Research Article

Approximately Ternary Homomorphisms and Derivations on $C^*$-Ternary Algebras

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1. Introduction

A $C^*$-ternary algebra is a complex Banach space $A$, equipped with a ternary product $(x, y, z) \mapsto [x, y, z]$ of $A^3$ into $A$, which is $C$-linear in the outer variables, conjugate $C$-linear in the middle variable, and associative in the sense that $[x, y, [z, w, v]] = [x, [w, z, y], v] = [[x, y, z], w, v]$, and satisfies $|[x, y, z]| \leq |x| \cdot |y| \cdot |z|$ and $||x, x, x|| = ||x||^3$. If a $C^*$-ternary algebra $(A, [\cdot, \cdot, \cdot])$ has an identity, that is, an element $e \in A$ such that $x = [x, e, e] = [e, e, x]$ for all $x \in A$, then it is routine to verify that $A$, endowed with $xoy := [x, e, y]$ and $x^* := [e, x, e]$, is a unital $C^*$-algebra. Conversely, if $(A, o)$ is a unital $C^*$-algebra, then $[x, y, z] := xoy^*oz$ makes $A$ into a $C^*$-ternary algebra. A $C$-linear mapping $H : A \to B$ is called a $C^*$-ternary algebra homomorphism if

$$H([x, y, z]) = [H(x), H(y), H(z)],$$

(1.1)
for all \( x, y, z \in A \). A \( \mathbb{C} \)-linear mapping \( \delta : A \to A \) is called a \( C^{*} \)-ternary algebra derivation if
\[
\delta([x,y,z]) = [\delta(x),y,z] + [x,\delta(y),z] + [x,y,\delta(z)],
\]
(1.2)

for all \( x, y, z \in A \).

Ternary structures and their generalization the so-called \( n \)-ary structures raise certain hopes in view of their applications in physics (see [1–8]).

We say a functional equation \( \zeta \) is stable if any function \( g \) satisfying the equation \( \zeta \) approximately is near to true solution of \( \zeta \). Moreover, \( \zeta \) is superstable if every approximately solution of \( \zeta \) is an exact solution of it.

The study of stability problems originated from a famous talk given by Ulam in 1940: “Under what condition does there exist a homomorphism near an approximate homomorphism?” In the next year 1941, Hyers answered affirmatively the question of Ulam for additive mappings between Banach spaces.

A generalized version of the theorem of Hyers for approximately additive maps was given by Rassias in 1978 as follows.

**Theorem 1.1.** Let \( f : E_1 \to E_2 \) be a mapping from a normed vector space \( E_1 \) into a Banach space \( E_2 \) subject to the inequality:
\[
\|f(x + y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p),
\]
(1.3)

for all \( x, y \in E_1 \), where \( \varepsilon \) and \( p \) are constants with \( \varepsilon > 0 \) and \( p < 1 \). Then, there exists a unique additive mapping \( T : E_1 \to E_2 \) such that
\[
\|f(x) - T(x)\| \leq \frac{2\varepsilon}{2 - 2^p}\|x\|^p,
\]
(1.4)

for all \( x \in E_1 \).

The stability phenomenon that was introduced and proved by Rassias is called Hyers-Ulam-Rassias stability. And then the stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [12–27]).

Throughout this paper, we assume that \( A \) is a \( C^{*} \)-ternary algebra with norm \( \| \cdot \|_A \) and that \( B \) is a \( C^{*} \)-ternary algebra with norm \( \| \cdot \|_B \). Moreover, we assume that \( n_0 \in \mathbb{N} \) is a positive integer and suppose that \( T_{1/n_0} := \{e^{i\theta}; 0 \leq \theta \leq 2\pi/n_0 \} \).

### 2. Superstability

In this section, first we investigate homomorphisms between \( C^{*} \)-ternary algebras. We need the following Lemma in the main results of the paper.

**Lemma 2.1.** Let \( f : A \to B \) be a mapping such that
\[
\|f\left(\frac{x_2 - x_1}{3}\right) + f\left(\frac{x_1 - 3x_3}{3}\right) + f\left(\frac{3x_1 + 3x_3 - x_2}{3}\right)\|_B \leq \|f(x_1)\|_B,
\]
(2.1)

for all \( x_1, x_2, x_3 \in A \). Then \( f \) is additive.
Proof. Letting $x_1 = x_2 = x_3 = 0$ in (2.1), we get

$$\|3f(0)\|_B \leq \|f(0)\|_B.$$  

(2.2)

So $f(0) = 0$. Letting $x_1 = x_2 = 0$ in (2.1), we get

$$\|f(-x_3) + f(x_3)\|_B \leq \|f(0)\|_B = 0,$$

(2.3)

for all $x_3 \in A$. Hence $f(-x_3) = -f(x_3)$ for all $x_3 \in A$. Letting $x_1 = 0$ and $x_2 = 6x_3$ in (2.1), we get

$$\|f(2x_3) - 2f(x_3)\|_B \leq \|f(0)\|_B = 0,$$

(2.4)

for all $x_3 \in A$. Hence

$$f(2x_3) = 2f(x_3),$$

(2.5)

for all $x_3 \in A$. Letting $x_1 = 0$ and $x_2 = 9x_3$ in (2.1), we get

$$\|f(3x_3) - f(x_3) - 2f(x_3)\|_B \leq \|f(0)\|_B = 0,$$

(2.6)

for all $x_3 \in A$. Hence

$$f(3x_3) = 3f(x_3),$$

(2.7)

for all $x_3 \in A$. Letting $x_1 = 0$ in (2.1), we get

$$\left\| f\left( \frac{x_2}{3}\right) + f\left(-x_3\right) + f \left( x_3 - \frac{x_2}{3}\right) \right\|_B \leq \|f(0)\|_B = 0,$$

(2.8)

for all $x_2, x_3 \in A$. So

$$f\left( \frac{x_2}{3}\right) + f\left(-x_3\right) + f \left( x_3 - \frac{x_2}{3}\right) = 0,$$

(2.9)

for all $x_2, x_3 \in A$. Let $t_1 = x_3 - (x_2/3)$ and $t_2 = x_2/3$ in (2.9). Then

$$f(t_2) - f(t_1 + t_2) + f(t_1) = 0,$$

(2.10)

for all $t_1, t_2 \in A$, this means that $f$ is additive. 

Now, we prove the first result in superstability as follows.
Theorem 2.2. Let $p \neq 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that
\[
\left\| f \left( \frac{x_2-x_1}{3} \right) + f \left( \frac{x_1-3\mu x_3}{3} \right) + \mu f \left( \frac{3x_1+3x_3-x_2}{3} \right) \right\|_B \leq \|f(x_1)\|_B, \quad \text{(2.11)}
\]
\[
\left\| f([x_1,x_2,x_3]) - [f(x_1),f(x_2),f(x_3)] \right\|_B \leq \theta \left( \|x_1\|_A^{3p} + \|x_2\|_A^{3p} + \|x_3\|_A^{3p} \right), \quad \text{(2.12)}
\]

for all $\mu \in \mathbb{T}_{1/n}$ and all $x_1, x_2, x_3 \in A$. Then, the mapping $f : A \to B$ is a $C^*$-ternary algebra homomorphism.

Proof. Assume $p > 1$.

Let $\mu = 1$ in (2.11). By Lemma 2.1, the mapping $f : A \to B$ is additive. Letting $x_1 = x_2 = 0$ in (2.11), we get
\[
\left\| f(-\mu x_3) + \mu f(x_3) \right\|_B = \|f(0)\|_B = 0, \quad \text{(2.13)}
\]
for all $x_3 \in A$ and $\mu \in \mathbb{T}$. So
\[
-f(\mu x_3) + \mu f(x_3) = f(-\mu x_3) + \mu f(x_3) = 0, \quad \text{(2.14)}
\]
for all $x_3 \in A$ and all $\mu \in \mathbb{T}$. Hence $f(\mu x_3) = \mu f(x_3)$ for all $x_3 \in A$ and all $\mu \in \mathbb{T}_{1/n}$. By same reasoning as proof of Theorem 2.2 of [28], the mapping $f : A \to B$ is $\mathbb{C}$-linear. It follows from (2.12) that
\[
\left\| f([x_1,x_2,x_3]) - [f(x_1),f(x_2),f(x_3)] \right\|_B
\]
\[
= \lim_{n \to \infty} 8^n \left\| f \left( \frac{[x_1,x_2,x_3]}{2^n} \right) - \left[ f \left( \frac{x_1}{2^n} \right), f \left( \frac{x_2}{2^n} \right), f \left( \frac{x_3}{2^n} \right) \right] \right\|_B \quad \text{(2.15)}
\]
\[
\leq \lim_{n \to \infty} \frac{8^n \theta}{8np} \left( \|x_1\|_A^{3p} + \|x_2\|_A^{3p} + \|x_3\|_A^{3p} \right) = 0,
\]
for all $x_1, x_2, x_3 \in A$. Thus,
\[
f([x_1,x_2,x_3]) = [f(x_1),f(x_2),f(x_3)], \quad \text{(2.16)}
\]
for all $x_1, x_2, x_3 \in A$. Hence, the mapping $f : A \to B$ is a $C^*$-ternary algebra homomorphism. Similarly, one obtains the result for the case $p < 1$.

Now, we establish the superstability of derivations on $C^*$-ternary algebras as follows.
Theorem 2.3. Let \( p \neq 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : A \to A \) be a mapping satisfying (2.11) such that

\[
\| f([x_1, x_2, x_3]) - [f(x_1), x_2, x_3] - [x_1, f(x_2), x_3] - [x_1, x_2, f(x_3)] \|_A \\
\leq \theta \left( \| x_1 \|_A^{3p} + \| x_2 \|_A^{3p} + \| x_3 \|_A^{3p} \right),
\]

for all \( x_1, x_2, x_3 \in A \). Then the mapping \( f : A \to A \) is a \( C^* \)-ternary derivation.

Proof. Assume \( p > 1 \).

By the Theorem 2.2, the mapping \( f : A \to A \) is \( \mathbb{C} \)-linear. It follows from (2.17) that

\[
\| f([x_1, x_2, x_3]) - [f(x_1), x_2, x_3] - [x_1, f(x_2), x_3] - [x_1, x_2, f(x_3)] \|_A \\
= \lim_{n \to \infty} 8^n \left\| \left[ f\left( \frac{x_1 + x_2 + x_3}{8^n} \right) \right] - \left[ f\left( \frac{x_1}{2^n}, \frac{x_2}{2^n}, \frac{x_3}{2^n} \right) \right] \\
- \left[ \frac{x_1}{2^n}, \frac{x_2}{2^n}, f\left( \frac{x_3}{2^n} \right) \right] \right\|_A \\
\leq \lim_{n \to \infty} 8^n \theta \left( \| x_1 \|_A^{3p} + \| x_2 \|_A^{3p} + \| x_3 \|_A^{3p} \right) = 0,
\]

for all \( x_1, x_2, x_3 \in A \). So

\[
f([x_1, x_2, x_3]) = [f(x_1), x_2, x_3] + [x_1, f(x_2), x_3] + [x_1, x_2, f(x_3)]
\]

for all \( x_1, x_2, x_3 \in A \). Thus, the mapping \( f : A \to A \) is a \( C^* \)-ternary derivation. Similarly, one obtains the result for the case \( p < 1 \). \( \square \)

3. Stability

First we prove the generalized Hyers-Ulam-Rassias stability of homomorphisms in \( C^* \)-ternary algebras.

Theorem 3.1. Let \( p > 1 \) and \( \theta \) be nonnegative real numbers, and let \( f : A \to B \) be a mapping such that

\[
\| f\left( \frac{x_2 - x_1}{3} \right) + f\left( \frac{x_1 - 3x_3}{3} \right) + \mu f\left( \frac{3x_1 + 3x_3 - x_2}{3} \right) - f(x_1) \|_B \\
\leq \theta \left( \| x_1 \|_A^p + \| x_2 \|_A^p + \| x_3 \|_A^p \right),
\]

\[
\| f([x_1, x_2, x_3]) - [f(x_1), f(x_2), f(x_3)] \|_B \leq \theta \left( \| x_1 \|_A^p + \| x_2 \|_A^p + \| x_3 \|_A^p \right),
\]
for all $\mu \in T^{1}_{1/m}$, and all $x_1, x_2, x_3 \in A$. Then there exists a unique C*-ternary homomorphism $H : A \to B$ such that

$$\|H(x_1) - f(x_1)\|_B \leq \frac{\theta(1 + 2^p)\|x_1\|_A^p}{1 - 3^{1-p}},$$

(3.3)

for all $x_1 \in A$.

Proof. Let us assume $\mu = 1$, $x_2 = 2x_1$ and $x_3 = 0$ in (3.1). Then we get

$$\left\|3f\left(\frac{x_1}{3}\right) - f(x_1)\right\|_B \leq \theta(1 + 2^p)\|x_1\|_A^p,$$

(3.4)

for all $x_1 \in A$. So by induction, we have

$$\left\|3^n f\left(\frac{x_1}{3^n}\right) - f(x_1)\right\|_B \leq \theta(1 + 2^p)\|x_1\|_A^p \sum_{i=0}^{n-1} 3^{i(1-p)},$$

(3.5)

for all $x_1 \in A$. Hence

$$\left\|3^{n+m} f\left(\frac{x_1}{3^{n+m}}\right) - 3^m f\left(\frac{x_1}{3^m}\right)\right\|_B \leq \theta(1 + 2^p)\|x_1\|_A^p \sum_{i=0}^{n-1} 3^{(i+m)(1-p)}$$

$$\leq \theta(1 + 2^p)\|x_1\|_A^p \sum_{i=m}^{n+m-1} 3^{i(1-p)},$$

(3.6)

for all nonnegative integers $m$ and $n$ with $n \geq m$, and all $x_1 \in A$. It follows that the sequence $\{3^n f(x_1/3^n)\}$ is a Cauchy sequence for all $x_1 \in A$. Since $B$ is complete, the sequence $\{3^n f(x_1/3^n)\}$ converges. Thus, one can define the mapping $H : A \to B$ by

$$H(x_1) := \lim_{n \to \infty} 3^n f\left(\frac{x_1}{3^n}\right),$$

(3.7)

for all $x_1 \in A$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (3.6), we get (3.3). It follows from (3.1) that

$$\left\|H\left(\frac{x_2 - x_1}{3}\right) + H\left(\frac{x_1 - 3\mu x_3}{3}\right) + \mu H\left(\frac{3x_1 + 3x_3 - x_2}{3}\right) - H(x_1)\right\|_B$$

$$= \lim_{n \to \infty} \left\|f\left(\frac{x_2 - x_1}{3^{n+1}}\right) + f\left(\frac{x_1 - 3\mu x_3}{3^{n+1}}\right) + f\left(\frac{3x_1 + 3x_3 - x_2}{3^{n+1}}\right) - f\left(\frac{x_1}{3^n}\right)\right\|_B$$

$$\leq \lim_{n \to \infty} \frac{3^n \theta}{3^n} \left(\|x_1\|_A^p + \|x_2\|_A^p + \|x_3\|_A^p\right) = 0,$$
for all μ ∈ ℤ\(^1_{1/n_0}\), and all x₁, x₂, x₃ ∈ A. So

\[
H \left( \frac{x_2 - x_1}{3} \right) + H \left( \frac{x_1 - 3\mu x_3}{3} \right) + \mu H \left( \frac{3x_1 + 3x_3 - x_2}{3} \right) = H(x_1),
\]

(3.9)

for all μ ∈ ℤ\(^1_{1/n_0}\) and all x₁, x₂, x₃ ∈ A. By the same reasoning as proof of Theorem 2.2 of [28], the mapping H : A → B is C-linear.

Now, let H' : A → B be another additive mapping satisfying (3.3). Then, we have

\[
\|H(x_1) - H'(x_1)\|_B = 3^n \left\| H \left( \frac{x_1}{3^n} \right) - H' \left( \frac{x_1}{3^n} \right) \right\|_B \\
\leq 3^n \left( \left\| H \left( \frac{x_1}{3^n} \right) - f \left( \frac{x_1}{3^n} \right) \right\|_B + \left\| H' \left( \frac{x_1}{3^n} \right) - f \left( \frac{x_1}{3^n} \right) \right\|_B \right) \\
\leq \frac{2 \cdot 3^n \theta (1 + 2^p)}{3^n (1 - 3^{1-p})} \|x_1\|_{A'}^p
\]

which tends to zero as n → ∞ for all x₁ ∈ A. So we can conclude that H(x₁) = H'(x₁) for all x₁ ∈ A. This proves the uniqueness of H.

It follows from (3.2) that

\[
\|H([x_1, x_2, x_3]) - [H(x_1), H(x_2), H(x_3)]\|_B \\
= \lim_{n \to \infty} 27^n \left\| f \left( \frac{[x_1, x_2, x_3]}{3^n \cdot 3^n \cdot 3^n} \right) - \left[ f \left( \frac{x_1}{3^n} \right), f \left( \frac{x_2}{3^n} \right), f \left( \frac{x_3}{3^n} \right) \right] \right\|_B \\
\leq \lim_{n \to \infty} \frac{27^n \theta}{27^{np}} \left( \|x_1\|_{A'}^{3p} + \|x_2\|_{A'}^{3p} + \|x_3\|_{A'}^{3p} \right) = 0,
\]

for all x₁, x₂, x₃ ∈ A.

Thus, the mapping H : A → B is a unique C*-ternary homomorphism satisfying (3.3). □

**Theorem 3.2.** Let p < 1 and θ be nonnegative real numbers, and let f : A → B be a mapping satisfying (3.1) and (3.2). Then, there exists a unique C*-ternary homomorphism H : A → B such that

\[
\|H(x_1) - f(x_1)\|_B \leq \frac{\theta (1 + 2^p) \|x_1\|_{A'}^p}{3^{1-p} - 1},
\]

(3.12)

for all x₁ ∈ A.

**Proof.** The proof is similar to the proof of Theorem 3.1. □

Now, we prove the generalized Hyers-Ulam-Rassias stability of derivations on C*-ternary algebras.
Theorem 3.3. Let $p > 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to A$ be a mapping such that

\[
\left\| f \left( \frac{x_2 - x_1}{3} \right) + f \left( \frac{x_1 - 3\mu x_3}{3} \right) + \mu f \left( \frac{3x_1 + 3x_3 - x_2}{3} \right) - f(x_1) \right\|_A \\
\leq \theta \left( \left\| x_1 \right\|_A^p + \left\| x_2 \right\|_A^p + \left\| x_3 \right\|_A^p \right),
\]

(3.13)

\[
\left\| f([x_1, x_2, x_3]) - [f(x_1), x_2, x_3] - [x_1, f(x_2), x_3] - [x_1, x_2, f(x_3)] \right\|_A \\
\leq \theta \left( \left\| x_1 \right\|_A^{3p} + \left\| x_2 \right\|_A^{3p} + \left\| x_3 \right\|_A^{3p} \right),
\]

(3.14)

for all $\mu \in \mathbb{T}_n^\theta$, and all $x_1, x_2, x_3 \in A$. Then, there exists a unique $C^*$-ternary derivation $D : A \to A$ such that

\[
\left\| D(x_1) - f(x_1) \right\|_A \leq \frac{\theta(1 + 2^p)\left\| x_1 \right\|_A^p}{1 - 3^{1-p}},
\]

(3.15)

for all $x_1 \in A$.

Proof. By the same reasoning as in the proof of the Theorem 3.1, there exists a unique $C$-linear mapping $D : A \to A$ satisfying (3.15). The mapping $D : A \to A$ is defined by

\[
D(x_1) := \lim_{n \to \infty} 3^n f \left( \frac{x_1}{3^n} \right),
\]

(3.16)

for all $x_1 \in A$. It follows from (3.14) that

\[
\left\| D([x_1, x_2, x_3]) - [D(x_1), x_2, x_3] - [x_1, D(x_2), x_3] - [x_1, x_2, D(x_3)] \right\|_A \\
= \lim_{n \to \infty} 27^n \left\| \frac{x_1, x_2, x_3}{3^n} - \left( f \left( \frac{x_1}{3^n} \right) \frac{x_2}{3^n} x_3 \right) - \left( x_1 \frac{x_2}{3^n} f \left( \frac{x_3}{3^n} \right) \frac{x_3}{3^n} \right) - \left( x_1 \frac{x_2}{3^n} x_3 \right) \right\|_A \\
\leq \lim_{n \to \infty} \frac{27^n \theta}{27^n p} \left( \left\| x_1 \right\|_A^{3p} + \left\| x_2 \right\|_A^{3p} + \left\| x_3 \right\|_A^{3p} \right) = 0,
\]

(3.17)

for all $x_1, x_2, x_3 \in A$. So

\[
D([x_1, x_2, x_3]) = [D(x_1), x_2, x_3] + [x_1, D(x_2), x_3] + [x_1, x_2, D(x_3)]
\]

(3.18)

for all $x_1, x_2, x_3 \in A$.

Thus, the mapping $D : A \to A$ is a unique $C^*$-ternary derivation satisfying (3.15). \[\Box\]
Theorem 3.4. Let $p < 1$ and $\theta$ be nonnegative real numbers, and let $f : A \rightarrow A$ be a mapping satisfying (3.13) and (3.14). Then, there exists a unique $C^*$-ternary derivation $D : A \rightarrow A$ such that

$$
\|D(x_1) - f(x_1)\|_A \leq \frac{\theta(1 + 2^p)\|x_1\|_A^p}{3^{1-p} - 1},
$$

(3.19)

for all $x_1 \in A$.

Proof. The proof is similar to the proof of Theorems 3.1 and 3.3.

4. Conclusions

In this paper, we have analyzed some detail $C^*$-ternary algebras and derivations on $C^*$-ternary algebras, associated with the following functional equation:

$$
f\left(\frac{x_2 - x_1}{3}\right) + f\left(\frac{x_1 - 3x_3}{3}\right) + f\left(\frac{3x_1 + 3x_3 - x_2}{3}\right) = f(x_1).
$$

(4.1)

A detailed study of how we can have the generalized Hyers-Ulam-Rassias stability of homomorphisms and derivations on $C^*$-ternary algebras is given.

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