Research Article

Subharmonics with Minimal Periods for Convex Discrete Hamiltonian Systems

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We consider the subharmonics with minimal periods for convex discrete Hamiltonian systems. By using variational methods and dual functional, we obtain that the system has a $pT$-periodic solution for each positive integer $p$, and solution of system has minimal period $pT$ as $H$ subquadratic growth both at 0 and infinity.

1. Introduction

Consider Hamiltonian systems

$$J\dot{u}(t) + VH(t, u(t)) = 0, \quad u(0) = u(pT),$$

where $u(t) \in \mathbb{R}^{2N}$, $t \in \mathbb{R}$, $VH$ stands for the gradient of $H$ with respect to the second variable, and $J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$ is the symplectic matrix with $I_N$ the identity in $\mathbb{R}^N$. Moreover, $H$ is $T$-periodic in the variable $t$, $p \in \mathbb{N} \setminus \{0\}$.

Classically, solutions for systems (1) are called subharmonics. The first result concerning the subharmonics problem traced back to Birkhoff and Lewis in 1933 (refer to [1]), in which there exists a sequence of subharmonics with arbitrarily large minimal period, using perturbation techniques. More results can be found in [1–5], where $H$ is convex with subquadratic growth both at 0 and infinity. Using $Z_p$ index theory and Clarke duality, Xu and Guo [1, 5] proved that the number of solutions for systems (1) with minimal period $pT$ tends towards infinity as $p \to \infty$.

For periodic and subharmonic solutions for discrete Hamiltonian systems, Guo and Yu [6, 7] obtained some existence results by means of critical point theory, where they introduced the action functional

$$F(u) = \frac{1}{2} \sum_{n=1}^{pT} (J\Delta u(n-1), u(n)) - \sum_{n=1}^{pT} H(n, Lu(n)).$$

Using Clarke duality, periodic solution of convex discrete Hamiltonian systems with forcing terms has been investigated in [8]. Clarke duality was introduced in 1978 by Clarke [9], and developed by Clarke, Ekeland, and others, see [10–12]. This approach is different from the direct method of variations; some scholars applied it to consider the periodic solutions, subharmonic solutions with prescribed minimal period of Hamiltonian systems; one can refer to [3, 5, 12–14] and references therein. The dynamical behavior of differential and difference equations was studied by using various methods; see [15–19]. We refer the reader to Agarwal [20] for a broad introduction to difference equations.

Motivated by the works of Mawhin and Willem [12] and Xu and Guo [21], we use variational methods and Clarke duality to consider the subharmonics with minimal periods for discrete Hamiltonian systems

$$J\Delta u(n) + VH(n, Lu(n)) = 0, \quad u(n) = u(n + pT),$$

where $u(n) = \begin{pmatrix} u_1(n) \\ u_2(n) \end{pmatrix}$, $Lu(n) = \begin{pmatrix} u_{1,1}(n) \\ u_{2,1}(n) \end{pmatrix}$, $u_i(n) \in \mathbb{R}^N$ ($i = 1, 2$) with $N$ a given positive integer, and $\Delta u(n) = u(n + 1) - u(n)$ is the forward difference operator. $p, T \in \mathbb{N} \setminus \{0\}$. Moreover, hamiltonian function $H$ satisfies the following assumption:

(A1) $H : \mathbb{Z} \times \mathbb{R}^{2N} \to \mathbb{R}$ is continuous differentiable on $\mathbb{R}^{2N}$, $H(n, \cdot)$ convex for each $n \in \mathbb{Z}$ and $H(n + T, u) = H(n, u)$ for each $u \in \mathbb{R}^{2N}$,
(A2) there exist constants $a_1 > 0, a_2 > 0$, $1 < \theta < 2$, such that
\[
\frac{a_1}{\theta} |u|^\theta \leq H(n, u) \leq \frac{a_2}{\theta} |u|^\theta, \quad u \in \mathbb{R}^{2N},
\]
which implies $H$ subquadratic growth both at 0 and infinity.

**Theorem 1.** Assume (A1) holds. $H(n, u) \to +\infty$, $H(n, u)/|u|^2 \to 0$, as $|u| \to \infty$ uniformly in $n \in \mathbb{Z}$. Then there exists a $p_T$-periodic solution $u_p$ of (3), such that
\[
\|u_p\|_{\infty} \to \infty, \quad \text{where } u_p \to \infty \quad \text{as } p \to \infty.
\]

**Theorem 2.** Under the assumptions (A1) and (A2), if
\[
\frac{a_2}{a_1} \leq \left\{ \begin{array}{ll}
\left( \frac{1}{4} \sin \frac{\pi}{pT} \right)^{\theta/2}, & \text{when } pT \text{ is even,} \\
\left( \frac{1}{2} \sin \frac{\pi}{2pT} \right)^{\theta/2}, & \text{when } pT \text{ is odd}
\end{array} \right.
\]
for given integer $p > 1$, then the solution of (3) has minimal period $pT$.

**2. Clarke Duality and Eigenvalue Problem**

First we introduce a space $E_{pT}$ with dimension $2NpT$ as follows:
\[
E_{pT} = \{ u = \{ u(n) \} \in S \mid u(n + pT) = u(n), \quad n \in \mathbb{Z} \},
\]
where
\[
S = \left\{ u = \{ u(n) \} \mid u(n) = \left( \begin{array}{c}
u_1(n) \\
u_2(n) \end{array} \right) \in \mathbb{R}^{2N}, \quad u_j(n) \in \mathbb{R}^N, \quad j = 1, 2, n \in \mathbb{Z} \right\}.
\]

Equipped with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ in $E_{pT}$ as
\[
\langle u, v \rangle = \sum_{n=1}^{pT} \langle u(n), v(n) \rangle, \\
\|u\| = \left( \sum_{n=1}^{pT} |u(n)|^2 \right)^{1/2}, \quad u, v \in E_{pT},
\]
where $\langle \cdot, \cdot \rangle$ and $| \cdot |$ denote the usual scalar product and corresponding norm in $\mathbb{R}^{2N}$, respectively. It is easy to see that $(E_{pT}, \langle \cdot, \cdot \rangle)$ is a Hilbert space with $2NpT$ dimension, which can be identified with $\mathbb{R}^{2NpT}$. Then for any $u \in E_{pT}$, it can be written as $u = (u(j), u(T), u(T + 1), \ldots, u(T + pT - 1))^T$, where $u(j) = (\overline{u(j)}_1, \overline{u(j)}_2, \overline{u(j)}_3)^T \in \mathbb{R}^{2N}$, $j \in Z[1, pT]$, the discrete interval $\{1, 2, \ldots, pT\}$ is denoted by $Z[1, pT]$, and $^T$ denotes the transpose of a vector or a matrix.

Denote the subspace $\overline{Y} = \{ u \in E_{pT} \mid u(1) = u(2) = \cdots = u(pT) \in \mathbb{R}^{2N} \}$. Let $\mathcal{Y}$ be the direct orthogonal complement of $E_{pT}$ to $\overline{Y}$. Thus $E_{pT}$ can be split as $E_{pT} = Y \oplus \overline{Y}$, and for any $u \in E_{pT}$, it can be expressed in the form $u = \overline{u} + \overline{u}$, where $\overline{u}, \overline{u} \in Y, \overline{Y}$.

Next we recall Clarke duality and some lemmas.

The Legendre transform (see [12]) $H^*(t, \cdot)$ of $H(t, \cdot)$ with respect to the second variable is defined by
\[
H^*(t, v) = \sup_{u \in \mathbb{R}^{2N}} \{ (v, u) - H(t, u) \},
\]
where $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^{2N}$. It follows from (A1) and (A2) that
\[
\begin{align*}
(B1) & \quad H^*(n, \cdot) \text{ is continuous differentiable on } \mathbb{R}^{2N}, \\
(B2) & \quad \text{for } \tau = \theta/(\theta - 1), \quad v \in \mathbb{R}^{2N}, \quad n \in \mathbb{Z}, \quad \text{one has}
\end{align*}
\]
\[
\left| \left( 1 - \frac{1}{\tau} \right) \frac{1}{|u|^\tau} \leq H^*(n, v) \leq \left( \frac{1}{\tau} \right) \frac{1}{|v|^\tau}.
\]

Associated with variational functional (2), the dual action functional is defined by
\[
\chi(v) = \sum_{n=1}^{pT} \frac{1}{2} \left( L \Delta \nu(n-1), v(n) \right) + \sum_{n=1}^{pT} H^*(n, \Delta \nu(n)), \quad v \in E_{pT}.
\]

Indeed, by (11), we have $\chi(v + \overline{u}) = \chi(v)$ for any $\overline{u} \in \overline{Y}$ and $v \in Y$. Therefore, $\chi$ can be restricted in subspace $Y$ of $E_{pT}$. Moreover, in terms of Lemma 2.6 in [8] and the following lemma, the critical points of (11) correspond to the subharmonic solutions of (3).

**Lemma 3** (see [8, Theorem 1]). Assume that
\[
\begin{align*}
(H1) & \quad H(n, \cdot) \in C^1(\mathbb{R}^{2N}, \mathbb{R}), \quad H(n, \cdot) \text{ is convex in the second variable for } n \in \mathbb{Z}, \\
(H2) & \quad \text{there exists } \beta : \mathbb{Z} \to \mathbb{R}^{2N} \text{ such that for all } (n, u) \in \mathbb{Z} \times \mathbb{R}^{2N}, \quad H(n, u) \geq (\beta(n), u), \text{ and } \beta(n + T) = \beta(n), \\
(H3) & \quad \text{there exist } \alpha \in (0, 2 \sin(\pi/pT)) \text{ and } \gamma : \mathbb{Z} \to \mathbb{R}^*, \text{ such that for any } (n, u) \in \mathbb{Z} \times \mathbb{R}^{2N}, \quad H(n, u) \leq (\alpha/2)|u|^2 + \gamma(n), \text{ and } \gamma(n + T) = \gamma(n), \\
(H4) & \quad \text{for each } u \in \mathbb{R}^{2N}, \quad \sum_{n=1}^{pT} H(n, u(n)) \to +\infty \text{ as } |u| \to \infty.
\end{align*}
\]

Then system (3) has at least one periodic solution $u$, such that $v = -J[u - (1/pT) \sum_{n=1}^{pT} u(n)]$ minimizes the dual action
\[
\chi(v) = \sum_{n=1}^{pT} \left( 1/2 \right) \left( L \Delta \nu(n-1), v(n) \right) + \sum_{n=1}^{pT} H^*(n, \Delta \nu(n)).
\]

The following lemmas will be useful in our proofs, where Lemma 4 can be proved by means of Euler formula $e^{i\theta} = \cos \theta + i \sin \theta$, and Lemma 5 is a Hölder inequality.

**Lemma 4.** For any $k \in \mathbb{Z}$, $\sum_{n=1}^{pT} \sin((2k\pi/pT)n) = \sum_{n=1}^{pT} \cos((2k\pi/pT)n) = 0$.

**Lemma 5.** For any $a_j > 0, \gamma_j > 0, k \in \mathbb{Z}$, one has
\[
\sum_{j=1}^{k} u_j \gamma_j \leq \left( \sum_{j=1}^{k} u_j^p \right)^{1/p} \left( \sum_{j=1}^{k} \gamma_j^{1/q} \right)^{1/q}, \quad \text{where } p > 1, q > 1 \text{ and } 1/p + 1/q = 1.
Lemma 6 (see [12, proposition 2.2]). Let $H : \mathbb{R}^m \rightarrow \mathbb{R}$ be of $C^1$ and convex functional, $-\beta \leq H(u) \leq \alpha q^{-1}|u|^{q} + \gamma$, where $u \in \mathbb{R}^m$, $\alpha > 0$, $q > 1$, $\beta > 0$, $\gamma > 0$. Then $\alpha^{-p/q}p^{-1} |\nabla H(u)|^p \leq (\nabla H(u), u) + \beta + \gamma$, where $1/p + 1/q = 1$.

In order to know the form of $u \in E_{pT}$, we consider eigenvalue problem

$$LJu(n-1) = \lambda u(n), \quad u(n+pT) = u(n), \quad (12)$$

where $u(n) = (u_{i(n)}), Lu(n-1) = (u_{i(n-1)}) \in \mathbb{R}^{2N}, n \in \mathbb{Z}, \lambda \in \mathbb{R}$. We can rewrite (12) as the following form:

$$u_1(n+1) = (1-\lambda^2)u_1(n) + \lambda u_2(n),$$
$$u_2(n+1) = -\lambda u_1(n) + u_2(n),$$

$$u_1(n+pT) = u_1(n), \quad u_2(n+pT) = u_2(n). \quad (13)$$

Denoting

$$M(\lambda) = \begin{pmatrix} (1-\lambda^2)I_N & \lambda I_N \\ -\lambda I_N & I_N \end{pmatrix}, \quad (14)$$

then problem (12) is equivalent to

$$u(n+1) = M(\lambda)u(n), \quad u(n+pT) = u(n). \quad (15)$$

Letting $u(n) = \mu^n c$ be the solution of (15), for some $c \in \mathbb{C}^{2N}$, we have $\mu c = M(\lambda)c$ and $\mu^{pT} = 1$. Consider the polynomial $|M(\lambda) - \mu I_{2N}| = 0$ and condition $\mu^{pT} = 1$; it follows that

$$\mu = e^{2k\pi i/pT}, \quad \lambda = 2\sin\frac{k\pi}{pT}, \quad k \in \mathbb{Z}[-pT+1, pT-1]. \quad (16)$$

In the following we denote by $\mu_k = e^{2k\pi i/pT}, \lambda_k = 2\sin(k\pi/pT), k \in \mathbb{Z}[-pT+1, pT-1]$, and $\rho \in \mathbb{R}^N$. By $(M(\lambda) - \mu I_{2N})c = 0$, it follows that

$$\xi_k(n) = \left( ie^{(-k\pi/pT)} \rho \right). \quad (17)$$

Thus

$$u_k(n) = \mu_k^n \xi_k = e^{2k\pi i/n} \left( ie^{(-k\pi i/pT)} \rho \right)$$

$$= \begin{pmatrix} \cos\left(\frac{2k\pi}{pT}n\right) \rho \\ -\sin\left(\frac{2k\pi}{pT}(n-\frac{1}{2})\right) \rho \end{pmatrix}$$

$$+ i \begin{pmatrix} \sin\left(\frac{2k\pi}{pT}n\right) \rho \\ \cos\left(\frac{2k\pi}{pT}(n-\frac{1}{2})\right) \rho \end{pmatrix}. \quad (18)$$

Let

$$\xi_k(n) = \begin{pmatrix} \cos\left(\frac{2k\pi}{pT}n\right) \rho \\ -\sin\left(\frac{2k\pi}{pT}(n-\frac{1}{2})\right) \rho \end{pmatrix},$$

$$\eta_k = \begin{pmatrix} \sin\left(\frac{2k\pi}{pT}n\right) \rho \\ \cos\left(\frac{2k\pi}{pT}(n-\frac{1}{2})\right) \rho \end{pmatrix}. \quad (19)$$

Obviously, $\xi_k(n)$ and $\eta_k(n)$ satisfy (15). Moreover $LJ\Delta \xi_k(n-1) = \lambda_k \xi_k(n), LJ\Delta \eta_k(n-1) = \lambda_k \eta_k(n), \xi_{2pT+k}(n) = \xi_k(n), \eta_{2pT+k}(n) = \eta_k(n)$, $\xi_{pT-k}(n) = \xi_k(n), \eta_{pT-k}(n) = -\eta_k(n)$.

For $k \neq pT/2$, subspace $Y_k$ is defined by

$$Y_{pT/2} = \text{span} \left\{\xi_{pT/2}(n), n \in \mathbb{Z}\right\},$$
$$Y_{-pT/2} = \text{span} \left\{\xi_{-pT/2}(n), n \in \mathbb{Z}\right\}. \quad (20)$$

Therefore,

$$Y = \oplus Y_k, \quad k \in \mathbb{Z}[-pT/2, pT/2] \setminus \{0\}, \text{ if } pT \text{ is even},$$

$$Y = \oplus Y_k, \quad k \in \mathbb{Z}[-pT/2, pT/2] \setminus \{0\}, \text{ if } pT \text{ is odd}. \quad (22)$$

Moreover, for any $u = \{u(n)\} \in E_{pT}$, we may express $u(n)$ as

$$u(n) = \sum_{k=-pT+1}^{pT-1} \begin{pmatrix} \cos\left(\frac{2k\pi}{pT}n\right) a_k \\ -\sin\left(\frac{2k\pi}{pT}(n-\frac{1}{2})\right) a_k \end{pmatrix}$$

$$+ \begin{pmatrix} \sin\left(\frac{2k\pi}{pT}n\right) b_k \\ \cos\left(\frac{2k\pi}{pT}(n-\frac{1}{2})\right) b_k \end{pmatrix}, \quad (23)$$

where $a_k, b_k \in \mathbb{R}^N$.

Since $(\Delta u(n), \Delta u(n)) = - (\Delta^2 u(n-1), u(n))$, we consider eigenvalue problem

$$-\Delta^2 u(n-1) = \lambda u(n), \quad u(n+pT) = u(n), \quad u(n) \in \mathbb{R}^N, \quad (24)$$
where \( \Delta^2 u(n-1) = \Delta u(n) - \Delta u(n-1) = u(n+1) - 2u(n) + u(n-1) \). The second order difference equation (24) has complexity solution \( u(n) = e^{\theta n} c \) for \( c \in \mathbb{C} \), where \( \theta = 2k\pi/pT \). Moreover, \( \lambda = 2 - e^{i\theta} - e^{i\theta} = 2(1 - \cos \theta) = 4\sin^2(\theta/2) \); that is, \( \lambda = 4\sin^2(k\pi/pT), k \in \mathbb{Z}_{[0, pT-1]} \).

By the previous, it follows Lemma 7.

**Lemma 7.** For any \( u \in E_{pT} \), one has \( -\lambda_{\max} \|u\|^2 \leq \sum_{n=1}^{pT} (LJ \Delta u(n-1), u(n)) \leq \lambda_{\max} \|u\|^2 \), and \( 0 \leq \sum_{n=1}^{pT} |\Delta u(n)|^2 \leq \lambda_{\max}^2 \|u\|^2 \), where

\[
\lambda_{\max} = \max_{k \in [0, pT-1]} \left\{ \frac{2 \sin \frac{k\pi}{pT}}{2} \right\}
\]

(25)

Moreover, if \( u \in Y \), then \( 4\sin^2(\pi/pT) \|u\|^2 \leq \sum_{n=1}^{pT} |\Delta u(n)|^2 \leq \lambda_{\max}^2 \|u\|^2 \).

### 3. Proofs of Main Results

**Lemma 8.** Consider

\[
\sum_{n=1}^{pT} (LJ \Delta u(n-1), u(n)) \geq -\left(2 \sin \frac{\pi}{pT}\right)^{-1} \sum_{n=1}^{pT} |\Delta u(n)|^2, \quad \forall u \in E_{pT}.
\]

(26)

Proof. Letting \( \bar{u}(n) = u(n) - (1/pT) \sum_{n=1}^{pT} u(n) \), then \( \bar{u} \in Y \). By Lemmas 5 and 7, we have

\[
\sum_{n=1}^{pT} (LJ \Delta u(n-1), u(n)) = \sum_{n=1}^{pT} (LJ \Delta u(n-1), \bar{u}(n))
\]

\[
\geq -\left(\sum_{n=1}^{pT} |LJ \Delta u(n-1)|^2\right)^{1/2} \cdot \left(\sum_{n=1}^{pT} |\bar{u}(n)|^2\right)^{1/2}
\]

\[
\geq -\left(\sum_{n=1}^{pT} |\Delta u(n)|^2\right)^{1/2} \cdot \left(2 \sin \frac{\pi}{pT}\right)^{-1} \left(\sum_{n=1}^{pT} |\Delta \bar{u}(n)|^2\right)^{1/2}
\]

\[
= -\left(2 \sin \frac{\pi}{pT}\right)^{-1} \sum_{n=1}^{pT} |\Delta u(n)|^2.
\]

(27)

**Lemma 9.** If there exist \( \alpha \in (0, \sin(\pi/pT)), \beta \geq 0 \) and \( \delta > 0 \), such that

\[
\delta |u| - \beta \leq H(n, u) \leq \frac{\alpha}{2} |u|^2 + \gamma
\]

(28)

for all \( n \in [1, pT] \) and \( u \in \mathbb{R}^{2N} \), then each solution of (3) satisfies the inequalities

\[
\sum_{n=1}^{pT} |\Delta u(n)|^2 \leq \frac{2\alpha(\beta + \gamma) pT \sin(\pi/pT)}{\sin(\pi/pT) - \alpha},
\]

\[
\sum_{n=1}^{pT} |L u(n)| \leq \frac{(\beta + \gamma) pT \sin(\pi/pT)}{\delta(\sin(\pi/pT) - \alpha)}.
\]

(29)

Proof. Let \( u \) be the solution of (3). By Lemma 6, we have

\[
\frac{1}{2\alpha} |\nabla H(n, L u(n))|^2 \leq \langle \nabla H(n, L u(n)), L u(n) \rangle + \beta + \gamma
\]

\[
= -\langle J \Delta u(n), L u(n) \rangle + \beta + \gamma.
\]

(30)

Obviously, \( |J \Delta u(n)|^2 = -\langle \nabla H(n, L u(n)), J \Delta u(n) \rangle = |\nabla H(n, L u(n))|^2 \) by (3), and it follows that \( (1/2\alpha) \sum_{n=1}^{pT} |\Delta u(n)|^2 + \sum_{n=1}^{pT} (J \Delta u(n), L u(n)) \leq (\beta + \gamma) pT \); that is,

\[
\frac{1}{2\alpha} \sum_{n=1}^{pT} |\Delta u(n)|^2 + \sum_{n=1}^{pT} (L J \Delta u(n-1), u(n)) \leq (\beta + \gamma) pT.
\]

(31)

By means of Lemma 8, we have

\[
\left[ \frac{1}{2\alpha} - \left(2 \sin \frac{\pi}{pT}\right)^{-1}\right] \sum_{n=1}^{pT} |\Delta u(n)|^2 \leq (\beta + \gamma) pT,
\]

(32)

which gives first conclusion.

Now, \( H(n, 0) \leq \gamma \) in view of (28); therefore by convex and Lemma 8, we have

\[
\delta \sum_{n=1}^{pT} |L u(n)| - \beta pT \leq \sum_{n=1}^{pT} H(n, L u(n)) \leq \sum_{n=1}^{pT} [H(n, 0) + \langle \nabla H(n, L u(n)), L u(n) \rangle].
\]
which gives the second conclusion. The proof is completed.

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Proof of Theorem 1. Let \( c_1 = \max_{n \in \mathbb{Z}} |H(n, 0)| \). By assumption in Theorem 1, there exists \( R > 0 \), such that \( H(n, u) \geq 1 + c_1 \), for \( n \in \mathbb{Z} \) and \( |u| \geq R \). Moreover, there exist \( \alpha \in (0, 2 \sin(\pi/pT)) \), \( \gamma > 0 \) such that

\[
H(n, u) \geq \frac{\alpha}{2} |u|^2 + \gamma, \quad \forall (n, u) \in \mathbb{Z} \times \mathbb{R}^{2N}.
\]

Thus, by convex of \( H \), for all \((n, u) \in \mathbb{Z} \times \mathbb{R}^{2N} \) with \( |u| \geq R \), we have

\[
1 + c_1 \leq H \left( n, \frac{R}{|u|} u \right) \leq H(n, 0) + \frac{R}{|u|} (H(n, u) - H(n, 0)) \leq \frac{R}{|u|} H(n, u) + c_1.
\]

Therefore there exist \( \beta > 0 \) and \( \delta > 0 \), such that

\[
H(n, u) \geq \delta |u| - \beta, \quad \forall (n, u) \in \mathbb{Z} \times \mathbb{R}^{2N}.
\]

Combining the previous argument, by Lemma 3, the system (3) has a \( pT \)-periodic solution \( u_p \) such that \( v_p = -f[u_p = (1/pT) \sum_{n=1}^{pT} u_p(n)] \in Y \) minimizes the dual action

\[
J_p \left( v_p \right) = \sum_{n=1}^{pT} \left( \frac{1}{2} \left( |J \Delta v_p(n-1), v_p(n) \right) + \sum_{n=1}^{pT} H^* \left( n, \Delta v_p(n) \right) \right) \quad \text{on} \quad E_{pT}.
\]

It follows that \( \Delta u_{p_k}(n) = J \Delta v_p(n) \) and \( J v_p(n) = u_p(n) - (1/pT) \sum_{n=1}^{pT} u_p(n) \).

We next prove that \( \|u_{p_k}\|_{\infty} \to \infty \) as \( p_k \to \infty \).

Suppose not, there exist \( c_2 > 0 \) and a subsequence \( \{p_k\} \) such that

\[
p_k \to \infty, \quad \|u_{p_k}\|_{\infty} \leq c_2 \quad \text{as} \quad k \to \infty.
\]

In terms of (3), it follows that \( \|\Delta u_{p_k}\|_{\infty} \leq c_3 \) for some \( c_3 > 0 \), and \( \|v_{p_k}\|_{\infty} \leq 2c_3, \|\Delta v_{p_k}\|_{\infty} \leq c_3 \). Consequently, by \( H^* (n, v) \geq -H(n, 0) \geq -c_1 \), we have

\[
c_{p_k} = \mathcal{H}_p \left( v_{p_k} \right) = \sum_{n=1}^{pT} \left( \frac{1}{2} \left( \frac{1}{pT} \sum_{n=1}^{pT} \left( |J \Delta v_{p_k}(n-1), v_{p_k}(n) \right) + \sum_{n=1}^{pT} H^* \left( n, \Delta v_{p_k}(n) \right) \right) \right) \leq 2c_3, \|\Delta v_{p_k}\|_{\infty} \leq c_3.\]

where \( n \in \mathbb{Z} \) and

\[
\left| J \Delta v_{p_k}(n-1) \right| = \left( \left| \Delta v_{p_k}(n-1) \right|^2 + \left| \Delta v_{p_k}(n-1) \right|^2 \right)^{1/2} \leq \sqrt{5} \|\Delta v_{p_k}\|_{\infty} \leq \sqrt{5} c_3.
\]

By (36), if \( |v| \leq \delta \), we have \( H(n, u) \leq (v, u) - \delta |u| + \beta \leq \beta \), and \( H^* (n, v) \leq \beta \). Letting \( \rho \in \mathbb{R}^N \) and \( |\rho| = 1 \), in terms of (12), \( h_p \) associated with \( \lambda_{-1} = -2 \sin(\pi/pT) \) is given by

\[
h_p(n) = \frac{\delta}{4 \sin(\pi/pT)} \left( \cos \frac{\pi}{pT} n - \sin \frac{\pi}{pT} n \right) \rho \left( \sin \frac{\pi}{pT} \left( n - \frac{1}{2} \right) + \cos \frac{\pi}{pT} \left( n - \frac{1}{2} \right) \right) \rho \quad \text{on} \quad E_{pT}.
\]

which belongs to \( E_{pT} \), and

\[
|\Delta h_p(n)|^2 \leq \left( \frac{\delta}{4 \sin(\pi/pT)} \right)^2 \left( 2 \sin \frac{\pi}{pT} \left( n + \frac{1}{2} \right) - \cos \frac{\pi}{pT} \left( n + \frac{1}{2} \right) \right) \rho \left( \cos \frac{2\pi}{pT} n - \sin \frac{2\pi}{pT} n \right) \rho \quad \text{on} \quad E_{pT}.
\]

\[
= \frac{1}{4} \left[ 2 \sin \frac{2\pi}{pT} (2n+1) - \sin \frac{2\pi}{pT} (2n) \right] \cdot |\rho|^2 \delta^2 \leq \delta^2.
\]
Moreover, by Lemma 4 we have

\[
\sum_{n=1}^{pT} |h_p(n)|^2 = \sum_{n=1}^{pT} \left( \frac{\delta}{4 \sin(\pi/pT)} \right)^2 \cdot \left( 2 + \sin \frac{2\pi}{pT} (2n - 1) - \sin \frac{2\pi}{pT} (2n) \right) |p|^2 = \left( \frac{\delta}{4 \sin(\pi/pT)} \right)^2 2|p|^2 pT = \frac{\delta^2 pT}{8 \sin^2 (\pi/pT)}.
\]

By Lemma 7 and (45), it follows that

\[
\|\bar u_{p_k}\|^2 = \sum_{n=1}^{T_{p_k}} |\bar u_{p_k}(n)|^2 \leq \left( \frac{2 \sin \pi}{T_{p_k}} \right)^{-1} \sum_{n=1}^{T_{p_k}} |\Delta u_{p_k}(n)|^2 \leq \frac{(\beta + \gamma) T}{\sin (\pi/T) - \alpha},
\]

which implies that \(\|\bar u_{p_k}\|_\infty\) is bounded, therefore \(\|u_{p_k}\|_\infty\) is bounded; a contradiction with the second claim \(\lim_{p \to \infty} u_p\|_\infty = \infty\). This completes the proof. \(\square\)

**Proof of Theorem 2.** Under the assumptions (A1) and (A2), all conditions in Theorem 1 are satisfied. Then, for each integer \(p > 1\), there exists a \(pT\)-periodic solution \(u\) of (3) such that \(v = -J[u-(1/pT)\sum_{n=1}^{pT} u(n)] \in Y\) minimizes the dual action

\[
\chi(v) = \frac{1}{2} \sum_{n=1}^{pT} (LJ\Delta v(n-1), v(n)) + \sum_{n=1}^{pT} H^*(n, \Delta v(n)) \quad \text{on } E_{pT}.
\]

If the critical point \(v\) of dual action functional \(\chi\) has minimal period \(pT/l \in \mathbb{N} \setminus \{0\}\), where \(l \in \mathbb{N} \setminus \{0\}\), then by Lemma 7 with \(pT\) replaced by \(pT/l\), we have the following estimate:

\[
4 \sin^2 \frac{\pi}{pT} \sum_{n=1}^{pT} |v(n)|^2 \leq \sum_{n=1}^{pT} |\Delta v(n)|^2.
\]

By Lemma 5 and the previous inequality, we have

\[
\sum_{n=1}^{pT} (LJ\Delta v(n-1), v(n)) \geq -\left( \sum_{n=1}^{pT} |LJ\Delta v(n-1)|^2 \right)^{1/2} \cdot \left( \sum_{n=1}^{pT} |v(n)|^2 \right)^{1/2} \geq -\left( \sum_{n=1}^{pT} |\Delta v(n)|^2 \right)^{1/2},
\]
\[
\begin{align*}
\cdot & \left(2 \sin \frac{l \pi}{p T}\right)^{-1} \left(\sum_{n=1}^{p T} |\Delta V(n)|^2 \right)^{1/2} \\
& = - \left(2 \sin \frac{l \pi}{p T}\right)^{-1} \frac{\sum_{n=1}^{p T} |\Delta V(n)|^2}{(p T)^{1-2/\tau} \left(\sum_{n=1}^{p T} |\Delta V(n)|^\tau \right)^{2/\tau}}, \\
& \geq - \left(2 \sin \frac{l \pi}{p T}\right)^{-1} \frac{\sum_{n=1}^{p T} |\Delta V(n)|^2}{ \left(\sum_{n=1}^{p T} |\Delta V(n)|^\tau \right)^{2/\tau}},
\end{align*}
\]

(51)

where \( \tau = \theta/(\theta - 1) > 2 \) for \( 1 < \theta < 2 \). It follows from assumption (B2) that

\[
H^* (n, \Delta V(n)) \geq \frac{1}{\tau} \left(\frac{1}{a_2}\right)^{\tau-1} |\Delta V(n)|^\tau,
\]

(52)

thus

\[
\chi(v) \geq - \left(2 \sin \frac{l \pi}{p T}\right)^{-1} (p T)^{1-2/\tau} \left(\sum_{n=1}^{p T} |\Delta V(n)|^\tau \right)^{2/\tau} \\
+ \frac{1}{\tau} \left(\frac{1}{a_2}\right)^{\tau-1} \sum_{n=1}^{p T} |\Delta V(n)|^\tau \\
\geq \frac{(1/\tau - 1/2) p T^{(\tau-1)/(\tau-2)}}{ \left(\sin (l \pi/p T)^{\tau/2}) \right)^{2/\tau}}.
\]

(53)

(54)

One can obtain the previous inequality by minimizing in (53) with respect to \( \sum_{n=1}^{p T} |\Delta V(n)|^\tau \), and the minimum is attained at \( (p T)^{1/\tau} (a_2)^{(\tau-1)/(\tau-2)}/(\sin (l \pi/p T))^{1/(\tau-2)}. \)

On the other hand, let

\[
v(n) = \frac{1}{\sqrt{p T}} \left( \cos \frac{2k \pi}{p T} n \cdot a_k - \sin \frac{2k \pi}{p T} \left(n - \frac{1}{2}\right) \cdot a_k \right),
\]

(55)

where \( a_k \in \mathbb{R}^N, k \in \mathbb{Z}[[-p T/2], [p T/2]] \setminus \{0\} \). Then \( v \in Y_k \), and

\[
\Delta V(n) = -2 \sin \frac{k \pi}{p T} \frac{1}{\sqrt{p T}} \left( \sin \frac{2k \pi}{p T} \left(n + \frac{1}{2}\right) \cdot a_k \right).
\]

(56)

Taking \( a_k = (d, 0, \ldots, 0)^T \in \mathbb{R}^N \), where \( d \in \mathbb{R} \), by Lemma 4, it follows that

\[
\sum_{n=1}^{p T} \left((- L J \Delta V(n-1), v(n)) \right) \\
= \sum_{n=1}^{p T} \left[ -\Delta V_2(n) v_1(n) + \Delta V_1(n-1) v_2(n) \right] \\
= \sum_{n=1}^{p T} \frac{1}{p T} \cdot 2 \sin \frac{k \pi}{p T} \cdot \left(\cos \frac{2k \pi}{p T} n \cdot |d| + \sin \frac{2k \pi}{p T} \left(n - \frac{1}{2}\right) \cdot |d| \right) \\
\leq \lambda_k |d|^2,
\]

where \( \lambda_k = 2 \sin(k \pi/p T) \) and

\[
\sum_{n=1}^{p T} |\Delta V(n)|^\tau \\
= \sum_{n=1}^{p T} |\lambda_k|^\tau (p T)^{-\tau/2} \\
\cdot \left(\sin \frac{2k \pi}{p T} \left(n + \frac{1}{2}\right) \right) \cdot \left(\cos \frac{2k \pi}{p T} \left(n - \frac{1}{2}\right) \right) \cdot |d|^\tau \\
\leq \lambda_{\max} \cdot (p T)^{1-(\tau/2)} \cdot 2^{\tau/2} |d|^\tau.
\]

(57)

(58)

Therefore, taking \( k = [-p T/2] \), by eigenvalue problem (24) and (B2), it follows that

\[
\chi(v) = \frac{1}{2} \sum_{n=1}^{p T} \left((- L J \Delta V(n-1), v(n)) \right) \\
+ \sum_{n=1}^{p T} H^* (n, \Delta V(n)) \\
\leq - \frac{1}{2} \lambda_{\max} |d|^2 \\
+ \frac{1}{\tau} \left(\frac{1}{a_1}\right)^{\tau-1} \sum_{n=1}^{p T} |\Delta V(n)|^\tau \\
\leq - \frac{1}{2} \lambda_{\max} |d|^2 + \frac{1}{\tau} \left(\frac{1}{a_1}\right)^{\tau-1} \lambda_{\max}^\tau \\
\cdot (p T)^{1-(\tau/2)} \cdot 2^{\tau/2} |d|^\tau.
\]

(59)

Let \( f(\rho) \) equal the right-hand side of (59) where \( \rho = |d| \). It is easy to see that the absolute minimum of \( f \) is attained at \( \rho_{\min} = (a_1)^{(\tau-1)/(\tau-2)}/(\lambda_{\max}^\tau \cdot 2^{\tau/2}) \) and given
by $f_{\min} = (1/\tau - 1/2)pT(a_2^{l-1/\tau}(r_2^{-1/\tau}))/\tau$. Hence, by (19), let
\[\xi(n) = \xi_{\lfloor pT/2 \rfloor}(n) = \begin{pmatrix} \cos \frac{2k\pi n}{pT} \cdot \rho \\ -\sin \frac{2k\pi}{2}(n - \frac{1}{2}) \cdot \rho \end{pmatrix}, \tag{60}\]
where $\rho \in \mathbb{R}^N$, $k = \lfloor pT/2 \rfloor$.

If $pT$ is even, then $\xi(n) = (1, 1)^T \cdot (-1)^k \rho$. Set
\[Y_{\rho_{\min}} = \{ v \in Y_{\lfloor pT/2 \rfloor} : v(n) = \xi(n), \rho = (d, 0, \ldots, 0)^T \in \mathbb{R}^N, d \in \mathbb{R} \}, \tag{61}\]
for $v \in Y_{\rho_{\min}},$ we have
\[\chi(v) \leq f_{\min}. \tag{62}\]
Combining (54), (59), and (62), we have
\[(1/\tau - 1/2) \frac{pT\left(\frac{2l}{\tau}\right)^{(r_2-1)/\tau}}{(\sin(\pi/pT))^{2/(r_2-1)}} \leq \frac{(1/\tau - 1/2) pT\left(\frac{2l}{\tau}\right)^{(r_2-1)/\tau}}{(2\lambda_{\max})^{2/(r_2-1)}}. \tag{63}\]
By $\tau > 2$, and $\theta = \tau/(\tau - 1)$, it follows that
\[
\sin(\pi/pT) \leq \frac{(\pi/2)^{2/\tau}}{(2\lambda_{\max})^{2/\tau}}. \tag{64}\]

For integer $p > 1, T \geq 1, l \in \mathbb{N} \setminus \{0\}, pT/l \in \mathbb{N} \setminus \{0\}$, we have $0 < \ln(pT) \leq \pi$, $0 < \pi/pT \leq \pi/2$.

If $pT$ is even, then $\lambda_{\max} = 2$. By assumption $a_2/a_1 \leq ((1/2)^{2/\tau}) \leq \sin(\pi/pT)$, which implies that $l = 1$ or $l = pT - 1$. If $pT > 2$, then $pT/l \notin \mathbb{N}$. So we have $l = 1$.

If $pT$ is odd, then $\lambda_{\max} = 2 \cos(\pi/pT)$. By assumption $a_2/a_1 \leq ((1/2)^{2/\tau}) \leq \sin(\pi/pT)$, we have $\sin(\pi/pT) \leq \sin(\pi/pT)$, so $l = 1$. This completes the proof. \square

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**References**


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