = \bigcup_{m=1}^{\infty} K_{m_i} \) whenever \( k \in \mathcal{K}_i \) for each \( i \geq 1 \). For example, by the definition of \( \mathcal{K}_1 \), we have that \( \mathcal{K}_1 = \{1, 2, 4, 7, 11, 16, \ldots\} \), and \( i_1 = i_2 = i_3 = i_4 = i_5 = i_6 = i_7 = \ldots = 1 \). Then, we have that
\[ \xi_k = \sup_{p \in F} \phi(p, x_k) + \mu_{m_i}^{(i)} \quad \forall k \in \mathcal{K}_i. \]
(28)

Note that \( \{m_k\}_{k \in \mathcal{K}_i} = \{i, i + 1, i + 2, \ldots\} \); that is, \( m_k \uparrow \infty \) as \( \mathcal{K}_i \ni k \to \infty \). It follows from (27) and (28) that
\[ \lim_{\mathcal{K}_i \ni k \to \infty} \xi_k = 0. \]
(29)

Since \( x_{n_{k+1}} \in C_{n+1} \), it follows from (14), (27), and (29) that
\[ \phi(x_{k+1}, y_k) \leq \phi(x_{k+1}, x_k) + \xi_k = \phi(p, x_k) + (1 - \alpha_k) \beta_k (1 - \beta_k) g \times (\|Jx_k - JT_{i_k}^m x_k\|) ; \]
(30)

that is,
\[ (1 - \alpha_k) \beta_k (1 - \beta_k) g \times (\|Jx_k - JT_{i_k}^m x_k\|) \]
\[ \leq \phi(p, x_k) + \xi_k - \phi(p, y_k) \to 0 \quad (\mathcal{K}_i \ni k \to \infty). \]
(33)

This, together with assumption conditions imposed on the sequences \( \{x_n\} \) and \( \{\beta_k\} \), shows that \( \lim_{\mathcal{K}_i \ni k \to \infty} g(\|Jx_k - JT_{i_k}^m x_k\|) \) is bounded. So, for each \( i \geq 1 \), from (17) and (18), for any \( p \in F \), we have that
\[ (1 - \alpha_k) \beta_k (1 - \beta_k) g \times (\|Jx_k - JT_{i_k}^m x_k\|) \]
(32)

This, together with assumption conditions imposed on the sequences \( \{x_n\} \) and \( \{\beta_k\} \), shows that \( \lim_{\mathcal{K}_i \ni k \to \infty} g(\|Jx_k - JT_{i_k}^m x_k\|) \) is bounded. So, for each \( i \geq 1 \), from (17) and (18), for any \( p \in F \), we have that
\[ (1 - \alpha_k) \beta_k (1 - \beta_k) g \times (\|Jx_k - JT_{i_k}^m x_k\|) \]
(33)

In addition, \( Jx_k \to Jx^* \) implies that \( \lim_{\mathcal{K}_i \ni k \to \infty} JT_{i_k}^m x_k = Jx^* \). From Remark 4 (ii), it yields that, as \( \mathcal{K}_i \ni k \to \infty \),
\[ T_{i_k}^m x_k \to x^*, \quad \forall i \geq 1. \]
(35)

Again, since for each \( i \geq 1 \), as \( \mathcal{K}_i \ni k \to \infty \),
\[ \|T_{i_k}^m x_k - x^*\| = \|JT_{i_k}^m x_k - Jx^*\| \leq \|Jx_k - JT_{i_k}^m x_k\| \to 0. \]
(36)

This, together with (35) and the Kadec-Klee property of \( E \), shows that
\[ \lim_{\mathcal{K}_i \ni k \to \infty} T_{i_k}^m x_k = x^*, \quad \forall i \geq 1. \]
(37)
On the other hand, by the assumptions that for each \( i \geq 1, T_i \) is uniformly \( L_i \)-Lipschitz continuous, and noting again that \( \{ m_k \}_{k \in \mathbb{K}} = \{ i, i+1, i+2, \ldots \} \), that is, \( m_{k+1} - 1 = m_k \) for all \( k \in \mathbb{K} \), we then have

\[
\| T_i^{m_k} x_k - T_i^{m_k} x_k \| \leq \| T_i^{m_k+1} x_k - T_i^{m_k+1} x_{k+1} \| \\
\quad + \| T_i^{m_k+1} x_{k+1} - x_{k+1} \| \\
\quad + \| x_{k+1} - x_k \| + \| x_k - T_i^{m_k} x_k \| \\
\leq (L_i + 1) \| x_k - x_{k+1} \| + \| T_i^{m_k+1} x_{k+1} - x_{k+1} \| \\
\quad + \| x_k - T_i^{m_k} x_k \|.
\]

(38)

From (37) and \( x_k \to x^* \), we have that \( \limsup_{k \to +\infty} \| T_i^{m_k} x_k - x^* \| = 0 \), and \( \lim_{k \to +\infty} T_i^{m_k} x_k = x^* \); that is, \( \lim_{k \to +\infty} T_i(T_i^{m_k-1} x_k) = x^* \). It then follows that, for each \( i \geq 1 \),

\[
\lim_{k \to +\infty} T_i(T_i^{m_k} x_k) = x^*.
\]

(39)

In view of the closeness of \( T_i \), it follows from (37) that \( T_i x^* = x^* \), namely, for each \( i \geq 1, x^* \in F(T_i) \), and hence, \( x^* \in F \).

(V) \( x^* = \Pi_E x_1 \), and so \( x_n \to \Pi_E x_1 \) as \( n \to +\infty \).

Put \( u = \Pi_E x_1 \). Since \( u \in F \subset C \) and \( x_n = \Pi_{C_n} x_1 \), we have that \( \phi(x_n, x_1) \leq \phi(u, x_1) \), for all \( n \geq 1 \). Then,

\[
\phi(x^*, x_1) = \lim_{n \to +\infty} \phi(x_n, x_1) \leq \phi(u, x_1),
\]

(40)

which implies that \( x^* = u \) since \( u = \Pi_E x_1 \), and hence, \( x_n \to x^* = \Pi_E x_1 \).

This completes the proof. \( \square \)

4. Applications

Let \( E \) be a smooth, strictly convex, and reflexive Banach space, and let \( C \) be a nonempty closed convex subset of \( E \). Let \( \{ B_i \}_{i=1}^{\infty} : C \to E^* \) be a sequence of \( \beta_i \)-inverse strongly monotone mappings, \( \{ \psi_i \}_{i=1}^{\infty} : C \to \mathbb{R} \) a sequence of lower semicontinuous and convex functions, and \( \{ \theta_i \}_{i=1}^{\infty} : C \times C \to \mathbb{R} \) a sequence of bifunctions satisfying the following conditions:

\begin{align*}
(A_1) \quad & \theta(x, x) = 0; \\
(A_2) \quad & \theta is monotone; that is, \( \theta(x, y) + \theta(y, x) \leq 0; \\
(A_3) \quad & \limsup_{t \to 0} \theta(x + t(z - x), y) \leq \theta(x, y); \\
(A_4) \quad & the mapping \( y \mapsto \theta(x, y) \) is convex and lower semicontinuous.
\end{align*}

A system of generalized mixed equilibrium problems (GMEPs), for \( \{ \theta_i \}_{i=1}^{\infty}, \{ B_i \}_{i=1}^{\infty}, \) and \( \{ \psi_i \}_{i=1}^{\infty} \) is to find an \( x^* \in C \) such that

\[
\theta_i(x^*, y) + \langle y - x^*, B_i x^* \rangle + \psi_i(y) - \psi_i(x^*) \geq 0, \quad \forall y \in C, \ i \geq 1,
\]

(41)

whose set of common solutions is denoted by \( \Omega := \bigcap_{i=1}^{\infty} \Omega_i \), where \( \Omega_i \) denotes the set of solutions to generalized mixed equilibrium problem for \( \theta_i, B_i, \) and \( \psi_i \).

Define a countable family of mappings \( \{ T_{r,i} \}_{i=1}^{\infty} : E \to C \) with \( r > 0 \) as follows:

\[
T_{r,i}(x) = \begin{cases}
\{ z \in C : \tau_i(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \}, & \forall i \geq 1,
\end{cases}
\]

(42)

where \( \tau_i(x, y) = \theta_i(x, y) + \langle y - x, B_i x \rangle + \psi_i(y) - \psi_i(x) \), for all \( x, y \in C, i \geq 1 \). It has been shown by Zhang [15] that

1. \( \{ T_{r,i} \}_{i=1}^{\infty} \) is a sequence of single-valued mappings;
2. \( \{ T_{r,i} \}_{i=1}^{\infty} \) is a sequence of closed quasi-\( \phi \)-nonexpansive mappings;
3. \( \bigcap_{i=1}^{\infty} F(T_{r,i}) = \Omega \).

Now, we have the following result.

**Theorem 9.** Let \( E \) be the same as that in Theorem 8, and let \( C \) be a nonempty closed convex subset of \( E \). Let \( \{ T_{r,i} \}_{i=1}^{\infty} : C \to C \) be a sequence of mappings defined by (42) with \( F := \bigcap_{i=1}^{\infty} F(T_{r,i}) \neq \emptyset \). Let \( \{ \alpha_n \} \) be a sequence in \( [0, 1] \) for some \( \varepsilon \in (0, 1) \), and let \( \{ \beta_n \} \) be a sequence in \( [0, 1] \) satisfying \( 0 < \liminf_{n \to +\infty} \beta_n (1 - \beta_n) \). Let \( \{ x_n \} \) be the sequence generated by

\[
x_1 \in C, \quad C_1 = C, \\
y_n = J^{-1} \left[ \alpha_n J x_n + (1 - \alpha_n) J z_n \right], \\
z_n = J^{-1} \left[ \beta_n J x_n + (1 - \beta_n) J T_{r,i} x_n \right], \\
C_{n+1} = \{ v \in C_n : \phi(v, x_n) \leq \phi(u, x_n) \}, \\
x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1,
\]

where \( i_n \) satisfies the positive integer equation: \( n = i + (m - 1)n/2, \) and \( m \geq i \) \((m \geq i, n = 1, 2, \ldots ) \). Then, \( \{ x_n \} \) converges strongly to \( \Pi_E x_1 \), which is some solution to the system of generalized mixed equilibrium problems for \( \{ T_{r,i} \}_{i=1}^{\infty} \).

**Proof.** Note that \( \{ T_{r,i} \}_{i=1}^{\infty} \) are quasi-\( \phi \)-nonexpansive mappings; so, they are obviously generalized quasi-\( \phi \)-asymptotically nonexpansive. Therefore, this conclusion can be obtained immediately from Theorem 8. \( \square \)

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**References**


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