Research Article
Various Heteroclinic Solutions for the Coupled Schrödinger-Boussinesq Equation

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Various closed-form heteroclinic breathersolutions including classical heteroclinic, heteroclinic breather and Akhmediev breathersolutions for coupled Schrödinger-Boussinesq equation are obtained using two-soliton and homoclinic test methods, respectively. Moreover, various heteroclinic structures of waves are investigated.

1. Introduction

The existence of the homoclinic and heteroclinic orbits is very important for investigating the spatiotemporal chaotic behavior of the nonlinear evolution equations (NEEs). In recent years, exact homoclinic and heteroclinic solutions were proposed for some NEEs like nonlinear Schrödinger equation, Sine-Gordon equation, Davey-Stewartson equation, Zakharov equation, and Boussinesq equation [1–7].

The coupled Schrödinger-Boussinesq equation is considered as

\[
\begin{align*}
\dot{E} + E_{xx} + \beta_1 E - NE &= 0, \\
3N_{tt} - N_{xxxx} + 3(N^2)_{xx} + \beta_2 N_{xx} - (|E|^2)_{xx} &= 0, \\
\end{align*}
\]

(1)

with the periodic boundary condition

\[
E(x, t) = E(x + l, t), \quad N(x, t) = N(x + l, t), \quad (2)
\]

where \(l, \beta_1, \beta_2\) are real constants, \(E(x, t)\) is a complex function, and \(N(x, t)\) is a real function. Equation (1) has also appeared in [8] as a special case of general systems governing the stationary propagation of coupled nonlinear upper-hybrid and magnetosonic waves in magnetized plasma. The complete integrability of (1) was studied by Chowdhury et al. [9], and \(N\)-soliton solution, homoclinic orbit solution, and rogue solution were obtained by Hu et al. [10], Dai et al. [11–13], and Mu and Qin [14].

2. Linear Stability Analysis

It is easy to see that \((e^{i\theta_0}, \beta_1)\) is a fixed point of (1), and \(\theta_0\) is an arbitrary constant. We consider a small perturbation of the form

\[
\begin{align*}
E &= e^{i\theta_0} (1 + \epsilon), \\
N &= \beta_1 (1 + \phi), \\
\end{align*}
\]

(3)

where \(|\epsilon(x, t)| \ll 1, |\phi(x, t)| \ll 1\). Substituting (3) into (1), we get the linearized equations

\[
\begin{align*}
\dot{\epsilon} + \epsilon_{xx} - \beta_1 \phi &= 0, \\
3\dot{\phi}_{tt} - \phi_{xxxx} + (\beta_2 + 2\beta_1^2) \phi_{xx} - \epsilon_{xx} - \epsilon_{xx} &= 0. \\
\end{align*}
\]

(4)

Assume that \(\epsilon\) and \(\phi\) have the following forms:

\[
\begin{align*}
\epsilon &= Ge^{i\mu x + \sigma x}, \\
\phi &= C e^{i\mu x + \sigma x}, \\
\end{align*}
\]

(5)

where \(G, H\) are complex constants, and \(C\) is a real number; \(\mu_n = 2m\pi/l\), and \(\sigma_n\) is the growth rate of the \(n\)th modes.
Substituting (5) into (4), we have
\[ G (i\sigma_n - \mu_n^2) = \beta_1 C, \]
\[ H (i\sigma_n - \mu_n^2) = \beta_1 C, \]
\[ (3\sigma_n^2 - \mu_n^2 - \mu_n^2 (\beta_2 + 2\beta_1^2)) C = -(G + H) v_n^2, \]
\[ (3\sigma_n^2 - \mu_n^2 - \mu_n^2 (\beta_2 + 2\beta_1^2)) C = -(H + G) \mu_n^2. \]
Solving (6), we obtain that
\[ \sigma_n^2 = \frac{\mu_n^2 (\beta_2 + 2\beta_1^2) - 2\mu_n^4 \pm \sqrt{\Delta}}{6}, \]
with
\[ \Delta = 4\mu_n^8 + \mu_n^4 (\beta_2 + 2\beta_1^2)^2 - 4\mu_n^2 (\beta_2 + 2\beta_1^2) \]
\[ + 12\mu_n^4 (\mu_n^2 (\beta_2 + 2\beta_1^2) - 2\beta_1). \]
Obviously, (7) implies that \( \mu_n^2 (\beta_2 + 2\beta_1^2) - 2\mu_n^4 > 0 \); then,
\[ \mu_n^2 < \frac{\beta_2 + 2\beta_1^2}{2}. \]

### 3. Various Heterclinic Breather Solutions

Set
\[ E (x, t) = e^{-i\sigma t} u (x, t), \quad N (x, t) = v_0 + v (x, t). \]
Substituting (10) into (1), we get
\[ i u_x + u_{xx} + (a + \beta_1 - v_0) u = u v, \]
\[ 3 v_x - v_{xxx} + (6 v_0 + \beta_2) v_x + 3 (v^2)_x x = (|u|^2)_{xx}. \]
Equation (11) can be reduced into the following bilinear form:
\[ (iD_t + D_x^3) g \cdot f = 0, \]
\[ (3D_x^2 + 6v_0 + \beta_2) D_x^2 - D_x^4 - \lambda) f \cdot f + g g^* = 0, \]
where \( g(x, t) \) is an unknown complex function and \( f(x, t) \) is a real function, \( g^* \) is conjugate function of \( g(x, t) \), and \( \lambda \) is an integration constant. The Hirota bilinear operators \( D_x^m D_t^n \) are defined by
\[ D_x^m D_t^n f (x, t) \cdot g (x, t) \]
\[ = \left( \frac{\partial}{\partial x} \right)^m \left( \frac{\partial}{\partial t} \right)^n \left[ f (x, t) g (x', t') \right]_{x = x', t = t}. \]

We use three test functions to investigate the variation of the heterclinic solution for the coupled Schrödinger-Boussinesq equation (1). (1) We seek the following forms of the heterclinic solution:
\[ g = 1 + b_1 \cos (px) e^{i\Omega t + \gamma} + b_2 e^{2i\Omega t + 2\gamma}, \]
\[ f = 1 + b_3 \cos (px) e^{i\Omega t + \gamma} + b_4 e^{2i\Omega t + 2\gamma}, \]
where \( b_1, b_2 \) are complex numbers and \( b_3, b_4 \) are real numbers. \( b_i \) \((i = 1, 2, 3, 4)\) will be determined later.
Choosing \( v_0 = \beta_1 \), then \( a = 0 \). Substituting (15) into the (13), we have the following relations among these constants:
\[ \lambda = 1, \quad b_i = \frac{i\Omega + p^2}{i\Omega - p^2} b_i, \]
\[ b_1 = \frac{4\beta_1 - 6\beta_1 + 6\beta_2 + 6\beta_2^2}{4\beta_1^2 - 6\beta_1 + 6\beta_2 + 6\beta_2^2} \]
\[ b_2 = (i\Omega + p^2)^2 b_1, \quad b_3 = \frac{4\beta_1 - 6\beta_1 + 6\beta_2 + 6\beta_2^2}{4\beta_1^2 - 6\beta_1 + 6\beta_2 + 6\beta_2^2} b_2. \]

Therefore, we have the heterclinic solution for (1) as:
\[ E (x, t) = \frac{e^{i\Omega t + \gamma} + b_1 \cos (px) + b_2 e^{i\Omega t + \gamma}}{\sqrt{b_1} (2 \cos (\Omega t + \gamma + \ln \sqrt{b_1} + b_2 \cos (px))}, \]
\[ N (x, t) = b_1 \frac{2b_3 p^2 (2 \sqrt{b_1} \cos (px) \cos (\Omega t + \gamma + \ln \sqrt{b_1} + b_2))}{b_4 (2 \cos (\Omega t + \gamma + \ln \sqrt{b_1} + b_3 \cos (px)))^2}. \]

It is easy to see that \((E, N) \rightarrow (1, \beta_1)\) as \( t \rightarrow -\infty \) and \((E, N) \rightarrow ((i\Omega + p^2)/(i\Omega - p^2))^2, \beta_1)\) as \( t \rightarrow +\infty \). After giving some constants in (17), we find that the shape of the heterclinic orbit for Schrödinger-Boussinesq equation lies the hook, and the orbits are heterclinic to two different fixed points (see Figure 1 with \( \beta_1 = 1, \beta_2 = -2, p = 1, \) and \( \gamma = 1 \)).

(2) We take ansatz of extended homoclinic test approach for (13) as follows:
\[ f (x, t) = e^{-p_i (x - \alpha t) - \eta_i} + b_3 \cos (p (x + \alpha t) + \eta_i) \]
\[ + b_4 e^{p_i (x - \alpha t) + \eta_i}, \]
\[ g (x, t) = e^{-\theta_i} (e^{-p_i (x - \alpha t) - \eta_i} + b_3 \cos (p (x + \alpha t) + \eta_i) \]
\[ + b_4 e^{p_i (x - \alpha t) + \eta_i}) \]
\[ = e^{-i\sigma_t} u (x, t), \quad v_0 = \beta_1, \]
we get the following relations among the parameters:
and in (21), this kind of heterclinic orbit looks like a spiral, and it is heterclinic to the points

It is easy to see that

where

\[ E(x, t) = e^{i \theta_0} \sinh(p_1(x - \alpha t) + \eta_0 + \ln(\sqrt{-b_4})) \cos(p(x + \alpha t) + \eta_1) + b_3 \cos(p(x + \alpha t) + \eta_1), \]

\[ N(x, t) = b_1 - \frac{8 \sqrt{-b_4} b_2 p_1^2 \sinh(p_1(x - \alpha t) + \eta_0 + \ln(\sqrt{-b_4})) \cos(p(x + \alpha t) + \eta_1) \left(2 \sqrt{-b_4} \sinh(p_1(x - \alpha t) + \eta_0 + \ln(\sqrt{-b_4})) - b_3 \cos(p(x + \alpha t) + \eta_1)\right)^2}{2 \sqrt{-b_4} \cosh(p_1(x - \alpha t) + \eta_0 + \ln(\sqrt{-b_4})) - b_3 \cos(p(x + \alpha t) + \eta_1)^2}, \]

From (19), we get the restrictive conditions with

\[-\sqrt{2} < \beta_2 + 6 \beta_1 < 0, \quad b_4 < 0. \tag{20}\]

Denote that \((ia - 2p_1)/(ia + 2p_1) = e^{i \theta_0}. Then, substituting (10) into (1) and employing (19), we obtain the solution of the coupled Schrödinger-Boussinesq equation as follows:

\[ g = b_1 \cosh(\alpha t) + b_2 \cos(px) + b_3 \sinh(\alpha t), \]

\[ f = b_4 \cosh(\alpha t) + b_5 \cos(px), \tag{22}\]

where \(b_1, b_2, b_3\) are complex numbers and \(b_4, b_5\) are real numbers. \(b_i (i = 1, 2, 3, 4, 5), p, \alpha\) will be determined later.
We also choose \( v_0 = \beta_1 \) and substitute (22) into (13). We have the following relations among these constants:

\[
ib_3b_\alpha = b_2 b_4 p^2, \\
b_5 (b_1 + b_3) (i\alpha - p^2) = b_2 b_4 (i\alpha + p^2), \\
b_2 b_4 (i\alpha - p^2) = b_5 (b_1 - b_3) (i\alpha + p^2), \\
- b_4^2 + 12\alpha^2 b_4^2 - 2b_5^2 \cos^2 (px) - 16b_2^2 p^4 - 4b_2^2 p^2 (6\beta_1 + \beta_2) + b_1 b_3 - b_2 b_4 + 2b_2 b_4^2 \cos^2 (px) = 0.
\]

Solving (23), we get

\[
b_1 = \frac{(p^4 - \alpha^2) b_2}{\alpha \sqrt{2 (\alpha^2 + p^4)}}, \quad b_3 = \pm i \frac{\sqrt{2} p^2 b_2}{\sqrt{\alpha^2 + p^4}}, \\
b_4^2 = \frac{(\alpha^2 + p^4) b_5^2}{2\alpha^2}.
\]

Therefore, we have the heterclinic solution for (1) as

\[
E (x, t) = \frac{b_1 \cosh (\alpha t) + b_2 \cos (px) + b_3 \sinh (\alpha t)}{b_4 \cosh (\alpha t) + b_5 \cos (px)}, \\
N (x, t) = \beta_1 + 2 \frac{b_2 p^2 (b_3 \cos (px) \cosh (\alpha t) + b_5)}{(b_1 \cosh (\alpha t) + b_5 \cos (px))^2}.
\]

Giving some special parameters in (25), we see that the shape of the heterclinic orbits looks like the arc (see Figure 4 with \( \beta_1 = 1, \alpha = \sqrt{3}, \) and \( p = \sqrt{2} \)). The fixed points are \((E, N) \rightarrow ((b_1 - b_3)/b_4, \beta_1)\) as \( t \rightarrow -\infty \) and \((E, N) \rightarrow ((b_1 + b_3)/b_4, \beta_1)\) as \( t \rightarrow +\infty \).

4. Conclusion

In this work, by using three special test functions in two-soliton method and homoclinic test method, we obtain three families of heteroclinic breather wave solution heteroclinic to two different fixed points, respectively. Moreover, we investigate different structures of these wave solutions. These results show that the Schrödinger-Boussinesq equation has the variety of heteroclinic structure. As the further work, we
will consider whether there exist the spatiotemporal chaos for the coupled Schrödinger-Boussinesq equation or not.

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References


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