Research Article

Solution and Hyers-Ulam-Rassias Stability of Generalized Mixed Type Additive-Quadratic Functional Equations in Fuzzy Banach Spaces


1 Department of Mathematics, Semnan University, Semnan 35131-19111, Iran
2 Department of Mathematics, Yasouj University, Yasouj 75918-74831, Iran
3 Department of Computer Hacking and Information Security, Daejeon University, Youngwoodong Donggu, Daejeon 300-716, Republic of Korea
4 Department of Mathematics, Kangnam University, Yongin, Gyeonggi 446-702, Republic of Korea

Correspondence should be addressed to G. H. Kim, ghkim@kangnam.ac.kr

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By using fixed point methods and direct method, we establish the generalized Hyers-Ulam stability of the following additive-quadratic functional equation

\[ f(x + ky) + f(x - ky) = f(x + y) + f(x - y) + (2k+1)/k f(ky) - 2(k+1) f(y) \]

for fixed integers \( k \) with \( k \neq 0, \pm 1 \) in fuzzy Banach spaces.

1. Introduction and Preliminaries

The stability problem of functional equations was originated from a question of Ulam [1] in 1940, concerning the stability of group homomorphisms. Let \((G_1, \cdot)\) be a group and let \((G_2, *, d)\) be a metric group with the metric \(d(., .)\). Given \( \epsilon > 0 \), does there exist a \( \delta > 0 \), such that if a mapping \( h : G_1 \to G_2 \) satisfies the inequality \( d(h(x \cdot y), h(x) * h(y)) < \delta \)
for all \( x, y \in G_1 \), then there exists a homomorphism \( H : G_1 \to G_2 \) with \( d(h(x), H(x)) < \epsilon \)
for all \( x \in G_1 \)? In other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. Let \( f : E \to E' \) be a mapping between Banach spaces such that

\[ \| f(x + y) - f(x) - f(y) \| \leq \delta, \]  

(1.1)
for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \leq \delta, \quad (1.2)$$

for all $x \in E$. Moreover if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then $T$ is linear. In 1978, Rassias [3] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded. In 1991, Gajda [4] answered the question for the case $p > 1$, which was raised by Rassias. This new concept is known as Hyers-Ulam-Rassias stability of functional equations (see [5–17]).

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \quad (1.3)$$

is related to a symmetric biadditive function. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.3) is said to be a quadratic function. It is well known that a function $f$ between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function $B$ such that $f(x) = B(x, x)$ for all $x$ (see [6, 18]). The biadditive function $B$ is given by

$$B(x, y) = \frac{1}{4}(f(x + y) - f(x - y)). \quad (1.4)$$

A Hyers-Ulam-Rassias stability problem for the quadratic functional equation (1.3) was proved by Skof for functions $f : A \to B$, where $A$ is normed space and $B$ Banach space (see [19–22]). Borelli and Forti [23] generalized the stability result of quadratic functional equations as follows (cf. [24, 25]): let $G$ be an Abelian group, and $X$ a Banach space. Assume that a mapping $f : G \to X$ satisfies the functional inequality:

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \varphi(x, y), \quad (1.5)$$

for all $x, y \in G$, and $\varphi : G \times G \to [0, \infty)$ is a function such that

$$\Phi(x, y) := \sum_{i=0}^{\infty} \frac{1}{4^{i+1}} \varphi(2^i x, 2^i y) < \infty, \quad (1.6)$$

for all $x, y \in G$. Then there exists a unique quadratic mapping $Q : G \to X$ with the property

$$\|f(x) - Q(x)\| \leq \Phi(x, x), \quad (1.7)$$

for all $x \in G$.

Now, we introduce the following functional equation for fixed integers $k$ with $k \neq 0, \pm 1$:

$$f(x + ky) + f(x - ky) = f(x + y) + f(x - y) + \frac{2(k + 1)}{k} f(ky) - 2(k + 1)f(y), \quad (1.8)$$
with \( f(0) = 0 \) in a non-Archimedean space. It is easy to see that the function \( f(x) = ax + bx^2 \) is a solution of the functional equation (1.8), which explains why it is called additive-quadratic functional equation. For more detailed definitions of mixed type functional equations, we can refer to [26–47].

**Definition 1.1** (see [48]). Let \( X \) be a real vector space. A function \( N : X \times \mathbb{R} \to [0, 1] \) is called a fuzzy norm on \( X \) if for all \( x, y \in X \) and all \( s, t \in \mathbb{R} \),

\[
\begin{align*}
(N1) \quad & N(x, t) = 0 \text{ for } t \leq 0; \\
(N2) \quad & x = 0 \text{ if and only if } N(x, t) = 1 \text{ for all } t > 0; \\
(N3) \quad & N(cx, t) = N(x, t/|c|) \text{ if } c \neq 0; \\
(N4) \quad & N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}; \\
(N5) \quad & N(x, \cdot) \text{ is a nondecreasing function of } \mathbb{R} \text{ and } \lim_{t \to \infty} N(x, t) = 1; \\
(N6) \quad & \text{for } x \neq 0, N(x, \cdot) \text{ is continuous on } \mathbb{R}.
\end{align*}
\]

The pair \((X, N)\) is called a fuzzy normed vector space.

**Example 1.2.** Let \((X, \| \cdot \|)\) be a normed linear space and \( \alpha, \beta > 0 \). Then

\[
N(x, t) = \begin{cases} 
\frac{\alpha t}{\alpha t + \beta \|x\|}, & t > 0, \ x \in X, \\
0, & t \leq 0, \ x \in X,
\end{cases}
\]

is a fuzzy norm on \( X \).

**Definition 1.3.** Let \((X, N)\) be a fuzzy normed vector space. A sequence \( \{x_n\} \) in \( X \) is said to be convergent or converge if there exists an \( x \in X \) such that \( \lim_{n \to \infty} N(x_n - x, t) = 1 \) for all \( t > 0 \). In this case, \( x \) is called the limit of the sequence \( \{x_n\} \) in \( X \) and one denotes it by \( N \lim_{n \to \infty} x_n = x \).

**Definition 1.4.** Let \((X, N)\) be a fuzzy normed vector space. A sequence \( \{x_n\} \) in \( X \) is called Cauchy if for each \( \epsilon > 0 \) and each \( t > 0 \) there exists an \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \) and all \( p > 0 \), one has \( N(x_{n+p} - x_n, t) > 1 - \epsilon \).

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

**Example 1.5.** Let \( N : \mathbb{R} \times \mathbb{R} \to [0, 1] \) be a fuzzy norm on \( \mathbb{R} \) defined by

\[
N(x, t) = \begin{cases} 
\frac{t}{t + |x|}, & t > 0, \\
0, & t \leq 0.
\end{cases}
\]

\[
(1.10)
\]
The $(\mathbb{R}, N)$ is a fuzzy Banach space. Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}$, $\delta > 0$, and $\epsilon = \delta/(1 + \delta)$. Then there exist $m \in \mathbb{N}$ such that for all $n \geq m$ and all $p > 0$, one has

$$\frac{1}{1 + |x_{n+p} - x_n|} \geq 1 - \epsilon. \quad (1.11)$$

So $|x_{n+p} - x_n| < \delta$ for all $n \geq m$ and all $p > 0$. Therefore $\{x_n\}$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$. Let $x_n \to x_0 \in \mathbb{R}$ as $n \to \infty$. Then $\lim_{n \to \infty} N(x_n - x_0, t) = 1$ for all $t > 0$.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces $X$ and $Y$ is continuous at a point $x \in X$ if for each sequence $\{x_n\}$ converging to $x_0 \in X$, the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on $X$ ([49]).

**Definition 1.6.** Let $X$ be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on $X$ if $d$ satisfies the following conditions:

1. $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

**Theorem 1.7.** Let $(X, d)$ be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either

$$d\left(J^n x, J^{n+1} x\right) = \infty, \quad (1.12)$$

for all nonnegative integers $n$, or there exists a positive integer $n_0$ such that

1. $d(J^n x, J^{n+1} x) < \infty$ for all $n_0 \geq n_0$;
2. the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;
3. $y^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X : d(J^n x, y) < \infty\}$;
4. $d(y, y^*) \leq 1/(1 - L)d(y, Jy)$ for all $y \in Y$.

We have the following theorem from [42], which investigates the solution of (1.8).

**Theorem 1.8.** A function $f : X \to Y$ with $f(0) = 0$ satisfies (1.8) for all $x, y \in X$ if and only if there exist functions $A : X \to Y$ and $Q : X \times X \to Y$, such that $f(x) = A(x) + Q(x, x)$ for all $x \in X$, where the function $Q$ is symmetric biadditive and $A$ is additive.

### 2. A Fixed Point Method

Using the fixed point methods, we prove the Hyers-Ulam stability of the additive-quadratic functional equation (1.8) in fuzzy Banach spaces. Throughout this paper, assume that $X$ is a vector space and that $(Y, N)$ is a fuzzy Banach space.
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Theorem 2.1. Let \( \varphi : X^2 \to [0, \infty) \) be a mapping such that there exists an \( \alpha < 1 \) with

\[
\varphi(x, y) \leq |k| \alpha \varphi \left( \frac{x}{k}, \frac{y}{k} \right),
\]

for all \( x, y \in X \). Let \( f : X \to Y \) be an odd function satisfying \( f(0) = 0 \) and

\[
N \left( f(k(x+y)) + f(k(x-y)) - f(kx+y) - f(kx-y) - \frac{2(k+1)}{k} f(ky) + 2(k+1)f(y), t \right) \geq \frac{t}{t + \varphi(x, y)},
\]

for all \( x, y \in X \) and all \( t > 0 \). Then \( A(x) := N - \lim_{n \to \infty} (f(k^n x) / k^n) \) exists for all \( x \in X \) and defines a unique additive mapping \( A : X \to Y \) such that

\[
N(f(x) - A(x), t) \geq \frac{(2|k+2| - |2k+2| \alpha) t}{t + \varphi(x, y)},
\]

for all \( x \in X \) and \( t > 0 \).

Proof. Note that \( f(0) = 0 \) and \( f(-x) = -f(x) \) for all \( x \in X \) since \( f \) is an odd function. Putting \( x = 0 \) in (2.2), we get

\[
N \left( \frac{f(ky)}{k} - f(y), \frac{t}{2k+2} \right) \geq \frac{t}{t + \varphi(0, y)},
\]

for all \( y \in X \) and all \( t > 0 \). Replacing \( y \) by \( x \) in (2.4), we have

\[
N \left( \frac{f(kx)}{k} - f(x), \frac{t}{2k+2} \right) \geq \frac{t}{t + \varphi(0, x)},
\]

for all \( x \in X \) and all \( t > 0 \). Consider the set \( S := \{ h : X \to Y; h(0) = 0 \} \) and introduce the generalized metric on \( S \):

\[
d(g, h) = \inf_{\mu \in (0, +\infty)} \left\{ N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(0, x)}, \forall x \in X \right\},
\]

where, as usual, \( \inf \varphi = +\infty \). It is easy to show that \( (S, d) \) is complete (see [50]). We consider the mapping \( J : (S, d) \to (S, d) \) as follows:

\[
Jg(x) := \frac{1}{k} g(kx),
\]
for all \( x \in X \). Let \( g, h \in S \) be given such that \( d(g, h) = \beta \). Then

\[
N(g(x) - h(x), \beta t) \geq \frac{t}{t + \varphi(0, x)},
\]

for all \( x \in X \) and all \( t > 0 \). Hence

\[
N(Jg(x) - Jh(x), a\beta t) = N\left(\frac{1}{k}g(kx) - \frac{1}{k}h(kx), |k|\alpha t\right)
\]

\[
\geq \frac{|k|\alpha t}{|k|\alpha t + \varphi(0, x)}
\]

\[
\geq \frac{|k|\alpha t}{|k|\alpha t + |k|\alpha \varphi(0, x)}
\]

\[
= \frac{t}{t + \varphi(0, x)}
\]

for all \( x \in X \) and all \( t > 0 \). So \( d(g, h) = \beta \) implies that \( d(Jg, Jh) \leq a\beta \). This means that \( d(Jg, Jh) \leq a\varphi(g, h) \) for all \( g, h \in S \). It follows from (2.5) that

\[
d(f, Jf) \leq \frac{1}{|2k + 2|}.
\]

By Theorem 1.7, there exists a mapping \( A : X \rightarrow Y \) satisfying the following.

1. \( A \) is a fixed point of \( J \), that is,

\[
kA(x) = A(kx),
\]

for all \( x \in X \). The mapping \( A \) is a unique fixed point of \( J \) in the set \( M = \{ g \in S : d(h, g) < \infty \} \).

This implies that \( A \) is a unique mapping satisfying (2.11) such that there exists a \( \mu \in (0, \infty) \) satisfying

\[
N(f(x) - A(x), \mu t) \geq \frac{t}{t + \varphi(0, x)}
\]

for all \( x \in X \).

2. \( d(J^n f, A) \rightarrow 0 \) as \( n \rightarrow \infty \). This implies the equality \( \lim_{n \rightarrow \infty} (f(k^n x)/k^n) = A(x) \), for all \( x \in X \).

3. \( d(f, A) \leq \frac{1}{|2k + 2| - 2k + 2}\alpha \), which implies the inequality

\[
d(f, A) \leq \frac{1}{|2k + 2| - 2k + 2\alpha}.
\]

This implies that the inequality (2.3) holds.
It follows from (2.1) and (2.2) that

\[
N \left( \frac{f(k^n(x + ky))}{k^n} + \frac{f(k^n(x - ky))}{k^n} - \frac{f(k^n(x + y))}{k^n} - \frac{f(k^n(x - y))}{k^n} \right. \\
- \frac{2(k + 1)}{k} \frac{f(k^{n+1}y)}{k^n} + \left. 2(k + 1) \frac{f(k^n y)}{k^n}, t \right)
\geq \frac{t}{t + \varphi(k^n x, k^n y)},
\]

for all \( x, y \in X, \) all \( t > 0, \) and all \( n \in \mathbb{N}. \) So

\[
N \left( \frac{f(k^n(x + ky))}{k^n} + \frac{f(k^n(x - ky))}{k^n} - \frac{f(k^n(x + y))}{k^n} - \frac{f(k^n(x - y))}{k^n} \right. \\
- \frac{2(k + 1)}{k} \frac{f(k^{n+1}y)}{k^n} + \left. 2(k + 1) \frac{f(k^n y)}{k^n}, t \right)
\geq \frac{|k|^n t}{|k|^n t + |k|^n \alpha_n \varphi(x, y)},
\]

for all \( x, y \in X, \) all \( t > 0, \) and all \( n \in \mathbb{N}. \) Since \( \lim_{n \to \infty} (|k|^n t / (|k|^n t + |k|^n \alpha_n \varphi(x, y))) = 1 \) for all \( x, y \in X \) and all \( t > 0, \) we obtain that

\[
N \left( A(k(x + y)) + A(k(x - y)) - A(kx + y) - A(kx - y) - \frac{2(k + 1)}{k} A(ky) \right. \\
+ 2(k + 1) A(y), t \right) = 1,
\]

for all \( x, y, z \in X \) and all \( t > 0. \) Hence the mapping \( A : X \to Y \) is additive, as desired. \( \square \)

**Corollary 2.2.** Let \( \theta \geq 0 \) and let \( r \) be a real positive number with \( r < 1. \) Let \( X \) be a normed vector space with norm \( \| \cdot \|. \) Let \( f : X \to Y \) be an odd mapping satisfying

\[
N \left( f(k(x + y)) + f(k(x - y)) - f(kx + y) - f(kx - y) - \frac{2(k + 1)}{k} f(ky) + 2(k + 1) f(y), t \right)
\geq \frac{t}{t + \theta(\|x\|^r + \|y\|^r)},
\]

for all \( x, y \in X \) and all \( t > 0. \) Then the limit \( A(x) := N - \lim_{n \to \infty} (f(k^n x) / k^n) \) exists for each \( x \in X \) and defines a unique additive mapping \( A : X \to Y \) such that

\[
N(f(x) - A(x), t) \geq \frac{|2k + 2|(|k| - |k|^r) t}{|2k + 2|(|k| - |k|^r) t + |k| \theta \|x\|^r},
\]

for all \( x \in X \) and all \( t > 0. \)
Proof. The proof follows from Theorem 2.1 by taking \( \varphi(x, y) := \theta(\|x\|^r + \|y\|^r) \) for all \( x, y \in X \). Then we can choose \( a = |k|^{r-1} \) and we get the desired result.

**Theorem 2.3.** Let \( \varphi : X^2 \to [0, \infty) \) be a mapping such that there exists an \( a < 1 \) with

\[
\varphi\left( \frac{x}{k}, \frac{y}{k} \right) \leq \frac{a}{|k|} \varphi(x, y),
\]

(2.19)

for all \( x, y \in X \). Let \( f : X \to Y \) be an odd mapping satisfying \( f(0) = 0 \) and (2.2). Then the limit \( A(x) := N - \lim_{n \to \infty} k^n f(x/k^n) \) exists for all \( x \in X \) and defines a unique additive mapping \( A : X \to Y \) such that

\[
N\left( f(x) - A(x), t \right) \geq \frac{(|2k+2|-|2k+2|a)t}{(2k+2)-|2k|a} t + \alpha \varphi(0, x),
\]

(2.20)

for all \( x \in X \) and all \( t > 0 \).

Proof. Let \((S, d)\) be the generalized metric space defined as in the proof of Theorem 2.1. Consider the mapping \( J : S \to S \) by

\[
Jg(x) := kg\left( \frac{x}{k} \right),
\]

(2.21)

for all \( g \in S \). Let \( g, h \in S \) be given such that \( d(g, h) = \beta \). Then

\[
N\left( g(x) - h(x), \beta t \right) \geq \frac{t}{t + \varphi(0, x)},
\]

(2.22)

for all \( x \in X \) and all \( t > 0 \). Hence

\[
N\left( Jg(x) - Jh(x), \alpha \beta t \right) = N\left( kg\left( \frac{x}{k} \right) - kh\left( \frac{x}{k} \right), \alpha \beta t \right)
\]

\[
= N\left( g\left( \frac{x}{k} \right) - h\left( \frac{x}{k} \right), \frac{\alpha \beta t}{|k|} \right)
\]

\[
\geq \frac{t}{\alpha \beta t + \varphi(0, x/k)} \frac{t}{t + \varphi(0, x)},
\]

(2.23)

for all \( x \in X \) and all \( t > 0 \). So \( d(g, h) = \beta \) implies that \( d(Jg, Jh) \leq \alpha \beta \). This means that \( d(Jg, Jh) \leq \alpha d(g, h) \) for all \( g, h \in S \). It follows from (2.5) that

\[
N\left( kf\left( \frac{x}{k} \right) - f(x), \frac{kt}{|2k+2|} \right) \geq \frac{t}{t + \varphi(0, x/k)} \geq \frac{t}{t + (\alpha/|k|) \varphi(0, x)},
\]

(2.24)
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for all \( x \in X \) and all \( t > 0 \). Therefore

\[
N \left( k f \left( \frac{x}{k} \right) - f(x) \right) \geq \frac{at}{|2k + 2|},
\]

(2.25)

So \( d(f, Jf) \leq a \). By Theorem 1.7, there exists a mapping \( A : X \to Y \) satisfying the following.

(1) \( A \) is a fixed point of \( J \), that is,

\[
A \left( \frac{x}{k} \right) = \frac{1}{k} A(x),
\]

(2.26)

for all \( x \in X \). The mapping \( A \) is a unique fixed point of \( f \) in the set \( \Omega = \{ h \in S : d(g, h) < \infty \} \).

This implies that \( A \) is a unique mapping satisfying (2.26) such that there exists \( \mu \in (0, \infty) \) satisfying

\[
N(f(x) - A(x), \mu t) \geq \frac{t}{|t + \varphi(0, x)|},
\]

(2.27)

for all \( x \in X \) and \( t > 0 \).

(2) \( d(J^n f, A) \to 0 \) as \( n \to \infty \). This implies the equality \( N - \lim_{n \to \infty} k^n f(x/k^n) = A(x) \)

for all \( x \in X \).

(3) \( d(f, A) \leq d(f, Jf)/(1 - L) \) with \( f \in \Omega \), which implies the inequality

\[
d(f, A) \leq \frac{\alpha}{|2k + 2| - |2k + 2|}.\]

(2.28)

This implies that the inequality (2.20) holds.

The rest of proof is similar to the proof of Theorem 2.1. \( \square \)

**Corollary 2.4.** Let \( \theta \geq 0 \) and let \( r \) be a real number with \( r > 1 \). Let \( X \) be a normed vector space with norm \( \| \cdot \| \). Let \( f : X \to Y \) be an odd mapping satisfying (2.17). Then \( A(x) := N - \lim_{n \to \infty} k^n f(x/k^n) \) exists for each \( x \in X \) and defines a unique additive mapping \( A : X \to Y \) such that

\[
N(f(x) - A(x), t) \geq \frac{|2k + 2|(|k|^r - |k|)t}{|2k + 2|(|k|^r - |k|)t + |k|\|x\|^r},
\]

(2.29)

for all \( x \in X \) and all \( t > 0 \).

**Proof.** The proof follows from Theorem 2.3 by taking \( \varphi(x, y) := \theta(\|x\|^r + \|y\|^r) \) for all \( x, y \in X \). Then we can choose \( \alpha = |k|^{1-r} \) and we get the desired result. \( \square \)

**Theorem 2.5.** Let \( \varphi : X^2 \to [0, \infty) \) be a function such that there exists an \( \alpha < 1 \) with

\[
\varphi(x, y) \leq k^2 \alpha \varphi \left( \frac{x}{k}, \frac{y}{k} \right),
\]

(2.30)
consider the linear mapping $J$ on $S$ for all $x, y \in X$. Let $f : X \to Y$ be an even mapping with $f(0) = 0$ and satisfying (2.2). Then $Q(x) := N - \lim_{n \to \infty} (f(k^n x)/k^{2n})$ exists for all $x \in X$ and defines a unique quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(2|k| - 2|k|\alpha)t}{(2|k| - 2|k|\alpha)t + \varphi(0, x)},$$

(2.31)

for all $x \in X$ and all $t > 0$.

Proof. Replacing $x$ by $kx$ in (2.2), we get

$$N \left(\frac{f(kx + y) + f(kx - y) - f(kx + y) - f(kx - y) - 2(k + 1)}{k} f(ky) + 2(k + 1) f(y), t\right) \geq \frac{t}{t + \varphi(kx, y)},$$

(2.32)

for all $x, y \in X$ and all $t > 0$. Putting $x = 0$ and replacing $y$ by $x$ in (2.32), we have

$$N \left(\frac{f(kx) - k f(x)}{k} - t/2, \right) \geq \frac{t}{t + \varphi(0, x)},$$

(2.33)

for all $x \in X$ and all $t > 0$. By (2.33), (N3), and (N4), we get

$$N \left(\frac{f(kx)}{k^2} - f(x), \frac{t}{2|k|}\right) \geq \frac{t}{t + \varphi(0, x)},$$

(2.34)

for all $x \in X$ and all $t > 0$. Consider the set $S^* := \{h : X \to Y; h(0) = 0\}$ and introduce the generalized metric on $S^*$:

$$d(g, h) = \inf_{0 < \mu < \infty} \left\{ N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(0, x)}, \forall x \in X \right\},$$

(2.35)

where, as usual, $\inf \varphi = +\infty$. It is easy to show that $(S^*, d)$ is complete (see [50]). Now we consider the linear mapping $J : (S^*, d) \to (S^*, d)$ such that

$$J g(x) := \frac{1}{k^2} g(kx),$$

(2.36)

for all $x \in X$. Proceeding as in the proof of Theorem 2.1, we obtain that $d(g, h) = \beta$ implies that $d(J g, J h) \leq \alpha \beta$. This means that $d(J g, J h) \leq \alpha d(g, h)$ for all $g, h \in S$. It follows from
(2.34) that

\[ d(f, Jf) \leq \frac{1}{2|k|}. \]  

(2.37)

By Theorem 1.7, there exists a mapping \( Q : X \rightarrow Y \) such that one has the following.

1. \( Q \) is a fixed point of \( J \), that is,

\[ k^2Q(x) = Q(kx), \]  

(2.38)

for all \( x \in X \). The mapping \( Q \) is a unique fixed point of \( J \) in the set \( M = \{ g \in S^* : d(h, g) < \infty \} \).

This implies that \( Q \) is a unique mapping satisfying (2.38) such that there exists a \( \mu \in (0, \infty) \) satisfying

\[ N(f(x) - Q(x), \mu t) \geq \frac{1}{1 - \alpha} d(f, Jf) \]

and defining a unique quadratic mapping \( Q : X \rightarrow Y \) such that

\[ N(f(x) - Q(x), t) \geq \frac{(2k^2 - 2k^2r)t}{(2k^2 - 2k^2r)t + |k|\|x\|^2}, \]

(2.39)

for all \( x \in X \) and all \( t > 0 \).

Corollary 2.6. Let \( \theta \geq 0 \) and let \( r \) be a real positive number with \( r < 1 \). Let \( X \) be a normed vector space with norm \( \| \cdot \| \). Let \( f : X \rightarrow Y \) be an even mapping with \( f(0) = 0 \) and satisfying (2.17). Then the limit \( Q(x) := N - \lim_{n \rightarrow \infty} (f(k^n x)/k^{2n}) \) exists for each \( x \in X \) and defines a unique quadratic mapping \( Q : X \rightarrow Y \) such that

\[ N(f(x) - Q(x), t) \geq \frac{(2|k| - 2|k|\alpha)t}{(2|k| - 2|k|\alpha)t + |k|\theta\|x\|^2}, \]

(2.40)

for all \( x, y \in X \). Let \( \phi : X^2 \rightarrow [0, \infty) \) be a function such that there exists an \( \alpha < 1 \) with

\[ \phi(x, y) \leq \alpha \phi(x, y), \]

(2.40)

for all \( x, y \in X \). Let \( f : X \rightarrow Y \) be an even mapping with \( f(0) = 0 \) and satisfying (2.2). Then the limit \( Q(x) := N - \lim_{n \rightarrow \infty} k^n f(x/k^n) \) exists for all \( x \in X \) and defines a unique quadratic mapping \( Q : X \rightarrow Y \) such that

\[ N(f(x) - Q(x), t) \geq \frac{(2|k| - 2|k|\alpha)t}{(2|k| - 2|k|\alpha)t + \alpha \phi(0, x)}, \]

(2.41)

for all \( x \in X \) and \( t > 0 \).
Proof. Let \((S^*, d)\) be the generalized metric space defined as in the proof of Theorem 2.5. It follows from (2.34) that
\[
N\left(k^2 \left( f\left(\frac{x}{k}\right) - f(x), \frac{|k|}{2} \right) \right) \geq \frac{t}{t + \varphi(0, x/k)} \geq \frac{t}{t + (\alpha/k^2)\varphi(0, x)},
\]
for all \(x \in X\) and \(t > 0\). So
\[
N\left(f(x) - k^2 \left( f\left(\frac{x}{k}\right), \frac{\alpha t}{2|k|} \right) \right) \geq \frac{t}{t + \varphi(0, x)}.
\]
The rest of the proof is similar to the proofs of Theorems 2.1 and 2.3.

\[\square\]

**Corollary 2.8.** Let \(\theta \geq 0\) and let \(r\) be a real number with \(r > 1\). Let \(X\) be a normed vector space with norm \(||\cdot||\). Let \(f : X \rightarrow Y\) be an even mapping with \(f(0) = 0\) and satisfying (2.17). Then \(Q(x) := N - \lim_{n \rightarrow \infty} k^n f(x/k^n)\) exists for each \(x \in X\) and defines a unique quadratic mapping \(Q : X \rightarrow Y\) such that
\[
N(f(x) - Q(x), t) \geq \frac{(2|k|^{2r+1} - 2|k|^3) t}{(2|k|^{2r+1} - 2|k|^3) t + k^2 \theta ||x||^r},
\]
for all \(x \in X\) and all \(t > 0\).

**Proof.** It follows from Theorem 2.7 by taking \(\varphi(x, y) := \theta(||x||^r + ||y||^r)\) for all \(x, y \in X\). Then we can choose \(\alpha = k^{2-2r}\) and we get the desired result.

\[\square\]

### 3. Direct Method

In this section, using direct method, we prove the Hyers-Ulam stability of functional equation (1.8) in fuzzy Banach spaces. Throughout this section, we assume that \(X\) is a linear space, \((Y, N)\) is a fuzzy Banach space, and \((Z, N')\) is a fuzzy normed space. Moreover, we assume that \(N(x, \cdot)\) is a left continuous function on \(\mathbb{R}\).

**Theorem 3.1.** Assume that a mapping \(f : X \rightarrow Y\) is an odd mapping with \(f(0) = 0\) satisfying the inequality
\[
N\left(f(k(x+y)) + f(k(x-y)) - f(kx+y) - f(kx-y) - \frac{2(k+1)}{k} f(ky) + 2(k+1) f(y), t\right) 
\geq N'(\varphi(x, y), t), 
\]
for all \(x, y \in X, t > 0,\) and \(\varphi : X^2 \rightarrow Z\) is a mapping for which there is a constant \(r \in \mathbb{R}\) satisfying \(0 < |r| < 1/|k|\) such that
\[
N'(\varphi\left(\frac{x}{k} \ , \ y \right), t) \geq N'(\varphi(x, y), \frac{t}{|r|}),
\]
(3.2)
for all \(x, y \in X\) and all \(t > 0\). Then there exists a unique additive mapping \(A : X \to Y\) satisfying (1.8) and the inequality

\[
N(f(x) - A(x), t) \geq N'(\varphi(0, x), \frac{\|x\|^2}{|r|}),
\]

(3.3)

for all \(x \in X\) and all \(t > 0\).

Proof. It follows from (3.2) that

\[
N'(\varphi(\frac{x}{k^j}, \frac{y}{k^j}), t) \geq N'(\varphi(x, y), \frac{t}{|r|}),
\]

(3.4)

for all \(x, y \in X\) and all \(t > 0\). Putting \(x = 0\) in (3.1) and then replacing \(y\) by \(x/k\), we get

\[
N(\frac{k}{k^j}f(\frac{x}{k}) - f(x), \frac{|k|t}{2k + 2}) \geq N'(\varphi(0, \frac{x}{k}), t),
\]

(3.5)

for all \(x \in X\) and all \(t > 0\). Replacing \(x\) by \(x/k^j\) in (3.5), we have

\[
N(\frac{k^{j+1}}{k^{j+1}}f(\frac{x}{k^{j+1}}) - k^j f(\frac{x}{k^j}), \frac{|k^{j+1}|t}{2k + 2}) \geq N'(\varphi(0, \frac{x}{k^{j+1}}), t) \geq N'(\varphi(0, x), \frac{t}{|r|}),
\]

(3.6)

for all \(x \in X\), all \(t > 0\), and all integer \(j \geq 0\). So

\[
N\left(f(x) - k^nf(\frac{x}{k^n}), \sum_{j=0}^{n-1} \frac{|k^{j+1}| r^{j+1} t}{2k + 2}\right)
\]

\[
= N\left(\sum_{j=0}^{n-1} k^{j+1}f(\frac{x}{k^{j+1}}) - k^j f(\frac{x}{k^j}), \sum_{j=0}^{n-1} \frac{|k^{j+1}| r^{j+1} t}{2k + 2}\right)
\]

\[
\geq \min_{0 \leq j \leq n-1} \left\{ N\left(k^{j+1}f(\frac{x}{k^{j+1}}) - k^j f(\frac{x}{k^j}), \frac{|k^{j+1}| r^{j+1} t}{2k + 2}\right) \right\}
\]

(3.7)

\[
\geq \min_{0 \leq j \leq n-1} \{ N'(\varphi(0, x), t) \}
\]

\[
= N'(\varphi(0, x), t),
\]

which yields

\[
N\left(k^{n+p}f(\frac{x}{k^{n+p}}) - k^p f(\frac{x}{k^p}), \sum_{j=0}^{n-1} \frac{|k^{j+p+1}| r^{j+1} t}{2k + 2}\right) \geq N'(\varphi(0, \frac{x}{2^p}), t) \geq N'(\varphi(0, x), \frac{t}{|r|^p}),
\]

(3.8)
for all \( x \in X, t > 0, \) and all integers \( n > 0, p \geq 0. \) So

\[
N\left( k^{n+p} f\left( \frac{x}{k^{n+p}} \right) - k^p f\left( \frac{x}{k^p} \right), \sum_{j=0}^{n-1} \frac{|k|^{j+p+1} |r|^{j+p+1} t}{|k + 2|} \right) \geq N'(\varphi(0, x), t), \tag{3.9}
\]

for all \( x \in X, t > 0, \) and any integers \( n > 0, p \geq 0. \) Hence one can obtain

\[
N\left( k^{n+p} f\left( \frac{x}{k^{n+p}} \right) - k^p f\left( \frac{x}{k^p} \right), t \right) \geq N'(\varphi(0, x), \frac{t}{\sum_{j=0}^{n-1} \frac{|k|^{j+p+1} |r|^{j+p+1} / |k + 2|}}), \tag{3.10}
\]

for all \( x \in X, t > 0, \) and any integers \( n > 0, p \geq 0. \) Since the series \( \sum_{j=0}^{+\infty} k^j |r|^j \) is a convergent series, we see by taking the limit \( p \to \infty \) in the last inequality that the sequence \( \{k^n f(x/k^n)\} \)

is a Cauchy sequence in the fuzzy Banach space \((Y, N)\) and so it converges in \( Y. \) Therefore a mapping \( A : X \to Y \) defined by \( A(x) := N - \lim_{n \to +\infty} k^n f(x/k^n) \) is well defined for all \( x \in X. \) This means that

\[
\lim_{n \to +\infty} N\left( A(x) - k^n f\left( \frac{x}{k^n} \right), t \right) = 1, \tag{3.11}
\]

for all \( x \in X \) and all \( t > 0. \) In addition, it follows from (3.10) that

\[
N\left( f(x) - k^n f\left( \frac{x}{k^n} \right), t \right) \geq N'(\varphi(0, x), \frac{t}{\sum_{j=0}^{n-1} \frac{|k|^{j+1} |r|^{j+1} / |k + 2|}}), \tag{3.12}
\]

for all \( x \in X \) and all \( t > 0. \) So

\[
N(f(x) - A(x), t) \geq \min\left\{ N\left( f(x) - k^n f\left( \frac{x}{k^n} \right), (1 - \varepsilon)t \right), N\left( A(x) - k^n f\left( \frac{x}{k^n} \right), \varepsilon t \right) \right\}
\]

\[
\geq N'(\varphi(0, x), \frac{\varepsilon t}{\sum_{j=0}^{n-1} \frac{|k|^{j+1} |r|^{j+1} / |k + 2|}})
\]

\[
\geq N'(\varphi(0, x), \frac{|k + 2|(1 - |k||r|)\varepsilon t}{|k||r|}), \tag{3.13}
\]

for sufficiently large \( n \) and for all \( x \in X, t > 0, \) and \( \varepsilon \) with \( 0 < \varepsilon < 1. \) Since \( \varepsilon \) is arbitrary and \( N' \) is left continuous, we obtain

\[
N(f(x) - A(x), t) \geq N'(\varphi(0, x), \frac{|k + 2|(1 - |k||r|)t}{|k||r|}), \tag{3.14}
\]
for all $x \in X$ and $t > 0$. It follows from (3.1) that

\[
N\left(\frac{f(k^n(x + ky))}{k^n} + \frac{f(k^n(x - ky))}{k^n} - \frac{f(k^n(x + y))}{k^n} - \frac{f(k^n(x - y))}{k^n}
- \frac{2(k + 1)f(k^{n+1}y)}{k^n} + 2(k + 1)\frac{f(k^ny)}{k^n}, t\right)
\geq N'(\varphi(k^nx, k^ny), \frac{t}{|k|^n}) \geq N'(\varphi(x, y), \frac{t}{|r|^n|k|^n}) \to 1 \quad \text{as } n \to +\infty,
\]  
(3.15)

for all $x, y \in X$ and all $t > 0$. Therefore, we obtain in view of (3.11)

\[
N\left(\frac{f(k^n(x + y))}{k^n} + \frac{f(k^n(x - y))}{k^n} - \frac{f(k^n(x + y))}{k^n} - \frac{f(k^n(x - y))}{k^n}
- \frac{2(k + 1)f(k^{n+1}y)}{k^n} + 2(k + 1)\frac{f(k^ny)}{k^n}, t\right)
\geq \min\left\{N\left(\frac{f(k^n(x + y))}{k^n} + \frac{f(k^n(x - y))}{k^n} - \frac{f(k^n(x + y))}{k^n} - \frac{f(k^n(x - y))}{k^n}
+ 2(k + 1)\frac{f(k^n(x + y))}{k^n} - \frac{f(k^n(x - y))}{k^n}
+ \frac{2(k + 1)f(k^{n+1}y)}{k^n} + 2(k + 1)\frac{f(k^ny)}{k^n}, t\right),
N\left(\frac{f(k^n(x + ky))}{k^n} + \frac{f(k^n(x - ky))}{k^n} - \frac{f(k^n(x + y))}{k^n} - \frac{f(k^n(x - y))}{k^n}
- \frac{2(k + 1)f(k^{n+1}y)}{k^n} + 2(k + 1)\frac{f(k^ny)}{k^n}, t\right)\right\}
= N\left(\frac{f(k^n(x + ky))}{k^n} + \frac{f(k^n(x - ky))}{k^n} - \frac{f(k^n(x + y))}{k^n} - \frac{f(k^n(x - y))}{k^n}
- \frac{2(k + 1)f(k^{n+1}y)}{k^n} + 2(k + 1)\frac{f(k^ny)}{k^n}, t\right)
\geq N'(\varphi(x, y), \frac{t}{2|k|^n|r|^n}) \to 1 \quad \text{as } n \to +\infty,
\]  
(3.16)

for all $x, y \in X$ and all $t > 0$, which implies that

\[
A(k(x + y)) + A(k(x - y)) = A(kx + y) + A(kx - y) + \frac{2(k + 1)}{k}A(ky) - 2(k + 1)A(y).
\]  
(3.17)

Hence the mapping $A : X \to Y$ is additive, as desired.
Let \( \theta \geq 0 \) and \( p > 1 \) such that an odd mapping \( f : X \to Y \) with \( f(0) = 0 \) satisfies the following inequality:

\[
N \left( f(k(x + y)) + f(k(x - y)) - f(kx + y) - f(kx - y) - \frac{2(k + 1)}{k} f(ky) + 2(k + 1)f(y), t \right) \\
\geq N' \left( \theta(||x||^p + ||y||^p), t \right),
\]

for all \( x, y \in X \) and \( t > 0 \). Then there is a unique additive mapping \( A : X \to Y \) satisfying (1.8) and the inequality

\[
N(f(x) - A(x), t) \geq N' \left( \frac{\theta ||x||^p}{2k + 2}, \left( \frac{|k|^p - |k|}{|k|} \right) t \right).
\]

Proof. Let \( \varphi(x, y) := \theta(||x||^p + ||y||^p) \) and \(|r| = |k|^{-p} \). Applying Theorem 3.1, we get desired results.

**Theorem 3.3.** Let \( f : X \to Y \) be an odd mapping with \( f(0) = 0 \) satisfying the inequality (3.1) and let \( \varphi : X^2 \to Z \) be a mapping for which there exists a constant \( r \in \mathbb{R} \) satisfying \( 0 < |r| < |k| \) such that

\[
N'(\varphi(x, y), |r|t) \geq N' \left( \varphi \left( \frac{x}{k^p}, \frac{y}{k} \right), t \right),
\]

for all \( t > 0 \). Therefore \( A(x) = L(x) \) for all \( x \in X \). This completes the proof. \( \square \)
for all \( x, y \in X \) and all \( t > 0 \). Then there exists a unique additive mapping \( A : X \to Y \) satisfying (1.8) and the following inequality:

\[
N(f(x) - A(x), t) \geq N'(\varphi(0, x), \frac{2k + 2(|k| - |r|)t}{|k|}),
\]

(3.22)

for all \( x \in X \) and all \( t > 0 \).

Proof. It follows from (3.5) that

\[
N\left(\frac{f(kx)}{k} - f(x), \frac{t}{2k + 2}\right) \geq N'(\varphi(0, x), t),
\]

(3.23)

for all \( x \in X \) and all \( t > 0 \). Replacing \( x \) by \( k^n x \) in (3.41), we obtain

\[
N\left(\frac{f(k^{n+1}x)}{k^{n+1}} - \frac{f(k^n x)}{k^n}, \frac{t}{2k + 2|k^n|}\right) \geq N'(\varphi(0, k^n x), t) \geq N'(\varphi(0, x), \frac{t}{|r|^n}).
\]

(3.24)

So

\[
N\left(\frac{f(k^{n+1}x)}{k^{n+1}} - \frac{f(k^n x)}{k^n}, \frac{|r|^n t}{2k + 2|k^n|}\right) \geq N'(\varphi(0, x), t),
\]

(3.25)

for all \( x \in X \) and all \( t > 0 \). Proceeding as in the proof of Theorem 3.1, we obtain that

\[
N\left(f(x) - \frac{f(k^n x)}{k^n}, \sum_{j=0}^{n-1} \frac{|r|^j t}{2k + 2|k|^j}\right) \geq N'(\varphi(0, x), t),
\]

(3.26)

for all \( x \in X \), all \( t > 0 \), and any integer \( n > 0 \). So

\[
N\left(f(x) - \frac{f(k^n x)}{k^n}, t\right) \geq N'(\varphi(0, x), \frac{t}{\sum_{j=0}^{n-1} |r|^j / |2k + 2|k|^j})).
\]

(3.27)

The rest of the proof is similar to the proof of Theorem 3.1. \( \square \)

**Corollary 3.4.** Let \( X \) be a normed space and let \((\mathbb{R}, N')\) be a fuzzy Banach space. Assume that there exist real numbers \( \theta \geq 0 \) and \( 0 < p < 1 \) such that an odd mapping \( f : X \to Y \) with \( f(0) = 0 \) satisfies (3.19). Then there exists a unique additive mapping \( A : X \to Y \) satisfying (1.8) and the inequality

\[
N(f(x) - A(x), t) \geq N'(\varphi(0, x), \frac{2k + 2(|k| - |k|^p)t}{|k|}).
\]

(3.28)
Proof. Let \( \varphi(x, y) := \theta(||x||^p + ||y||^p) \) and \( |r| = |k|^p \). Applying Theorem 3.3, we get the desired results.

\[ \textbf{Theorem 3.5.} \] Let \( f : X \to Y \) be an even mapping with \( f(0) = 0 \) satisfying the inequality (3.1) and let \( \varphi : X^2 \to Z \) be a mapping for which there exists a constant \( r \in \mathbb{R} \) such that \( 0 < |r| < 1/k^2 \) and that
\[
N'(\frac{x}{k}, \frac{y}{k}, t) \geq N'(\varphi(x, y), \frac{t}{|r|}),
\]
for all \( x, y \in X \) and all \( t > 0 \). Then there exists a unique quadratic mapping \( Q : X \to Y \) satisfying (1.8) and the inequality
\[
N(f(x) - Q(x), t) \geq N'(\varphi(0, x), \frac{2(1 - |k^2r|)t}{|k^2r|}),
\]
for all \( x \in X \) and all \( t > 0 \).

Proof. Replacing \( x \) by \( kx \) in (3.1), we get
\[
N\left( f(k(x + y)) + f(k(x - y)) - f(kx + y) - f(kx - y) - \frac{2(k + 1)}{k} f(ky) + 2(k + 1)f(y), t \right) \geq N'(\varphi(kx, y), t),
\]
for all \( x, y \in X \) and all \( t > 0 \). Putting \( x = 0 \) and replacing \( y \) by \( x \) in (3.31), we have
\[
N\left( \frac{f(kx)}{k^2} - f(x), \frac{t}{|k^2|} \right) \geq N'(\varphi(0, x), t),
\]
for all \( x \in X \) and all \( t > 0 \). Replacing \( x \) by \( x/k \) in (3.32), we find
\[
N\left( k^2f\left( \frac{x}{k} \right) - f(x), \frac{|k|t}{2} \right) \geq N'(\varphi\left( 0, \frac{x}{k} \right), t),
\]
for all \( x \in X \) and all \( t > 0 \). Also, replacing \( x \) by \( x/k^n \) in (3.33), we obtain
\[
N\left( k^{2n+2}f\left( \frac{x}{k^n} \right) - k^{2n}f\left( \frac{x}{k^n} \right), \frac{|k|^{2n+1}t}{2} \right) \geq N'(\varphi\left( 0, \frac{x}{k^{n+1}} \right), t) \geq N'(\varphi(0, x), \frac{t}{|r|^{n+1}}).
\]

So
\[
N\left( k^{2n+2}f\left( \frac{x}{k^n} \right) - k^{2n}f\left( \frac{x}{k^n} \right), \frac{|k|^{2n+1}t}{2} \right) \geq N'(\varphi(0, x), t),
\]
for all $x \in X$ and all $t > 0$. Proceeding as in the proof of Theorem 3.1, we obtain that

$$N\left(f(x) - k^{2n}f\left(\frac{x}{k^n}\right), \sum_{j=0}^{n-1} \frac{|k|^{j+1}|r|^{j+1}t}{2}\right) \geq N'(\varphi(0, x), t),$$

(3.36)

for all $x \in X$, all $t > 0$, and any integer $n > 0$. So

$$N\left(f(x) - k^{2n}f\left(\frac{x}{k^n}\right), t\right) \geq N'(\varphi(0, x), \frac{t}{\sum_{j=0}^{n-1} (|k|^{j+1}|r|^{j+1}t/2)}).$$

(3.37)

The rest of the proof is similar to the proof of Theorem 3.1.

**Corollary 3.6.** Let $X$ be a normed space and let $(\mathbb{R}, N')$ be a fuzzy Banach space. Assume that there exist real numbers $\theta \geq 0$ and $p > 1$ such that an even mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.19). Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (1.8) and the inequality

$$N(f(x) - Q(x), t) \geq N'(\theta \|x\|^p, \frac{2(k^{2p} - k^2)t}{|k|}).$$

(3.38)

**Proof.** Let $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ and $|r| = |k|^{-2p}$. Applying Theorem 3.5, we get the desired results.

**Theorem 3.7.** Assume that an even mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.1) and $\varphi : X^2 \to Z$ is a mapping for which there is a constant $r \in \mathbb{R}$ satisfying $0 < |r| < k^2$ such that

$$N'(\varphi(x, y), |r|t) \geq N'(\varphi\left(\frac{x}{k}, \frac{y}{k}\right), t),$$

(3.39)

for all $x, y \in X$ and all $t > 0$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (1.8) and the following inequality

$$N(f(x) - Q(x), t) \geq N'(\varphi(0, x), \frac{2(k^2 - |r|)t}{|k|}),$$

(3.40)

for all $x \in X$ and all $t > 0$.

**Proof.** It follows from (3.32) that

$$N\left(\frac{f(kt)}{k^2} - f(x), \frac{t}{2|k|}\right) \geq N'(\varphi(0, x), t),$$

(3.41)
for all $x \in X$ and all $t > 0$. Replacing $x$ by $k^n x$ in (3.41), we obtain

$$N\left( \frac{f(k^{n+1}x)}{k^{2n+2}} - \frac{f(k^n x)}{k^{2n}} , \frac{t}{2|k|^{2n+1}} \right) \geq N'\left( \varphi(0, k^n x), \frac{t}{|r|^n} \right), \quad (3.42)$$

for all $x \in X$ and all $t > 0$. So

$$N\left( \frac{f(k^{n+1}x)}{k^{2n+2}} - \frac{f(k^n x)}{k^{2n}} , \frac{|r|^n t}{2|k|^{2n+1}} \right) \geq N'\left( \varphi(0, x), t \right), \quad (3.43)$$

for all $x \in X$ and all $t > 0$. So

$$N\left( f(x) - \frac{f(k^n x)}{k^{2n}}, t \right) \geq N'\left( \varphi(0, x), \frac{t}{\sum_{j=0}^{n-1} (|r| t/2|k|^{2j+1})} \right). \quad (3.44)$$

The rest of the proof is similar to the proof of Theorem 3.1. \qed

**Corollary 3.8.** Let $X$ be a normed space and let $(\mathbb{R}, N')$ be a fuzzy Banach space. Assume that there exist real numbers $\theta \geq 0$ and $0 < p < 1$ such that an even mapping $f : X \to Y$ with $f(0) = 0$ satisfies (3.19). Then there is a unique quadratic mapping $Q : X \to Y$ satisfying (1.8) and the inequality

$$N\left( f(x) - Q(x), t \right) \geq N'\left( \varphi(0, x), \frac{2(k^2 - k^{2p})t}{|k|} \right), \quad (3.45)$$

for all $x \in X$, all $t > 0$.

**Proof.** Let $\varphi(x, y) := \theta(||x||^p + ||y||^p)$ and $|r| = k^{2p}$. Applying Theorem 3.7, we get the desired results. \qed

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**References**

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