Research Article

Some Geometric Properties of the Domain of the Double Sequential Band Matrix \( B(\tilde{r}, \tilde{s}) \) in the Sequence Space \( \ell(p) \)

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The sequence space \( \ell(p) \) was introduced by Maddox (1967). Quite recently, the sequence space \( \ell(\tilde{B}, p) \) of nonabsolute type has been introduced and studied which is the domain of the double sequential band matrix \( B(\tilde{r}, \tilde{s}) \) in the sequence space \( \ell(p) \) by Nergiz and Başar (2012). The main purpose of this paper is to investigate the geometric properties of the space \( \ell(\tilde{B}, p) \), like rotundity and Kadec-Klee and the uniform Opial properties. The last section of the paper is devoted to the conclusion.

1. Introduction

By \( \omega \), we denote the space of all real-valued sequences. Any vector subspace of \( \omega \) is called a sequence space. We write \( \ell_{\infty} \), \( c \), and \( c_0 \) for the spaces of all bounded, convergent, and null sequences, respectively. Also by \( bs \), \( cs \), \( \ell_1 \), and \( \ell_p \); we denote the spaces of all bounded, convergent, absolutely convergent, and \( p \)-absolutely convergent series, respectively, where \( 1 < p < \infty \).

Assume here and after that \( (p_k) \) is a bounded sequence of strictly positive real numbers with \( \sup p_k = H \) and \( M = \max\{1, H\} \). Then, the linear space \( \ell(p) \) was defined by Maddox [1] (see also Simons [2] and Nakano [3]) as follows:

\[
\ell(p) = \{ x = (x_k) \in \omega : \sum_{k} |x_k|^{p_k} < \infty \} \quad (0 < p_k \leq H < \infty) \tag{1}
\]

which is complete paranormed space paranormed by

\[
g(x) = \left( \sum_{k} |x_k|^{p_k} \right)^{1/M}. \tag{2}
\]

For simplicity in notation, here and in what follows, the summation without limits runs from 1 to \( \infty \).

Quite recently, Nergiz and Başar [4] have introduced the space \( \ell(\tilde{B}, p) \) of nonabsolute type which consists of all sequences whose \( B(\tilde{r}, \tilde{s}) \)-transforms are in the space \( \ell(p) \), where \( B(\tilde{r}, \tilde{s}) = \{ b_{nk}(r_k, s_k) \} \) is defined by

\[
b_{nk}(r_k, s_k) = \begin{cases} r_k, & k = n, \\ s_k, & k = n - 1, \\ 0, & \text{otherwise} \end{cases} \tag{3}
\]

for all \( k, n \in \mathbb{N} \), where \( \tilde{r} = (r_k) \) and \( \tilde{s} = (s_k) \) are the convergent sequences. We should record that the double sequential band matrices were used for determining its fine spectrum over some sequence spaces by Kumar and Srivastava in [5, 6], Panigrahi and Srivastava in [7], and Akhmedov and El-Shabrawy in [8]. The reader may refer to Nergiz and Başar [4, 9] for relevant terminology and additional references on the space \( \ell(\tilde{B}, p) \), since the present paper is a natural continuation of them. Here and after, for short we write \( \tilde{B} \) instead of \( B(\tilde{r}, \tilde{s}) \).

In the special case \( p_k = p \) for all \( k \in \mathbb{N} \), the space \( \ell(\tilde{B}, p) \) is reduced to the space \( \ell(p)_{\tilde{B}} \), that is,

\[
\ell(p)_{\tilde{B}} := \{ (x_k) \in \omega : \sum_{k} |s_{k-1} x_{k-1} + r_k x_k|^{p_k} < \infty \} \quad (0 < p < \infty). \tag{4}
\]
2. The Rotundity of the Space $\ell(\tilde{B}, p)$

The rotundity of Banach spaces is one of the most important geometric property in functional analysis. For details, the reader may refer to [10–12]. In this section, we characterize the rotundity of the space $\ell(\tilde{B}, p)$ and give some results related to this concept.

Definition 1. Let $S(X)$ be the unit sphere of a Banach space $X$. Then, a point $x \in S(X)$ is called an extreme point if $2x = y + z$ implies $y = z$ for every $y, z \in S(X)$. A Banach space $X$ is said to be rotund (strictly convex) if every point of $S(X)$ is an extreme point.

Definition 2. A Banach space $X$ is said to have Kadec-Klee property (or property $(\mathcal{H})$) if every weakly convergent sequence on the unit sphere is convergent in norm.

Definition 3. A Banach space $X$ is said to have

(i) the Opial property if every sequence $(x_n)$ weakly convergent to $x_0 \in X$ satisfies

$$\lim_{n \to \infty} \|x_n - x\| < \lim_{n \to \infty} \|x_n + x\|$$

for every $x \in X$ with $x \neq x_0$;

(ii) the uniform Opial property if for each $\epsilon > 0$, there exists an $r > 0$ such that

$$1 + r \leq \lim_{n \to \infty} \|x_n + x\|$$

for each $x \in X$ with $\|x\| \geq \epsilon$ and each sequence $(x_n)$ in $X$ such that $x_n \to 0$ and $\liminf_{n \to \infty}\|x_n\| \geq 1$.

Definition 4. Let $X$ be a real vector space. A functional $\sigma : X \to [0, \infty)$ is called a modular if

(i) $\sigma(x) = 0$ if and only if $x = \theta$;

(ii) $\sigma(\alpha x) = \sigma(x)$ for all scalars $\alpha$ with $|\alpha| = 1$;

(iii) $\sigma(\alpha x + \beta y) \leq \alpha \sigma(x) + \beta \sigma(y)$ for all $x, y \in X$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$;

(iv) the modular $\sigma$ is called convex if $\sigma(\alpha x + \beta y) \leq \alpha \sigma(x) + \beta \sigma(y)$ for all $x, y \in X$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$.

A modular $\sigma$ on $X$ is called

(a) right continuous if $\lim_{\alpha \to 1^+} \sigma(\alpha x) = \sigma(x)$ for all $x \in X_{\sigma}$.

(b) left continuous if $\lim_{\alpha \to 1^-} \sigma(\alpha x) = \sigma(x)$ for all $x \in X_{\sigma}$.

(c) continuous if it is both right and left continuous, where

$$X_{\sigma} = \left\{ x \in X : \lim_{\alpha \to 0^+} \sigma(\alpha x) = 0 \right\}. $$

We define $\sigma_p$ on $\ell(\tilde{B}, p)$ by $\sigma_p(x) = \sum_{k=1}^{\infty} \|s_{k-1} - x_{k-1} + r_k x_k\|^{p_k}$. If $p_k \geq 1$ for all $k \in \mathbb{N}$, by the convexity of the function $t \mapsto \|t\|^{p_k}$ for each $k \in \mathbb{N}$, $\sigma_p$ is a convex modular on $\ell(\tilde{B}, p)$.

Proposition 5. The modular $\sigma_p$ on $\ell(\tilde{B}, p)$ satisfies the following properties with $p_k \geq 1$ for all $k \in \mathbb{N}$:

(i) If $0 < \alpha \leq 1$, then $\alpha^M \sigma_p(\alpha x) \leq \sigma_p(x)$ and $\sigma_p(\alpha x) \leq \alpha \sigma_p(x)$.

(ii) If $\alpha \geq 1$, then $\sigma_p(x) \leq \alpha^M \sigma_p(x/\alpha)$.

(iii) If $\alpha \geq 1$, then $\sigma_p(x) \leq \alpha \sigma_p(x/\alpha)$.

(iv) The modular $\sigma_p$ is continuous on the space $\ell(\tilde{B}, p)$.

Proof. Consider the modular $\sigma_p$ on $\ell(\tilde{B}, p)$.

(i) Let $0 < \alpha \leq 1$, then $\alpha^M / \alpha^{p_k} \leq 1$. So, we have

$$\alpha^M \sigma_p \left( \frac{x}{\alpha} \right) = \alpha^M \sum_{k=1}^{\infty} \frac{1}{\alpha^{p_k}} \|s_{k-1} - x_{k-1} + r_k x_k\|^{p_k}$$

$$= \sum_{k=1}^{\infty} \alpha^M \|s_{k-1} - x_{k-1} + r_k x_k\|^{p_k}$$

$$\leq \sum_{k=1}^{\infty} \|s_{k-1} - x_{k-1} + r_k x_k\|^{p_k} = \sigma_p(x),$$

(8)

$$\sigma_p(\alpha x) = \sum_{k=1}^{\infty} \alpha^{p_k} \|s_{k-1} - x_{k-1} + r_k x_k\|^{p_k}$$

$$\leq \alpha \sum_{k=1}^{\infty} \|s_{k-1} - x_{k-1} + r_k x_k\|^{p_k} = \alpha \sigma_p(x).$$

(9)

(ii) Let $\alpha \geq 1$. Then, $\alpha^M / \alpha^{p_k} \geq 1$ for all $p_k \geq 1$. So, we have

$$\sigma_p(x) \leq \alpha^M \sigma_p \left( \frac{x}{\alpha} \right) = \alpha^M \sigma_p \left( \frac{x}{\alpha} \right).$$

(9)

(iii) Let $\alpha \geq 1$. Then, $\alpha / \alpha^{p_k} \geq 1$ for all $p_k \geq 1$. So, we have

$$\sigma_p(x) = \sum_{k=1}^{\infty} \|s_{k-1} - x_{k-1} + r_k x_k\|^{p_k}$$

$$\leq \sum_{k=1}^{\infty} \alpha \|s_{k-1} - x_{k-1} + r_k x_k\|^{p_k} = \alpha \sigma_p \left( \frac{x}{\alpha} \right).$$

(iii) By (ii) and (iii), one can immediately see for $\alpha > 1$

$$\sigma_p(x) \leq \alpha \sigma_p(x) \leq \sigma_p(\alpha x) \leq \alpha^M \sigma_p(x).$$

(11)

By passing to limit as $\alpha \to 1^+$ in (11), we have

$$\lim_{\alpha \to 1^+} \sigma_p(\alpha x) = \sigma_p(x).$$

Hence, $\sigma_p$ is right continuous. If $0 < \alpha < 1$, by (i) we have

$$\alpha^M \sigma_p(x) \leq \sigma_p(\alpha x) \leq \alpha \sigma_p(x).$$

(12)

By letting $\alpha \to 1^-$ in (12), we observe that

$$\lim_{\alpha \to 1^-} \sigma_p(\alpha x) = \sigma_p(x).$$

Hence, $\sigma_p$ is also left continuous, and so, it is continuous. 

\qed
Proposition 6. For any \( x \in \ell(\tilde{B}, p) \), the following statements hold:

(i) If \( \| x \| < 1 \), then \( \sigma_p(x) \leq \| x \| \).

(ii) If \( \| x \| > 1 \), then \( \sigma_p(x) \geq \| x \| \).

(iii) \( \| x \| = 1 \) if and only if \( \sigma_p(x) = 1 \).

(iv) \( \| x \| < 1 \) if and only if \( \sigma_p(x) < 1 \).

(v) \( \| x \| > 1 \) if and only if \( \sigma_p(x) > 1 \).

Proof. Let \( x \in \ell(\tilde{B}, p) \).

(i) Let \( \epsilon > 0 \) be such that \( 0 < \epsilon < 1 - \| x \| \). By the definition of \( \| \cdot \| \), there exists an \( \alpha > 0 \) such that \( \| x \| + \epsilon > \alpha \) and \( \sigma_p(x) \leq 1 \). From Parts (i) and (ii) of Proposition 5, we obtain

\[
\sigma_p(x) \leq \sigma_p \left( \frac{\| x \| + \epsilon}{\alpha} \right) \leq \frac{1}{\| x \|} \| x \| + \epsilon .
\]  

(13)

Since \( \epsilon \) is arbitrary, we have (i).

(ii) If we choose \( \epsilon > 0 \) such that \( 0 < \epsilon < 1 - (1/\| x \|) \), then \( 1 < (1 - \epsilon)\| x \| < \| x \| \). By the definition of \( \| \cdot \| \) and Part (i) of Proposition 5, we have

\[
1 < \sigma_p \left( \frac{x}{(1 - \epsilon)\| x \|} \right) = \frac{1}{(1 - \epsilon)\| x \|} \| x \| + \epsilon .
\]  

(14)

So, \( (1 - \epsilon)\| x \| < \sigma_p(x) \) for all \( \epsilon \in (0, 1 - (1/\| x \|)) \). This implies that \( \| x \| < \sigma_p(x) \).

(iii) Since \( \sigma_p \) is continuous, by Theorem 1.4 of [12] we directly have (iii).

(iv) This follows from Parts (i) and (iii).

(v) This follows from Parts (ii) and (iii).

Now, we consider the space \( \ell(\tilde{B}, p) \) equipped with the Luxemburg norm given by

\[
\| x \| = \inf \left\{ \alpha > 0 : \sigma_p \left( \frac{x}{\alpha} \right) \leq 1 \right\} .
\]  

(15)

Theorem 7. \( \ell(\tilde{B}, p) \) is a Banach space with Luxemburg norm.

Proof. Let \( S_x = \{ \alpha > 0 : \sigma_p(x/\alpha) \leq 1 \} \) and \( \| x \| = \inf S_x \) for all \( x \in \ell(\tilde{B}, p) \). Then, \( S_x \subset (0, \infty) \). Therefore, \( \| x \| \geq 0 \) for all \( x \in \ell(\tilde{B}, p) \).

For \( x = \theta \), \( \sigma_p(\theta) = 0 \) for all \( \alpha > 0 \). Hence, \( S_0 = (0, \infty) \) and \( \| \theta \| = \inf S_0 = \inf(0, \infty) = 0 \).

Let \( x \neq \theta \) and \( Y = \{ kx : k \in \mathbb{C} \text{ and } x \in \ell(\tilde{B}, p) \} \) be a nonempty subset of \( \ell(\tilde{B}, p) \). Since \( \mathbb{Y} \subset S(\ell(\tilde{B}, p)) \), there exists \( k_1 \in \mathbb{C} \) such that \( k_1x \notin S(\ell(\tilde{B}, p)) \). Obviously, \( k_1 \neq 0 \). We assume that \( 0 < \alpha < 1/k_1 \) and \( \alpha \in S_x \). Then, \( (x/\alpha) \in S(\ell(\tilde{B}, p)) \). Since \( |k_1/\alpha| < 1 \), we get

\[
k_1x = k_1\alpha x = \tilde{x} \in \ell(\tilde{B}, p).
\]  

(16)

which contradicts the assumption. Hence, we obtain that \( \alpha \in S_x \), then \( \alpha > 1/|k_1| \). This means that \( \| x \| \geq 1/|k_1| > 0 \). Thus, we conclude that \( \| x \| = 0 \) if and only if \( x = \theta \).

Now, let \( k \neq 0 \) and \( \alpha \in S_{kx} \). Then, we have

\[
\sigma_p \left( \frac{kx}{\alpha} \right) \leq 1, \quad \frac{kx}{\alpha} \in S \left[ \ell \left( \tilde{B}, p \right) \right].
\]  

(17)

Therefore, we obtain

\[
\frac{|k|}{\alpha} \frac{x}{k} \in S \left[ \ell \left( \tilde{B}, p \right) \right], \quad \frac{\alpha}{|\alpha|} \in S_x.
\]  

(18)

That is, \( \| x \| \leq \alpha/|k| \) and \( \| k \| \| x \| \leq \alpha \) for all \( \alpha \in S_{kx} \). So, \( |k| \| x \| \leq \| k \| \| x \| \).

If we take \( 1/k \) and \( kx \) instead of \( k \) and \( x \), respectively, then we obtain that

\[
\frac{1}{k} \frac{kx}{x} \leq \frac{1}{k} kx = \| x \|, \quad \| kx \| \leq \| k \| \| x \| \leq \| k \| \| x \| .
\]  

(19)

Hence, we get \( \| kx \| = |k| \| x \| \). This also holds when \( k = 0 \).

To prove the triangle inequality, let \( x, y \in \ell(\tilde{B}, p) \) and \( \epsilon > 0 \) be given. Then, there exist \( \alpha \in S_x \) and \( \beta \in S_y \) such that \( \alpha < \| x \| + \epsilon \) and \( \beta < \| y \| + \epsilon \). Since \( S[\ell(\tilde{B}, p)] \) is convex,

\[
\frac{x}{\alpha} + \frac{y}{\beta} \in S \left[ \ell \left( \tilde{B}, p \right) \right],
\]  

(20)

\[
\frac{x+y}{\alpha + \beta} = \frac{\alpha}{\alpha + \beta} \frac{x}{\alpha} + \frac{\beta}{\alpha + \beta} \frac{y}{\beta} \in S \left[ \ell \left( \tilde{B}, p \right) \right].
\]  

(21)}

Therefore, \( \alpha + \beta \in S_{x+y} \). Then, we have \( \| x + y \| \leq \alpha + \beta < \| x \| + \| y \| + 2\epsilon \). Since \( \epsilon > 0 \) was arbitrary, we obtain \( \| x + y \| \leq \| x \| + \| y \| \). Hence, \( \| x \| = \inf \{ \alpha > 0 : \sigma_p(x/\alpha) \leq 1 \} \) is a norm on \( \ell(\tilde{B}, p) \).

Now, we need to show that every Cauchy sequence in \( \ell(\tilde{B}, p) \) is convergent according to the Luxemburg norm. Let \( \{ x^{(n)} \} \) be a Cauchy sequence in \( \ell(\tilde{B}, p) \) and \( \epsilon \in (0, 1) \). Thus, there exists \( n_0 \) such that \( \| x^{(n)} - x^{(m)} \| < \epsilon \) for all \( n, m \geq n_0 \). By Part (i) of Proposition 6, we have

\[
\sigma_p \left( x^{(n)} - x^{(m)} \right) \leq \| x^{(n)} - x^{(m)} \| < \epsilon
\]  

(21)

for all \( n, m \geq n_0 \). This implies that

\[
\sum_k \left\| \tilde{B} \left( x^{(n)} - x^{(m)} \right) \right\|_{k} < \epsilon.
\]  

(22)

Then, for each fixed \( k \) and for all \( n, m \geq n_0 \),

\[
\left\| \tilde{B} \left( x^{(n)} - x^{(m)} \right) \right\|_{k} = \left( B x^{(n)} \right)_{k} - \left( B x^{(m)} \right)_{k} < \epsilon.
\]  

(23)

Hence, the sequence \( \{ \tilde{B} x^{(n)} \} \) is a Cauchy sequence in \( \mathbb{R} \). Since \( \mathbb{R} \) is complete, there is a \( (\tilde{B} x)_{k} \in \mathbb{R} \) such that \( (\tilde{B} x^{(n)})_{k} \rightarrow (\tilde{B} x)_{k} \) as \( m \rightarrow \infty \). Therefore, as \( m \rightarrow \infty \) by (22), we have

\[
\sum_k \left\| \tilde{B} \left( x^{(n)} - x \right) \right\|_{k} < \epsilon
\]  

(24)

for all \( n \geq n_0 \).
Now, we have to show that \((x_k)\) is an element of \(\ell(\tilde{B}, p)\). Since \((\tilde{B}x^{(m)})_k \to (\tilde{B}x)_k\) as \(m \to \infty\), we have
\[
\lim_{m \to \infty} \sigma_p \left( x^{(n)} - x^{(m)} \right) = \sigma_p \left( x^{(n)} - x \right). \tag{25}
\]

Then, we see by \((21)\) that \(\sigma_p (x^{(n)} - x) \leq \|x^{(n)} - x\| < \epsilon\) for all \(n \geq n_0\). This implies that \(x^n \to x\) as \(n \to \infty\). So, we have \(x = x^{(n)} - (x^{(n)} - x) \in \ell(\tilde{B}, p)\). Therefore, the sequence space \(\ell(\tilde{B}, p)\) is complete with respect to Luxemburg norm. This completes the proof.

**Theorem 8.** The space \(\ell(\tilde{B}, p)\) is rotund if and only if \(p_k > 1\) for all \(k \in \mathbb{N}\).

**Proof.** Let \(\ell(\tilde{B}, p)\) be rotund and choose \(k \in \mathbb{N}\) such that \(p_k = 1\) for \(k < 3\). Consider the following sequences given by
\[
x = \left( 0, \frac{1}{r_1}, \frac{-s_1}{r_1 r_2}, \frac{s_1 s_2}{r_1 r_2 r_3}, \ldots \right),
\]
\[
y = \left( 0, 0, \frac{-s_2}{r_2}, \frac{s_2 s_3}{r_2 r_3}, \frac{s_2 s_3 s_4}{r_2 r_3 r_4}, \ldots \right). \tag{26}
\]

Then, obviously \(x \neq y\) and
\[
\sigma_p(x) = \sigma_p(y) = \sigma_p \left( \frac{x + y}{2} \right) = 1. \tag{27}
\]

By Part (iii) of Proposition 6, \(x, y, (x + y)/2 \in S[\ell(\tilde{B}, p)]\) which leads us to the contradiction that the sequence space \(\ell(\tilde{B}, p)\) is not rotund. Hence, \(p_k > 1\) for all \(k \in \mathbb{N}\).

Conversely, let \(x \in S[\ell(\tilde{B}, p)]\) and \(v, z \in S[\ell(\tilde{B}, p)]\) with \(x = (v + z)/2\). By convexity of \(\sigma_p\) and Part (ii) of Proposition 6, we have
\[
1 = \sigma_p(x) \leq \frac{\sigma_p(v) + \sigma_p(z)}{2} \leq \frac{1}{2} + \frac{1}{2} = 1, \tag{28}
\]

which gives that \(\sigma_p(v) = \sigma_p(z) = 1\), and
\[
\sigma_p(x) = \frac{\sigma_p(v) + \sigma_p(z)}{2}. \tag{29}
\]

Also, we obtain from \((29)\) that
\[
\sum_k |s_{k-1} x_{k-1} + r_k x_k|^{p_k} = \frac{1}{2} \left( \sum_k |s_{k-1} v_{k-1} + r_k v_k|^{p_k} \right) + \sum_k |s_{k-1} z_{k-1} + r_k z_k|^{p_k}. \tag{30}
\]

Since \(x = (v + z)/2\), we have
\[
\sum_k |s_{k-1} (v_{k-1} + z_{k-1}) + r_k (v_k + z_k)|^{p_k} = \frac{1}{2} \left( \sum_k |s_{k-1} v_{k-1} + r_k v_k|^{p_k} + \sum_k |s_{k-1} z_{k-1} + r_k z_k|^{p_k} \right). \tag{31}
\]

This implies that
\[
|s_{k-1} (v_{k-1} + z_{k-1}) + r_k (v_k + z_k)|^{p_k} = \frac{1}{2} \left( |s_{k-1} v_{k-1} + r_k v_k|^{p_k} + |s_{k-1} z_{k-1} + r_k z_k|^{p_k} \right). \tag{32}
\]

for all \(k \in \mathbb{N}\). Since the function \(t \mapsto |t|^{p_k}\) is strictly convex for all \(k \in \mathbb{N}\), it follows by \((32)\) that \(v_k = z_k\) for all \(k \in \mathbb{N}\). Hence, \(v = z\). That is, the sequence space \(\ell(\tilde{B}, p)\) is rotund.

**Theorem 9.** Let \(x \in \ell(\tilde{B}, p)\). Then, the following statements hold:

(i) \(0 < \alpha < 1\) and \(\|x\| > \alpha\) imply \(\sigma_p(x) > \alpha^M\).

(ii) \(\alpha > 1\) and \(\|x\| < \alpha\) imply \(\sigma_p(x) < \alpha^M\).

**Proof.** Let \(x \in \ell(\tilde{B}, p)\).

(i) Suppose that \(\|x\| > \alpha\) with \(0 < \alpha < 1\). Then, \(\|x/\alpha\| > 1\). By Part (ii) of Proposition 6, \(\|x/\alpha\| > 1\) implies \(\sigma_p(x/\alpha) \geq \alpha\). That is, \(\sigma_p(x/\alpha) > 1\). Since \(0 < \alpha < 1\), by Part (i) of Proposition 5, we get \(\alpha^M \sigma_p(x/\alpha) < \sigma_p(x)\). Thus, we have \(\alpha^M < \sigma_p(x)\).

(ii) Let \(\|x\| < \alpha\) and \(\alpha > 1\). Then, \(\|x/\alpha\| < 1\). By Part (i) of Proposition 6, \(\|x/\alpha\| < 1\) implies \(\sigma_p(x/\alpha) < \|x\| < \alpha\). That is, \(\sigma_p(x/\alpha) < 1\). If \(\alpha = 1\), then \(\sigma_p(x/\alpha) = \sigma_p(x) < 1 = 1\). If \(\alpha > 1\), then by Part (ii) of Proposition 5, we have \(\sigma_p(x) \leq \alpha^M \sigma_p(x/\alpha)\). This means that \(\sigma_p(x) < \alpha^M\).

**Theorem 10.** Let \((x_n)\) be a sequence in \(\ell(\tilde{B}, p)\). Then, the following statements hold:

(i) \(\lim_{n \to \infty} \|x_n\| = 1\) implies \(\lim_{n \to \infty} \sigma_p(x_n) = 1\).

(ii) \(\lim_{n \to \infty} \sigma_p(x_n) = 0\) implies \(\lim_{n \to \infty} \|x_n\| = 0\).

**Proof.** Let \((x_n)\) be a sequence in \(\ell(\tilde{B}, p)\).

(i) Let \(\lim_{n \to \infty} \|x_n\| = 1\) and \(\epsilon \in (0, 1)\). Then, there exists \(n_0 \in \mathbb{N}\) such that \(1 - \epsilon < \|x_n\| < 1 + \epsilon\) for all \(n \geq n_0\). By Parts (i) and (ii) of Theorem 9, \(1 - \epsilon < \|x_n\|\) implies \(\sigma_p(x_n) > (1 - \epsilon)^M\) and \(\|x_n\| < \epsilon + 1\) implies \(\sigma_p(x_n) < (1 + \epsilon)^M\) for all \(n \geq n_0\). This means \(\epsilon \in (0, 1)\) and for all \(n \geq n_0\), there exists \(n_0 \in \mathbb{N}\) such that \((1 - \epsilon)^M < \sigma_p(x_n) < (1 + \epsilon)^M\). That is, \(\lim_{n \to \infty} \sigma_p(x_n) = 1\).

(ii) We assume that \(\lim_{n \to \infty} \|x_n\| \neq 0\) and \(\epsilon \in (0, 1)\). Then, there exists a subsequence \((x_{n_k})\) of \((x_n)\) such that \(\|x_{n_k}\| > \epsilon\) for all \(k \in \mathbb{N}\). By Part (i) of Theorem 9, \(0 < \epsilon < 1\) and \(\|x_{n_k}\| > \epsilon\) imply \(\sigma_p(x_{n_k}) > \epsilon^M\). Thus, \(\lim_{n \to \infty} \sigma_p(x_{n_k}) \neq 0\) for all \(k \in \mathbb{N}\). Hence, we obtain that \(\lim_{n \to \infty} \sigma_p(x_n) = 0\) implies \(\lim_{n \to \infty} \|x_n\| = 0\).

**Theorem 11.** Let \(x \in \ell(\tilde{B}, p)\) and \((x^{(n)}) \subset \ell(\tilde{B}, p)\). If \(\sigma_p(x^{(n)}) \to \sigma_p(x)\) as \(n \to \infty\) and \(x_k \to x_k\) as \(n \to \infty\) for all \(k \in \mathbb{N}\), then \(x^{(n)} \to x\) as \(n \to \infty\).
Proof. Let \( \epsilon > 0 \) be given. Since \( \sigma_p(x) = \sum_k |(Bx)_k|^{p_k} < \infty \), there exists \( k_0 \in \mathbb{N} \) such that

\[
\sum_{k=k_0+1}^{\infty} |(Bx)_k|^{p_k} < \frac{\epsilon}{3(2^{M+1})}.
\]

(33)

It follows from the fact

\[
\lim_{n \to \infty} \left[ \sigma_p(x^{(n)}) - \sum_{k=1}^{k_0} |(Bx^{(n)})_k|^{p_k} \right] = \sigma_p(x) - \sum_{k=1}^{k_0} |(Bx)_k|^{p_k}
\]

(34)

that there exists \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \) and for all \( k \in \mathbb{N} \),

\[
\sigma_p(x_{n_k}) - \sum_{k=1}^{k_0} |(Bx^{(n)})_k|^{p_k} < \sigma_p(x) - \sum_{k=1}^{k_0} |(Bx)_k|^{p_k} + \frac{\epsilon}{3(2^{M})},
\]

(35)

and for all \( n \geq n_0 \),

\[
\sum_{k=1}^{k_0} |B(x^{(n)} - x)|_k^{p_k} < \frac{\epsilon}{3}.
\]

(36)

Therefore, we obtain from (33), (35), and (36) that

\[
\sigma_p(x_n - x) = \sum_{k=1}^{\infty} |B(x^{(n)} - x)|_k^{p_k} < \frac{\epsilon}{3} + 2^M \left[ \sum_{k=k_0+1}^{\infty} |(Bx^{(n)})_k|^{p_k} + \sum_{k=k_0+1}^{\infty} |(Bx)_k|^{p_k} \right]
\]

\[
\leq \frac{\epsilon}{3} + 2^M \left[ \sum_{k=k_0+1}^{\infty} |(Bx^{(n)})_k|^{p_k} + \sum_{k=k_0+1}^{\infty} |(Bx)_k|^{p_k} \right]
\]

(37)

This means that \( \sigma_p(x^{(n)} - x) \to 0 \) as \( n \to \infty \). By Part (ii) of Theorem 10, \( \sigma_p(x^{(n)} - x) \to 0 \) as \( n \to \infty \) implies \( \|x_n - x\| \to 0 \) as \( n \to \infty \). Hence, \( x_n \to x \) as \( n \to \infty \). \( \Box \)

Theorem 12. The sequence space \( \ell(B, p) \) has the Kadec-Klee property.

Proof. Let \( x \in S[\ell(B, p)] \) and \( (x^{(n)}) \subset \ell(B, p) \) such that \( \|x^{(n)}\| \to 1 \) and \( x^{(n)} \wedge x \) are given. By Part (ii) of Theorem 10, we have \( \sigma_p(x^{(n)}) \to 1 \) as \( n \to \infty \). Also \( x \in S[\ell(B, p)] \) implies \( \|x\| \leq \sigma_p(x) = 1 \). By Part (iii) of Proposition 6, we obtain \( \sigma_p(x) = 1 \). Therefore, we have \( \sigma_p(x^{(n)}) \to \sigma_p(x) \) as \( n \to \infty \).

Since \( x^{(n)} \wedge x \) and \( q_k : \ell(B, p) \to \mathbb{R} \) defined by \( q_k(x) = x_k \) is continuous, \( x^{(n)} \to x \) as \( n \to \infty \) for all \( k \in \mathbb{N} \). Therefore, \( x^{(n)} \to x \) as \( n \to \infty \).

Since any weakly convergent sequence in \( \ell(B, p) \) is convergent, the sequence space \( \ell(B, p) \) has the Kadec-Klee property. \( \Box \)

Theorem 13. For any \( 1 < p < \infty \), the space \( \ell_p(B) \) has the uniform Opial property.

Proof. Let \( \epsilon > 0 \) and \( \epsilon_0 \in (0, \epsilon) \) be given such that \( 1 + (\epsilon_0^p / 2) > (1 + \epsilon_0)^p \). Also let \( x \in (\ell_p(B) \) and \( \|x\| \geq \epsilon \). There exists \( k_1 \in \mathbb{N} \) such that

\[
\sum_{k=k_1+1}^{\infty} |(Bx)_k|^{p_k} < \left( \frac{\epsilon_0}{4} \right)^p.
\]

(38)

Hence, we have

\[
\left\| \sum_{k=k_1+1}^{\infty} x^k \right\| < \frac{\epsilon_0}{4}.
\]

(39)
Furthermore, we have
\[
\|x^{(m)} + x\| \geq \left\| \sum_{k=1}^{k_1} (\bar{B}x)_k e_k + \sum_{k=k_1+1}^{\infty} (\bar{B}x)_k (m) e_k \right\| 
\]
\[
\geq \left\| \sum_{k=1}^{k_1} x_k e_k + \sum_{k=k_1+1}^{\infty} x_k^{(m)} e_k \right\| 
\]
\[
- \left\| \sum_{k=1}^{k_1} x_k^{(m)} e_k \right\| - \left\| \sum_{k=k_1+1}^{\infty} x_k e_k \right\| 
\]
\[
\geq \left\| \sum_{k=1}^{k_1} x_k e_k + \sum_{k=k_1+1}^{\infty} x_k^{(m)} e_k \right\| - \frac{\epsilon^p}{4} - \frac{\epsilon^p}{4}. 
\]

Moreover,
\[
\left\| \sum_{k=1}^{k_1} x_k e_k + \sum_{k=k_1+1}^{\infty} x_k^{(m)} e_k \right\|^p 
\]
\[
= \sum_{k=1}^{k_1} (\bar{B}x)_k e_k |^p + \sum_{k=k_1+1}^{\infty} (\bar{B}x)_k (m) e_k |^p 
\]
\[
\geq \frac{3\epsilon^p}{4} + \left( 1 - \frac{\epsilon^p}{4} \right) 
\]
\[
= 1 + \frac{\epsilon^p}{2} 
\]
\[
> (1 + \epsilon_0)^p. 
\]

Then, we have
\[
\|x^{(m)} + x\| \geq \left\| \sum_{k=1}^{k_1} x_k e_k + \sum_{k=k_1+1}^{\infty} x_k^{(m)} e_k \right\| - \frac{\epsilon^p}{2} 
\]
\[
\geq 1 + \epsilon_0 - \frac{\epsilon^p}{2} 
\]
\[
> 1 + \epsilon_0. 
\]

This means that \((\ell_p)_B\) has the uniform Opial property. \(\square\)

3. Conclusion

The sequence spaces \(bv(u, p)\) and \(bv_{\text{co}}(u, p)\) of nonabsolute type consisting of all sequences \(x = (x_k)\) such that \(|u_k(x_k - x_{k-1})|\) is in the Maddox’ spaces \(\ell(p)\) and \(\ell_{\text{co}}(p)\) were introduced by Başar et al. [13], where \(u = (u_k)\) is a sequence such that \(u_k \neq 0\) for all \(k \in \mathbb{N}\) and the rotundity of the space \(bv(u, p)\) was examined.

The sequence space \(a'(u, p)\) of nonabsolute type consisting of all sequences \(x = (x_k)\) such that \(a'(x_k)\) is a sequence \(x_k = (\sum_{k=0}^{n} (1 + \frac{r^k}{n+1}) x_k(n + 1) \in \ell(p)\) was studied by Aydın and Başar [14], and some results related to the rotundity of the space \(a'(u, p)\) were given.

Quite recently, the sequence space \(\ell(p)\) of nonabsolute type consisting of all sequences \(x = (x_k)\) such that \(B(r, s)x = (s_k x_k + r x_k) \in \ell(p)\) was defined by Aydın and Başar [15], and emphasized the rotundity of the space \(\ell(p)\) together with some related results.

Although the sequence spaces \(a'(u, p)\) and \(\ell(B, p)\) are not comparable, since the double sequential band matrix \(B(r, s)\) reduces to the generalized difference matrix \(B(r, s)\) in the special case \(\bar{r} = re\) and \(\bar{s} = se\), the new space \(\ell(B, p)\) is more general than the space \(\ell(p)\). Similarly, the sequence space \(\ell(B, p)\) is also reduced to the space \(bv(u, p)\) in the case \(\bar{r} = (u_k)\) and \(\bar{s} = (-u_k)\). So, the results on the space \(\ell(B, p)\) are much more comprehensive than the results on the space \(bv(u, p)\). Additionally, the corresponding theorems on the Kadec-Klee property of the space \(\ell(B, p)\) and the uniform Opial property of the space \((\ell_p)_B\) were not given by Başar et al. [13] and Aydın and Başar [15] which make the present paper significant.

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References


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