Research Article

Ψ-Stability of Nonlinear Volterra Integro-Differential Systems with Time Delay

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We give some sufficient conditions for Ψ-uniform stability of the trivial solutions of a nonlinear differential system and of nonlinear Volterra integro-differential systems with time delay.

1. Introduction

Akiyama [1] introduced the notion of Ψ-stability of the degree k with respect to a function Ψ ∈ C(ℝ+, ℝ+), increasing and differentiable on ℝ and such that Ψ(t) ≥ 1 for t ≥ 0 and limt→∞ Ψ(t) = b, b ∈ [1, ∞). Constantin [2] introduced the notions of degree of stability and degree of boundedness of solutions of an ordinary differential equation, with respect to a continuous positive and nondecreasing function Ψ : ℝ+ → ℝ+; some criteria for these notions are proved there too.

Morchało [3] introduced the notions of Ψ-stability, Ψ-uniform stability, and Ψ-asymptotic stability of trivial solution of the nonlinear system x′ = f(t, x). Several new and sufficient conditions for the mentioned types of stability are proved for the linear system x′ = A(t)x; in this paper Ψ is a scalar continuous function. In [4, 5], Diamandescu gives some sufficient conditions for Ψ-asymptotic stability and Ψ-(uniform) stability of the nonlinear Volterra integro-differential system x′ = A(t)x + ∫₀ᵗ F(t, s, x(s))ds; in these papers Ψ is a matrix function. Furthermore, in [6], sufficient conditions are given for the uniform Lipschitz stability of the system x′ = f(t, x) + g(t, x).

In paper [7], for the nonlinear system

y′ = f(t, y) + g(t, y)

and the nonlinear Volterra integro-differential system

z′ = f(t, z) + ∫₀ᵗ F(t, s, z(s))ds,

by using the knowledge of fundamental matrix and nonlinear variation of constants, we give some sufficient conditions for Ψ-(uniform) stability of trivial solution for the system. The purpose of this paper is to provide sufficient conditions for Ψ-uniform stability of trivial solutions for the nonlinear delayed system

x′(t) = f(t, x(t)) + g(t, x(t − τ(t)))

and the nonlinear delayed Volterra integro-differential systems

x′(t) = f(t, x(t)) + g(t, x(t − τ(t)))

+ p(t, x(t)) ∫₀ᵗ q(s, x(s − τ(s)))ds,

x′(t) = f(t, x(t)) + g(t, x(t − τ(t)))

+ p(t, x(t − τ(t))) ∫₀ᵗ q(s, x(s))ds,

where f, g, p, q ∈ C(ℝ+ × ℝ⁺, ℝ⁺), f(t, 0) = g(t, 0) = p(t, 0) = q(t, 0) = 0 for t ∈ ℝ⁺, and τ ∈ C¹(ℝ⁺, ℝ⁺) with
\( \tau(t) \leq t \) on \( \mathbb{R}_+ \). The systems studied in [7] do not include time delay, whereas all the systems studied in this paper have time delay.

In this paper, we investigate conditions on the functions \( f, g, p, q \) under which the trivial solutions of systems (3), (4), and (5) are \( \Psi \)-stability on \( \mathbb{R}_+ \); the main tool used is the integral inequalities and the integral technique. Here \( \Psi \) is a matrix function whose introduction allows us to obtain a mixed behavior for the components of solutions.

Let \( \mathbb{R}^n \) denote the Euclidean \( n \)-space. For \( x = (x_1, x_2, x_3, \ldots, x_n)^T \in \mathbb{R}^n \), let \( \|x\| = \max \{|x_1|, |x_2|, \ldots, |x_n|\} \) be the norm of \( x \). For a \( n \times n \) matrix \( A = (a_{ij}) \), we define the norm \( |A| = \sup_{\|x\| \leq 1} \|Ax\| \). It is well known that

\[
|A| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| \quad \text{(6)}
\]

Let \( \Psi_t : \mathbb{R}_+ \to (0, \infty), \; i = 1, 2, \ldots, n \), be continuous functions and \( \Psi = \text{diag} \{\Psi_1, \Psi_2, \ldots, \Psi_n\} \).

Now we give the definitions of \( \Psi \)-(uniform) stability that we will need in the sequel.

**Definition 1** (see [4, 8]). The trivial solution of (3) ((4) or (5)) is said to be \( \Psi \)-stable on \( \mathbb{R}_+ \) if for every \( \varepsilon > 0 \) and any \( t_0 \in \mathbb{R}_+ \), there exists \( \delta = \delta(\varepsilon, t_0) > 0 \) such that any solution \( x(t) \) of (3) ((4) or (5)), which satisfies the inequality \( \|\Psi(t_0)x(t_0)\| < \delta \), exists and satisfies the inequality \( \|\Psi(x(t))\| < \varepsilon \) for all \( t \geq t_0 \).

**Definition 2** (see [4, 8]). The trivial solution of (3) ((4) or (5)) is said to be \( \Psi \)-uniformly stable on \( \mathbb{R}_+ \) if it is \( \Psi \)-stable on \( \mathbb{R}_+ \) and the previous \( \delta \) is independent of \( t_0 \).

### 2. \( \Psi \)-Stability of the Systems

To prove our theorems, we need the following lemmas.

**Lemma 3.** Let \( h, k, p, q \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+) \) with \( (t, s) \mapsto \partial_t h(t, s), \partial_k h(t, s), \partial_t p(t, s), \partial_t q(t, s) \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+) \).

Assume, in addition, that \( b \in C(\mathbb{R}_+, \mathbb{R}_+) \) and \( \alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+) \) are nondecreasing functions and \( \alpha(t) \leq t \) for \( t \geq 0 \). If \( u \in C(\mathbb{R}_+, \mathbb{R}_+) \) satisfies

\[
\begin{align*}
    u(t) &\leq b(t) + \int_0^t h(t, s) u(s) ds + \int_0^{\alpha(t)} k(t, s) u(s) ds \\
    &\quad + \int_0^t p(t, s) u(s) \left( \int_0^{\alpha(s)} q(s, v) u(v) dv \right) ds,
\end{align*}
\]

for \( t \geq 0 \), and \( b(t) \int_0^t R(s)Q(s) ds < 1 \), then

\[
    u(t) \leq \frac{b(t) Q(t)}{1 - b(t) \int_0^t R(s)Q(s) ds}, \quad t \geq 0,
\]

where \( Q(t) = \exp(\int_0^t h(t, s) ds + \int_0^{\alpha(t)} k(t, s) ds), R(t) = (d/dt) \int_0^t p(t, s) \left( \int_0^{\alpha(s)} q(s, v) dv \right) ds. \)

**Proof.** Let \( T \geq 0 \) be fixed and denote

\[
    x(t) = \int_0^t h(t, s) u(s) ds + \int_0^{\alpha(t)} k(t, s) u(s) ds \\
    + \int_0^t p(t, s) u(s) \left( \int_0^{\alpha(s)} q(s, v) u(v) dv \right) ds, \quad t \geq 0,
\]

then \( u(t) \leq b(t) + x(t) \), and \( x \) is nondecreasing on \( \mathbb{R}_+ \). For \( t \in [0, T] \), by calculations we get the following:

\[
    x'(t) = \left[ h(t, t) u(t) + \int_0^t \partial_t h(t, s) u(s) ds \right] \\
    + \left[ k(t, \alpha(t)) u(\alpha(t)) \alpha'(t) + \int_0^{\alpha(t)} \partial_t k(t, s) u(s) ds \right] \\
    + \left[ p(t, t) u(t) + \int_0^{\alpha(t)} \partial_t p(t, s) u(s) ds \right] \\
    + \left[ \int_0^{\alpha(t)} \partial_t q(t, s) u(s) \left( \int_0^{\alpha(s)} q(s, v) u(v) dv \right) ds \right] \\
    \leq \left[ b(T) + x(t) \right] \left[ \frac{d}{dt} \left( \int_0^t h(t, s) ds + \int_0^{\alpha(t)} k(t, s) ds \right) \right] \\
    + \left[ b(T) + x(t) \right] \frac{d}{dt} \int_0^t p(t, s) \left( \int_0^{\alpha(s)} q(s, v) dv \right) ds.
\]

Suppose that \( b(0) > 0 \) (if \( b(0) = 0 \), carry out the following arguments with \( b(t) + \varepsilon \) instead of \( b(t) \), where \( \varepsilon > 0 \) is an arbitrary small constant, and subsequently pass to the limit as \( \varepsilon \to 0 \) to complete the proof), then we get

\[
    \frac{x'(t)}{[b(T) + x(t)]^2} = 1 - \frac{1}{b(T) + x(t)} \frac{d}{dt} \left( \int_0^t h(t, s) ds + \int_0^{\alpha(t)} k(t, s) ds \right) \\
    \leq \frac{d}{dt} \int_0^t p(t, s) \left( \int_0^{\alpha(s)} q(s, v) dv \right) ds.
\]

Let

\[
    x(t) = \frac{1}{b(T) + x(t)}, \\
    q(t) = 1 - b(T) + x(t), \\
    Q(t) = \exp(q(t)) \\
    = \exp\left( \int_0^t h(t, s) ds + \int_0^{\alpha(t)} k(t, s) ds \right), \\
    R(t) = \frac{d}{dt} \int_0^t p(t, s) \left( \int_0^{\alpha(s)} q(s, v) dv \right) ds,
\]

\[
    \frac{x'(t)}{[b(T) + x(t)]^2} = \frac{d}{dt} \left( \int_0^t h(t, s) ds + \int_0^{\alpha(t)} k(t, s) ds \right).
\]
then, we have
\[ z'(t) + z(t) \left( \frac{d}{dt} q(t) \right) \geq -R(t). \] (13)

Multiplying the above inequality by \( e^{\sigma(t)} = Q(t) \), we get
\[ \frac{d}{dt} \left( e^{\sigma(t)} Q(t) f(t) \right) \geq -Q(t) R(t). \] (14)

Consider now the integral on the interval \([0,t]\) to obtain
\[ z(t) Q(t) \geq z(0) - \int_0^t Q(s) R(s) ds, \quad 0 \leq t \leq T, \] (15)
so
\[ z(t) = \frac{1}{b(T) + x(t)} \]
\[ \geq \left[ \frac{1}{b(T)} - \int_0^t Q(s) R(s) ds \right] \frac{1}{Q(t)} \]
\[ = \frac{1 - b(T) \int_0^t Q(s) R(s) ds}{b(T) Q(t)} \] (16)
for \( 0 \leq t \leq T \). Let \( t = T \), since \( b(T) \int_0^T Q(s) R(s) ds < 1 \), then we have
\[ b(T) + x(T) \leq \frac{b(T) Q(T)}{1 - b(T) \int_0^T Q(s) R(s) ds}. \] (17)
Since \( T \geq 0 \) was arbitrarily chosen, considering \( u(t) \leq b(t) + x(t) \), we get (8).

**Lemma 4.** Let \( h, k, p, q, b, \alpha \) be as in Lemma 3. If \( u \in C(\mathbb{R}_+, \mathbb{R}_+) \) satisfies
\[ u(t) \leq b(t) + \int_0^t h(t, s) u(s) ds + \int_0^t k(t, s) u(s) ds + \int_0^t p(t, s) u(s) \left( \int_0^s q(s, v) u(v) dv \right) ds, \] (18)
for \( t \geq 0 \), and \( b(t) \int_0^t R(s) Q(s) ds < 1 \), then
\[ u(t) \leq \frac{b(t) Q(t)}{1 - b(t) \int_0^t R(s) Q(s) ds}, \quad t \geq 0, \] (19)
where \( Q(t) = \exp \left( \int_0^t h(t, s) ds + \int_0^t k(t, s) ds \right), R(t) = (d/dt) \int_0^t p(t, s) \left( \int_0^s q(s, v) dv \right) ds. \)

The proof is similar to the proof of Lemma 3, we omit the details.

**Theorem 5.** If there exist functions \( a(t, s), b(t, s) \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+) \) with \( (t, s) \mapsto \partial_t a(t, s), \partial_s b(t, s) \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+) \) such that
\[ \|\Psi(t) f(s, x)\| \leq a(t, s) \|\Psi(s) x\|, \]
\[ \|\Psi(t) g(s, x)\| \leq b(t, s) \|\Psi(s) x\|, \] (20)
for \( 0 \leq s \leq t \) and for all \( x \in \mathbb{R}^n \). Moreover,
\[ \lim_{t \to \infty} \sup_{s \leq t} \left( \int_0^t (a(t, s) + b(t, s)) ds \right) = L_1, \]
\[ \|\Psi(t) e^{-t} \| \leq L_2 \] (21)
of for \( 0 \leq s \leq t \), and \( |\Psi(t) x(\alpha(t))| \leq |\Psi(\alpha(t)) x(\alpha(t))| \), where \( L_1, L_2 \) are nonnegative constants. If \( \alpha(t) = t - \tau(t) \) is an increasing diffeomorphism of \( \mathbb{R}_+ \), then, the trivial solution of system (3) is \( \Xi \)-unifomly stable on \( \mathbb{R}_+ \).

**Proof.** Suppose that \( x(t, t_0, x_0) := x(t) \) is the unique solution of system (3) which satisfies \( x(t_0) = x_0 \), since
\[ x(t) = x_0 + \int_{t_0}^t f(s, x(s)) ds + \int_{t_0}^t g(s, x(s)) ds \]
\[ = x_0 + \int_{t_0}^t f(s, x(s)) ds + \int_{t_0}^t \frac{g(\alpha^{-1}(r), x(r))}{\alpha'(\alpha^{-1}(r))} dr, \] (22)
after performing the change of variables \( r = \alpha(s) \) in the second integral, and \( \alpha^{-1} \) is the inverse of the diffeomorphism \( \alpha \) then, it follows that
\[ \|\Psi(t) x(t)\| \leq \|\Psi(t) \Psi^{-1}(t_0) \Psi(t_0) x_0\| \]
\[ + \int_{t_0}^t \|\Psi(t) f(s, x(s))\| ds \]
\[ + \int_{t_0}^t \frac{\|\Psi(t) g(\alpha^{-1}(r), x(r))\| ds}{\alpha'(\alpha^{-1}(r))} \]
\[ \leq L_2 \|\Psi(t_0) x_0\| + \int_{t_0}^t \alpha(t) b(t, \alpha^{-1}(r)) \|\Psi(r) x(r)\| dr, \] (23)
this implies by Lemma 3 that
\[ \|\Psi(t) x(t)\| \leq L_2 \|\Psi(t_0) x_0\| \exp \]
\[ \times \left( \int_{t_0}^t a(t, s) ds + \int_{t_0}^t \frac{b(t, \alpha^{-1}(r)) dr}{\alpha'(\alpha^{-1}(r))} \right) \]
\[ = L_2 \|\Psi(t_0) x_0\| \exp \left( \int_{t_0}^t (a(t, s) + b(t, s)) ds \right) \]
\[ \leq L_2 e^{L_1} \|\Psi(t_0) x_0\|, \] (24)
so for every \( \varepsilon > 0 \), choose \( \delta = \varepsilon /(L_2 e^{L_1}) \), then
\[ \|\Psi(t) x(t)\| \leq L_2 e^{L_1} \|\Psi(t_0) x_0\| < \varepsilon \] (25)
for \( \|\Psi(t_0) x_0\| < \delta \) and for all \( 0 \leq t_0 \leq t < \infty \). Hence, the conclusion of the theorem follows. 

\]
Theorem 6. Let all the conditions in Theorem 5 hold. Suppose further that there exist functions \( m(t,s), n(t,s) \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+) \) with \((t,s) \mapsto \partial_t m(t,s), \partial_t n(t,s) \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)\) such that

\[
\|\Psi(t) p(s,x)\| \leq m(t,s) \|\Psi(s)x\|, \\
\|\Psi(t) q(s,x)\| \leq n(t,s) \|\Psi(s)x\|,
\]

for \(0 \leq s \leq t\) and for all \(x \in \mathbb{R}^n\), moreover,

\[
\lim_{t \to \infty} \sup_{s \leq t} \int_0^s m(t,s) \left( \int_0^s n(s,u) \, du \right) \, ds = L_3,
\]

where \(L_3\) is a nonnegative constant. Then, the trivial solutions of systems (4) and (5) are \(\Psi\)-uniformly stable on \(\mathbb{R}_+\).

**Proof.** For that system (4), suppose \(x(t, t_0, x_0) := x(t)\) is the unique solution of system (4) which satisfies \(x(t_0) = x_0\), since

\[
x(t) = x_0 + \int_{t_0}^t f(s, x(s)) \, ds + \int_{t_0}^t g(s, x(\alpha(s))) \, ds \\
+ \int_{t_0}^t p(s, x(s)) \int_0^s q(u, x(\alpha(u))) \, du \, ds,
\]

it follows that

\[
\|\Psi(t)x(t)\| \leq \|\Psi(t)\| \|\Psi(t_0)x_0\| \\
+ \int_{t_0}^t \|\Psi(t)f(s, x(s))\| \, ds \\
+ \int_{t_0}^t \|\Psi(t)g(\alpha^{-1}(r), x(r))\| \, dr \\
+ \int_{t_0}^t \|\Psi(t)p(s, x(s))\| \, ds \\
\times \left( \int_0^s \|\Psi(s)q(\alpha^{-1}(r), x(r))\| \, dr \right) \, ds
\]

\[
\leq L_2 \|\Psi(t_0)x_0\| + \int_{t_0}^t a(t, s) \|\Psi(s)x(s)\| \, ds \\
+ \int_{t_0}^t b(t, \alpha^{-1}(r)) \|\Psi(r)\| \|x(r)\| \, dr \\
+ \int_{t_0}^t m(t, s) \|\Psi(s)x(s)\| \\
\times \left( \int_0^s \frac{n(s, \alpha^{-1}(r)) \|\Psi(r)x(r)\| \, dr}{\alpha'(\alpha^{-1}(r))} \right) \, ds
\]

after performing the change of variables \(r = \alpha(s)\) (or \(r = \alpha(u)\)) at some intermediate step, and \(\alpha^{-1}\) is the inverse of the diffeomorphism \(\alpha\). Denote

\[
Q(t) = \exp \left( \int_{t_0}^t a(t, s) \, ds + \int_{t_0}^t b(t, \alpha^{-1}(r)) \frac{\alpha'(\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))} \, dr \right)
\]

\[
= \exp \left( \int_{t_0}^t (a(t, s) + b(t, s)) \, ds \right),
\]

\[
R(t) = \frac{d}{dt} \left[ \int_{t_0}^t m(t, s) \left( \int_{t_0}^s n(s, u) \, du \right) \, ds \right]
\]

This implies by Lemma 3 that

\[
\|\Psi(t)x(t)\| \\
\leq L_2 \|\Psi(t_0)x_0\| \frac{Q(t)}{1 - L_2 \|\Psi(t_0)x_0\|} \int_0^t Q(v) R(v) \, dv
\]

\[
\leq \frac{L_2 e^{L_1}}{1 - L_2 \|\Psi(t_0)x_0\|} \int_0^t \|\Psi(t_0)x_0\| e^{L_1} \, dv
\]

\[
= \frac{L_2 e^{L_1}}{1 - L_2 \|\Psi(t_0)x_0\|} \int_0^t \|\Psi(t_0)x_0\| e^{L_1} \, dv
\]

\[
\leq \frac{L_2 e^{L_1}}{1 - L_2 \|\Psi(t_0)x_0\|} \int_0^t \|\Psi(t_0)x_0\| e^{L_1} \, dv
\]

for \(L_2, L_3 \|\Psi(t_0)x_0\| e^{L_1} < 1 \) and \(0 \leq t_0 \leq t\). So, for every \(\epsilon > 0\) and \(t_0 \geq 0\), let \(0 < q < 1/L_2 L_3 e^{L_1}\) be a constant and choose \(\delta = \min(|q_i|, ((1 - q L_2 L_3 e^{L_1}) \epsilon)/L_2 e^{L_1})\), then

\[
\|\Psi(t)x(t)\| < \frac{1 - q L_2 L_3 e^{L_1}}{1 - q L_2 L_3 e^{L_1}} \epsilon \times \frac{L_2 e^{L_1}}{1 - q L_2 L_3 e^{L_1}} = \epsilon
\]

for \(\|\Psi(t_0)x_0\| < \delta\) and for all \(0 \leq t_0 \leq t < \infty\). This proves that the trivial solution of system (4) is \(\Psi\)-uniformly stable on \(\mathbb{R}_+\).

Using Lemma 4, the proof of system (5) is similar to that of system (4) and the details are left to the readers. \(\square\)

Remark 7. For \(\psi_i = 1, i = 1, 2, \ldots, n\), we obtain the theorems of classical stability and uniform stability.

### 3. Examples

**Example 8.** Consider the nonlinear differential system

\[
x_1'(t) = x_1(t) + x_1 \left( \frac{t}{2} \right) \sin t,
\]

\[
x_2'(t) = -x_2(t) + x_2 \left( \frac{t}{2} \right) \cos t.
\]
In (33), \( f(t,x(t)) = (x_1(t), -x_2(t))^T \), \( g(t,x(t/2)) = (x_1(t/2) \sin t, x_2(t/2) \cos t)^T \). Let \( \Psi(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \), then \( a(t,s) = b(t,s) = e^{-(t-s)} \) for \( 0 \leq s \leq t \leq \infty \), it is easy to verify that \( L_1 = 2, L_2 = 1 \), and all the assumptions in Theorem 5 satisfied, so the trivial solution of system (33) is \( \psi \)-uniformly stable on \( \mathbb{R}_+ \).

**Example 9.** Consider the nonlinear Volterra integro-differential system as follows:

\[
\begin{align*}
    x_1'(t) &= x_1(t) + x_1(t) e^{-t} \int_0^t x_1(\frac{s}{2}) \cos s \, ds, \\
    x_2'(t) &= -x_2(t) + x_2(t) e^{-t} \int_0^t x_2(\frac{s}{2}) \sin s \, ds.
\end{align*}
\]

In (34), \( f(t,x(t)) = (x_1(t), -x_2(t))^T \), \( g \equiv 0 \), \( p(t,x(t)) = (x_1(t) e^{-t}, x_2(t) e^{-t})^T \), \( q(s,x(s/2)) = (x_1(s/2) \cos s, x_2(s/2) \sin s)^T \). Choose the same matrix function \( \Psi(t) \), then \( a(t,s) = n(t,s) = e^{-(t-s)} \), \( b(t,s) \equiv 0 \), \( m(t,s) = e^{-2(t-s)} \) for \( 0 \leq s \leq t \leq \infty \), it is easy to verify that \( L_1 = L_2 = 1, L_3 = 1/2 \), and all the assumptions in Theorem 6 are satisfied, so the trivial solution of system (34) is \( \psi \)-uniformly stable on \( \mathbb{R}_+ \).

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