Research Article

Analysis of Stability of Traveling Wave for Kadomtsev-Petviashvili Equation

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This paper presents the boundedness and uniform boundedness of traveling wave solutions for the Kadomtsev-Petviashvili (KP) equation. They are discussed by means of a traveling wave transformation and Lyapunov function.

1. Introduction

We consider the Kadomtsev-Petviashvili (KP) equation:

\[ u_t + 6u_x u_{xx} + u_{xxx} + u_y + cu = 0. \]  

(1)

It is well known that Kadomtsev-Petviashvili equation arises in a number of remarkable nonlinear problems both in physics and mathematics. By using various methods and techniques, exact traveling wave solutions, solitary wave solutions, doubly periodic solutions, and some numerical solutions have been obtained in [1–6].

In this paper, (1) can be changed into an ordinary differential equation by using traveling wave transformation; the boundedness and uniform boundedness of solution for the resulting ordinary differential equation are discussed using the method of Lyapunov function.

2. The Boundedness

Taking a traveling wave transformation \( \xi = \alpha x + \beta y + \gamma t \) in (1), then (1) can be transformed into the following form:

\[ u^{(4)} + \left( \frac{\gamma}{\alpha^2} + \frac{\beta^2}{\alpha^4} + \frac{6}{\alpha^2} u \right) u'' + \frac{6}{\alpha^2} u'^2 + \frac{c}{\alpha^2} u = 0. \]  

(2)

In general, we use the following system, which is equivalent to (2):

\[ u^{(4)} + au''' + f(t, u, u'') + g(u') + du = p(t, u, u', u'', u'''), \]  

(3)

where

\[ f(t, u, u') = \left( \frac{\gamma}{\alpha^2} + \frac{\beta^2}{\alpha^4} + \frac{6}{\alpha^2} u \right) u'', \quad g(u') = \frac{6}{\alpha^2} u'^2, \]

\[ p(t, u, u', u'', u''') = -au'', \quad d = \frac{c}{\alpha^2}. \]  

(4)

We consider the following system, which is equivalent to (3):

\[ x_1' = x_2, \quad x_2' = x_3, \quad x_3' = x_4, \]

\[ x_4' = -ax_4 - f(t, x_1, x_2, x_3) - g(x_2) - dx_1 \]

\[ + p(t, x_1, x_2, x_3, x_4). \]  

(5)

Theorem 1. If the following conditions hold for the system (5):

(i) there are positive constants \( a, b, d, \delta, k, \) and \( \lambda \) such that

\[ k \leq b^2 \lambda, \quad b^2 a \frac{g(x_2)}{x_2} - \left[ \frac{g(x_2)}{x_2} \right]^2 - a^2 d \geq \delta, \quad (x_2 \neq 0). \]  

(6)
(ii) \( f(t, x_1, x_2, 0) = 0, 0 \leq f(t, x_1, x_2, x_3)/x_3 - b \leq 2\delta\lambda/k \) (\( x_2 \neq 0 \)).

(iii) \( x_3 f_1(t, x_1, x_2, x_3) + x_2 f_2(t, x_1, x_2, x_3) + x_3 f_3(t, x_1, x_2, x_3) \leq 0 \).

(iv) \( |p(t, x_1, x_2, x_3, x_4)| \leq q(t)(\lambda x_1^2 + \lambda x_2^2 + \lambda x_3^2 + x_4^2)^{1/2} \), where \( q(t) \) is a nonnegative continuous function and \( \int_0^\infty q(t)dt < \infty \).

Then, all the solutions of system (5) are bounded.

Proof. We first construct the Lyapunov function \( V = V(t, x_1, x_2, x_3, x_4) \) defined by

\[
V = b^2 (2x_4 + ax_3 + bx_2)^2 + 2bd(2x_3 + ax_2 + bx_1)^2 \\
+ \left( b^2 - 4d \right) (ax_4 + bx_2)^2 + 4ab^2 \\
\times \int_0^{x_3} \left[ \frac{g(x_2)}{x_2} - \frac{ad}{b} \right] x_2dx_2 \\
+ 4b^2 \int_0^{x_3} \left[ \frac{f(t, x_1, x_2, x_3)}{x_3} - b \right] x_3dx_3.
\] (7)

It follows from conditions (i) and (ii) that

\[
b^2 - 4d \geq 0, \quad 0 \leq \int_0^{x_3} \left[ \frac{g(x_2)}{x_2} - \frac{ad}{b} \right] x_2dx_2 \leq \frac{a(b^2 - d)}{2b} x_2^2, \quad 0 \leq \int_0^{x_3} \left[ \frac{f(t, x_1, x_2, x_3)}{x_3} - b \right] x_3dx_3 \leq \frac{\delta\lambda}{k} x_3^2.
\] (8)

Summing up the above discussions, we get

\[
V \geq 2b \left( b^2 - 4d \right) x_2^2 + 4adx_2^2.
\] (9)

Taking the total derivative of (7) with respect to \( t \) along the trajectory of (5), we obtain

\[
\frac{dV}{dt} = -2ab^2 \left[ x_4 + \frac{1}{a} g(x_2)^2 \right] \\
- 2b^3 x_3 \left[ \frac{f(t, x_1, x_2, x_3)}{x_3} - b \right] x_3^2 - 2b^2 \\
\times \left[ \frac{a b}{x_2} g(x_2) - g^2(x_2) - a^2 d \right] x_2^2 \\
+ 4b^2 \left( x_3 + \int_0^{x_3} f_1(t, x_1, x_2, x_3) dx_3 \right) \\
+ 2b^2 (bx_2 + ax_3 + 2x_4) p(t, x, x_2, x_3, x_4).
\] (11)

By using conditions (i) and (iii), it follows that

\[
\frac{dV}{dt} \leq -2b^2 \delta \frac{x_4}{a} - 2b^3 x_3 \left[ \frac{f(t, x_1, x_2, x_3)}{x_3} - b \right] \\
- 2ab^2 \left[ \frac{f(t, x_1, x_2, x_3)}{x_3} - b \right] x_3^2 \\
+ 2b^2 (bx_2 + ax_3 + 2x_4) p(t, x, x_2, x_3, x_4).
\] (12)

According to (ii), we have

\[
2b^3 x_3 \left[ \frac{f(t, x_1, x_2, x_3)}{x_3} - b \right] \\
+ 2ab^2 \left[ \frac{f(t, x_1, x_2, x_3)}{x_3} - b \right] x_3^2 \\
= -\frac{b^4}{2a} \left[ \frac{f(t, x_1, x_2, x_3)}{x_3} - b \right] x_3^2 \\
+ 2b^2 \left[ \frac{f(t, x_1, x_2, x_3)}{x_3} - b \right] \left( ax_3 + \frac{b}{2} x_2^2 \right)^2 \\
\geq -\frac{b^4}{2a} \left[ \frac{f(t, x_1, x_2, x_3)}{x_3} - b \right] x_3^2 \\
= -\frac{b^4}{2a} \frac{2\delta\lambda}{k} x_3^2 = -\frac{b^4}{2a} \frac{\delta\lambda}{ak} x_3^2.
\] (13)
Hence,
\[
\frac{dV}{dt} \leq -\frac{2b^2\delta}{a}x_2^2 + \frac{b^4\delta}{a}x_2^2
\]
\[
+ 2b^2 (bx_2 + ax_3 + 2x_4) p(t, x_2, x_3, x_4)
\]
\[
= -\frac{b^2\delta}{a}x_2^2 + \frac{b^2\lambda}{ak} (b^2\lambda - k)x_2^2
\]
\[
+ 2b^2 \left(4 + a^2 + b^2\right)^{1/2} \left(x_2^2 + x_3^2 + x_4^2\right)^{1/2}
\]
\[
\times \left(x_2^2 + x_3^2 + x_4^2\right)^{1/2} q(t)
\]
\[
\leq -\frac{b^2\delta}{a}x_2^2 + 2b^2 \left(4 + a^2 + b^2\right)^{1/2} \left(x_2^2 + x_3^2 + x_4^2\right) q(t)
\]
\[
\leq -\frac{b^2\delta}{a}x_2^2 + 2b^2 \left(4 + a^2 + b^2\right)^{1/2} \cdot q(t) \cdot \frac{V}{\epsilon}
\]
\[
\leq 2b^2 \left(4 + a^2 + b^2\right)^{1/2} \cdot \frac{q(t)}{\epsilon} \cdot V \equiv \varphi (V, t).
\]
(14)

Thus, all the solutions of system (5) are bounded.

Theorem 2. Let conditions (i)-(iv) of Theorem 1 be satisfied for the system (5), and let the following condition hold:
\[
\left(4 + a^2 + b^2\right)^{1/2} \cdot \frac{q(t)}{\epsilon} \cdot V - \frac{\delta}{a}x_2^2 \leq 0.
\]
(15)

Then, all the solutions of system (5) are uniformly bounded.

Proof. It is clear that the function \(V(t, x_1, x_2, x_3, x_4)\) defined in (7) satisfies the conditions (15), therefore, all the solutions of system (5) are uniformly bounded [7].

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References
