Research Article

On the Hyers-Ulam Stability of a General Mixed Additive and Cubic Functional Equation in $n$-Banach Spaces

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The objective of the present paper is to determine the generalized Hyers-Ulam stability of the mixed additive-cubic functional equation in $n$-Banach spaces by the direct method. In addition, we show under some suitable conditions that an approximately mixed additive-cubic function can be approximated by a mixed additive and cubic mapping.

1. Introduction and Preliminaries

A basic question in the theory of functional equations is as follows: when is it true that a function, which approximately satisfies a functional equation, must be close to an exact solution of the equation?

If the problem accepts a unique solution, we say the equation is stable (see [1]). The study of stability problems for functional equations is related to a question of Ulam [2] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [3]. The result of Hyers was generalized by Aoki [4] for approximate additive mappings and by Rassias [5] for approximate linear mappings by allowing the Cauchy difference operator $CDf(x, y) = f(x + y) - [f(x) + f(y)]$ to be controlled by $\epsilon(\|x\|^{p} + \|y\|^{p})$. In 1994, a generalization of Rassias’ theorem was obtained by Găvruţa [6], who replaced $\epsilon(\|x\|^{p} + \|y\|^{p})$ by a general control function $\varphi(x, y)$. On the other hand, several further interesting discussions, modifications, extensions, and generalizations of the original problem of Ulam have been proposed (see, e.g. [7–12] and the references therein).

Recently, Park [9] investigated the approximate additive mappings, approximate Jensen mappings, and approximate quadratic mappings in 2-Banach spaces and proved the
generalized Hyers-Ulam stability of the Cauchy functional equation, the Jensen functional equation, and the quadratic functional equation in 2-Banach spaces. This is the first result for the stability problem of functional equations in 2-Banach spaces.

In [11, 12], we introduced the following mixed additive-cubic functional equation for fixed integers $k$ with $k \neq 0, \pm 1$:

$$f(kx + y) + f(kx - y) = kf(x + y) + kf(x - y) + 2f(kx) - 2kf(x),$$

(1.1)

with $f(0) = 0$, and investigated the generalized Hyers-Ulam stability of (1.1) in quasi-Banach spaces and non-Archimedean fuzzy normed spaces, respectively.

In this paper, we investigate, approximate mixed additive-cubic mappings in $n$-Banach spaces. That is, we prove the generalized Hyers-Ulam stability of a general mixed additive-cubic equation (1.1) in $n$-Banach spaces by the direct method.

The concept of 2-normed spaces was initially developed by Gähler [13, 14] in the middle of 1960s, while that of $n$-normed spaces can be found in [15, 16]. Since then, many others have studied this concept and obtained various results; see for instance [15, 17–19].

We recall some basic facts concerning $n$-normed spaces and some preliminary results.

**Definition 1.1.** Let $n \in \mathbb{N}$, and let $X$ be a real linear space with dim $X \geq n$ and $\|\cdot,\ldots,\cdot\| : X^n \to \mathbb{R}$ a function satisfying the following properties:

(N1) $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,

(N2) $\|x_1, x_2, \ldots, x_n\|$ is invariant under permutation,

(N3) $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha|\|x_1, x_2, \ldots, x_n\|$, 

(N4) $\|x + y, x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|y, x_2, \ldots, x_n\|$

for all $\alpha \in \mathbb{R}$ and $x, y, x_1, x_2, \ldots, x_n \in X$. Then the function $\|\cdot,\ldots,\cdot\|$ is called an $n$-norm on $X$ and the pair $(X, \|\cdot,\ldots,\cdot\|)$ is called an $n$-normed space.

**Example 1.2.** For $x_1, x_2, \ldots, x_n \in \mathbb{R}^n$, the Euclidean $n$-norm $\|x_1, x_2, \ldots, x_n\|_E$ is defined by

$$\|x_1, x_2, \ldots, x_n\|_E = |\det(x_{ij})| = \text{abs} \left( \begin{array}{ccc} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{array} \right),$$

(1.2)

where $x_i = (x_{i1}, \ldots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, n$.

**Example 1.3.** The standard $n$-norm on $X$, a real inner product space of dimension dim $X \geq n$, is as follows:

$$\|x_1, x_2, \ldots, x_n\|_S = \left( \begin{array}{c} \langle x_1, x_1 \rangle \\ \vdots \\ \langle x_n, x_1 \rangle \end{array}, \begin{array}{c} \langle x_1, x_n \rangle \\ \vdots \\ \langle x_n, x_n \rangle \end{array} \right)^{1/2},$$

(1.3)

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $X$. If $X = \mathbb{R}^n$, then this $n$-norm is exactly the same as the Euclidean $n$-norm $\|x_1, x_2, \ldots, x_n\|_E$ mentioned earlier. For $n = 1$, this $n$-norm is the usual norm $\|x_1\| = (x_1, x_1)^{1/2}$.
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Definition 1.4. A sequence \( \{x_k\} \) in an \( n \)-normed space \( X \) is said to converge to some \( x \in X \) in the \( n \)-norm if

\[
\lim_{k \to \infty} \|x_k - x, y_2, \ldots, y_n\| = 0,
\]

for every \( y_2, \ldots, y_n \in X \).

Definition 1.5. A sequence \( \{x_k\} \) in an \( n \)-normed space \( X \) is said to be a Cauchy sequence with respect to the \( n \)-norm if

\[
\lim_{k,l \to \infty} \|x_k - x_l, y_2, \ldots, y_n\| = 0,
\]

for every \( y_2, \ldots, y_n \in X \). If every Cauchy sequence in \( X \) converges to some \( x \in X \), then \( X \) is said to be complete with respect to the \( n \)-norm. Any complete \( n \)-normed space is said to be an \( n \)-Banach space.

Now we state the following results as lemma (see [9] for the details).

Lemma 1.6. Let \( X \) be an \( n \)-normed space. Then,

1. For \( x_i \in X (i = 1, \ldots, n) \) and \( \gamma \), a real number,

\[
\|x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n\| = \|x_1, \ldots, x_i, \ldots, x_j \cdot \gamma, \ldots, x_n\| \quad (1.6)
\]

for all \( 1 \leq i \neq j \leq n \).

2. \( \|x, y_2, \ldots, y_n\| - \|y, y_2, \ldots, y_n\| \leq \|x - y, y_2, \ldots, y_n\| \) for all \( x, y, y_2, \ldots, y_n \in X \),

3. if \( \|x, y_2, \ldots, y_n\| = 0 \) for all \( y_2, \ldots, y_n \in X \), then \( x = 0 \),

4. for a convergent sequence \( \{x_j\} \) in \( X \),

\[
\lim_{j \to \infty} \|x_j, y_2, \ldots, y_n\| = \left\| \lim_{j \to \infty} x_j, y_2, \ldots, y_n \right\| \quad (1.7)
\]

for all \( y_2, \ldots, y_n \in X \).

2. Approximate Mixed Additive-Cubic Mappings

In this section, we investigate the generalized Hyers-Ulam stability of the generalized mixed additive-cubic functional equation in \( n \)-Banach spaces. Let \( X \) be a linear space and \( Y \) an \( n \)-Banach space. For convenience, we use the following abbreviation for a given mapping \( f : X \to Y \):

\[
Df(x, y) := f(kx + y) + f(kx - y) - kf(x + y) - kf(x - y) - 2f(kx) + 2kf(x) \quad (2.1)
\]

for all \( x, y \in X \).
Theorem 2.1. Let $X$ be a linear space and $Y$ an $n$-Banach space. Let $f: X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi: X^{n+1} \to [0, \infty)$ such that

$$\sum_{j=0}^{\infty} \frac{1}{2^j} \varphi\left(2^j x, 2^j y, u_2, \ldots, u_n\right) < \infty,$$

$$\|Df(x, y), u_2, \ldots, u_n\|_Y \leq \varphi(x, y, u_2, \ldots, u_n)$$

for all $x, y, u_2, \ldots, u_n \in X$. Then, there is a unique additive mapping $A: X \to Y$ such that

$$\|f(2x) - 8f(x) - A(x), u_2, \ldots, u_n\|_Y \leq \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \tilde{\varphi}\left(2^j x, u_2, \ldots, u_n\right)$$

for all $x, u_2, \ldots, u_n \in X$, where

$$\tilde{\varphi}(x, u_2, \ldots, u_n)$$

$$= \frac{1}{|k^3 - k|} \left\{ |k| + 1 \left[ \varphi(x, (2k + 1)x, u_2, \ldots, u_n) + \varphi(x, (2k - 1)x, u_2, \ldots, u_n) \right] + \varphi(3x, x, u_2, \ldots, u_n) + \left(8k^2 + 1\right) \varphi(x, x, u_2, \ldots, u_n) + \varphi(x, 3kx, u_2, \ldots, u_n) + \varphi(x, kx, u_2, \ldots, u_n) + k^2 \varphi(2x, 2x, u_2, \ldots, u_n) + \varphi(2x, 2kx, u_2, \ldots, u_n) + 2\varphi(x, (k + 1)x, u_2, \ldots, u_n) + 2\varphi(x, (k - 1)x, u_2, \ldots, u_n) + 2\varphi(x, x, u_2, \ldots, u_n) + 8\varphi\left(\frac{x}{2}, \frac{kx}{2}, u_2, \ldots, u_n\right) + 8|k|\varphi\left(\frac{x}{2}, \frac{(2k - 1)x}{2}, u_2, \ldots, u_n\right) + 8|k|\varphi\left(\frac{x}{2}, \frac{(2k + 1)x}{2}, u_2, \ldots, u_n\right) + 8|k|\varphi\left(\frac{x}{2}, \frac{3kx}{2}, u_2, \ldots, u_n\right) + |k| + 1 \varphi(0, (k + 1)x, u_2, \ldots, u_n) + \frac{8k^2 + 1}{|k - 1|} \varphi(0, (k - 1)x, u_2, \ldots, u_n) + \frac{|k|}{|k - 1|} \varphi(0, 3(k - 1)x, u_2, \ldots, u_n) + \frac{k^2}{|k - 1|} \varphi(0, 2(k - 1)x, u_2, \ldots, u_n) + \frac{k^2 + |k| - 1}{|k - 1|} \varphi(0, 2kx, u_2, \ldots, u_n) + \frac{8|k|}{|k - 1|} \varphi\left(0, \frac{(3k - 1)x}{2}, u_2, \ldots, u_n\right) + \frac{8|k|}{|k - 1|} \varphi\left(0, \frac{(3k - 1)x}{2}, u_2, \ldots, u_n\right) + \frac{8|k|}{|k - 1|} \varphi\left(0, \frac{(k + 1)x}{2}, u_2, \ldots, u_n\right) + \frac{8|k|}{|k - 1|} \varphi\left(0, \frac{(k + 1)x}{2}, u_2, \ldots, u_n\right) + \frac{8k^2 + 2|k| - 8}{|k - 1|} \varphi(0, kx, u_2, \ldots, u_n) \right\}. \tag{2.5}$$
Proof. Letting \( x = 0 \) in (2.3), we get

\[
\|f(y) + f(-y), u_2, \ldots, u_n\|_Y \leq \frac{1}{|k - 1|} \varphi(0, y, u_2, \ldots, u_n) \quad (2.6)
\]

for all \( y, u_2, \ldots, u_n \in X \). Putting \( y = x \) in (2.3), we have

\[
\|f((k + 1)x) + f((k - 1)x) - kf(2x) - 2f(kx) + 2k f(x), u_2, \ldots, u_n\|_Y \leq \varphi(x, x, u_2, \ldots, u_n) \quad (2.7)
\]

for all \( x, u_2, \ldots, u_n \in X \). Thus

\[
\|f(2(k + 1)x) + f(2(k - 1)x) - kf(4x) - 2f(2kx) + 2k f(2x), u_2, \ldots, u_n\|_Y \\
\leq \varphi(2x, 2x, u_2, \ldots, u_n) \quad (2.8)
\]

for all \( x, u_2, \ldots, u_n \in X \). Letting \( y = kx \) in (2.3), we get

\[
\|f((k + 1)x) + f((k - 1)x) - kf(-kx) - 2f(kx) + 2k f(x), u_2, \ldots, u_n\|_Y \leq \varphi(x, kx, u_2, \ldots, u_n) \quad (2.9)
\]

for all \( x, u_2, \ldots, u_n \in X \). Letting \( y = (k + 1)x \) in (2.3), we have

\[
\|f((k + 1)x) + f(-x) - kf((k + 2)x) - kf(-kx) - 2f(kx) + 2k f(x), u_2, \ldots, u_n\|_Y \\
\leq \varphi(x, (k + 1)x, u_2, \ldots, u_n) \quad (2.10)
\]

for all \( x, u_2, \ldots, u_n \in X \). Letting \( y = (k - 1)x \) in (2.3), we have

\[
\|f((k - 1)x) - (k + 2)f(kx) - kf(-(k - 2)x) + (2k + 1)f(x), u_2, \ldots, u_n\|_Y \\
\leq \varphi(x, (k - 1)x, u_2, \ldots, u_n) \quad (2.11)
\]

for all \( x, u_2, \ldots, u_n \in X \). Replacing \( x \) and \( y \) by \( 2x \) and \( x \) in (2.3), respectively, we get

\[
\|f((2k + 1)x) + f((2k - 1)x) - 2f(2kx) - kf(3x) + 2k f(2x) - kf(x), u_2, \ldots, u_n\|_Y \\
\leq \varphi(2x, x, u_2, \ldots, u_n) \quad (2.12)
\]

for all \( x, u_2, \ldots, u_n \in X \). Replacing \( x \) and \( y \) by \( 3x \) and \( x \) in (2.3), respectively, we get

\[
\|f((3k + 1)x) + f((3k - 1)x) - 2f(3kx) - kf(4x) - kf(2x) + 2k f(3x), u_2, \ldots, u_n\|_Y \\
\leq \varphi(3x, x, u_2, \ldots, u_n) \quad (2.13)
\]
for all \( x, u_2, \ldots, u_n \in X \). Replacing \( x \) and \( y \) by \( 2x \) and \( kx \) in (2.3), respectively, we have

\[
\| f(3kx) + f(kx) - kf((k + 2)x) - kf(-(k - 2)x) - 2f(2kx) + 2kf(2x), u_2, \ldots, u_n \|_Y
\leq \phi(2x, kx, u_2, \ldots, u_n)
\]  

(2.14)

for all \( x, u_2, \ldots, u_n \in X \). Setting \( y = (2k + 1)x \) in (2.3), we have

\[
\| f((3k + 1)x) + f(-(k + 1)x) - kf(2(k + 1)x) - kf(-2kx) - 2f(kx) + 2kf(x), u_2, \ldots, u_n \|_Y
\leq \phi(x, (2k + 1)x, u_2, \ldots, u_n)
\]  

(2.15)

for all \( x, u_2, \ldots, u_n \in X \). Letting \( y = (2k - 1)x \) in (2.3), we have

\[
\| f((3k - 1)x) + f(-(k - 1)x) - kf(-2(k - 1)x) - kf(2kx) - 2f(kx) + 2kf(x), u_2, \ldots, u_n \|_Y
\leq \phi(x, (2k - 1)x, u_2, \ldots, u_n)
\]  

(2.16)

for all \( x, u_2, \ldots, u_n \in X \). Letting \( y = 3kx \) in (2.3), we have

\[
\| f(4kx) + f(-2kx) - kf((3k + 1)x) - kf(-(3k - 1)x) - 2f(kx) + 2kf(x), u_2, \ldots, u_n \|_Y
\leq \phi(x, 3kx, u_2, \ldots, u_n)
\]  

(2.17)

for all \( x, u_2, \ldots, u_n \in X \). By (2.6), (2.7), (2.13), (2.15), and (2.16), we get

\[
\| kf(2(k + 1)x) + kf(-2(k - 1)x) + 6f(kx) - 2f(3kx) - kf(4x) + 2kf(3x) - 6kf(x), u_2, \ldots, u_n \|_Y
\leq \phi(x, (2k + 1)x, u_2, \ldots, u_n) + \phi(x, (2k - 1)x, u_2, \ldots, u_n) + \phi(3x, x, u_2, \ldots, u_n)
\]

\[
+ \phi(x, x, u_2, \ldots, u_n) + \frac{1}{|k - 1|} \phi(0, (k + 1)x, u_2, \ldots, u_n)
\]

\[
+ \frac{1}{|k - 1|} \phi(0, (k - 1)x, u_2, \ldots, u_n) + \frac{|k|}{|k - 1|} \phi(0, 2kx, u_2, \ldots, u_n)
\]

(2.18)

for all \( x, u_2, \ldots, u_n \in X \). By (2.6), (2.10), and (2.11), we have

\[
\| f((2k + 1)x) + f((2k - 1)x) - kf((k + 2)x) - kf(-(k - 2)x) - 4f(kx) + 4kf(x), u_2, \ldots, u_n \|_Y
\leq \phi(x, (k + 1)x, u_2, \ldots, u_n) + \phi(x, (k - 1)x, u_2, \ldots, u_n)
\]

\[
+ \frac{k}{|k - 1|} \phi(0, kx, u_2, \ldots, u_n)
\]

(2.19)
for all $x, u_2, \ldots, u_n \in X$. It follows from (2.12) and (2.19) that

$$
\begin{align*}
\|k f((k + 2)x) &+ k f(-(k - 2)x) - 2f(2kx) + 4f(kx) - k f(3x) + 2k f(2x) - 5k f(x), u_2, \ldots, u_n\|_Y \\
&\leq \varphi(x, (k + 1)x, u_2, \ldots, u_n) + \varphi(x, (k - 1)x, u_2, \ldots, u_n) + \varphi(2x, x, u_2, \ldots, u_n) \\
&+ \frac{1}{|k - 1|}\varphi(0, x, u_2, \ldots, u_n) + \left|\frac{k}{k - 1}\right|\varphi(0, kx, u_2, \ldots, u_n)
\end{align*}
$$

(2.20)
for all $x, u_2, \ldots, u_n \in X$. Hence,

\[
\|f(2kx) - 2f(kx) - k^3 f(2x) + 2k^3 f(x), u_2, \ldots, u_n\|_Y \\
\leq |k| \varphi\left(\frac{x}{2}, \frac{(2k+1)x}{2}, u_2, \ldots, u_n\right) + |k| \varphi\left(\frac{x}{2}, \frac{(2k-1)x}{2}, u_2, \ldots, u_n\right) + \varphi\left(\frac{x}{2}, \frac{3kx}{2}, u_2, \ldots, u_n\right) \\
+ \varphi\left(\frac{x}{2}, \frac{kx}{2}, u_2, \ldots, u_n\right) + k^2 \varphi(x, x, u_2, \ldots, u_n) + \frac{k}{k-1} \varphi\left(0, \frac{(k-1)x}{2}, u_2, \ldots, u_n\right) \\
+ \frac{k^2 - 1}{k-1} \varphi(0, kx, u_2, \ldots, u_n)
\] (2.24)

for all $x, u_2, \ldots, u_n \in X$. By (2.9), we have

\[
\|f(4kx) - k f(2(k+1)x) - k f(-2(k-1)x) - 2f(2kx) + 2kf(2x), u_2, \ldots, u_n\|_Y \\
\leq \varphi(2x, 2kx, u_2, \ldots, u_n)
\] (2.25)

for all $x, u_2, \ldots, u_n \in X$. From (2.23) and (2.25), we have

\[
\|k f(2k+1)x + k f(-2(k-1)x) - k f(3x) + (2k^3 - 2k) f(2x)\|_Y \\
\leq |k| \varphi(x, (2k+1)x, u_2, \ldots, u_n) + |k| \varphi(x, (2k-1)x, u_2, \ldots, u_n) + \varphi(x, 3kx, u_2, \ldots, u_n) \\
+ \varphi(x, kx, u_2, \ldots, u_n) + k^2 \varphi(2x, 2x, u_2, \ldots, u_n) + \varphi(2x, 2kx, u_2, \ldots, u_n) \\
+ \frac{k}{k-1} \varphi(0, (3k-1)x, u_2, \ldots, u_n) + \frac{k}{k-1} \varphi(0, (k+1)x, u_2, \ldots, u_n) \\
+ \frac{k^2 - 1}{k-1} \varphi(0, 2(k-1)x, u_2, \ldots, u_n) + \frac{k^2 - 1}{k-1} \varphi(0, 2kx, u_2, \ldots, u_n)
\] (2.26)

for all $x, u_2, \ldots, u_n \in X$. Also, from (2.18) and (2.26), we get

\[
\|2f(3kx) - 6f(kx) + (k - k^3) f(4x) - 2k f(3x) + (2k^3 - 2k) f(2x) + 6kf(x), u_2, \ldots, u_n\|_Y \\
\leq (|k| + 1) \left[ \varphi(x, (2k+1)x, u_2, \ldots, u_n) + \varphi(x, (2k-1)x, u_2, \ldots, u_n) \right] + \varphi(3x, x, u_2, \ldots, u_n) \\
+ \varphi(x, x, u_2, \ldots, u_n) + \varphi(x, 3kx, u_2, \ldots, u_n) + \varphi(x, kx, u_2, \ldots, u_n) \\
+ k^2 \varphi(2x, 2x, u_2, \ldots, u_n) + \varphi(2x, 2kx, u_2, \ldots, u_n) + \frac{|k| + 1}{|k-1|} \varphi(0, (k+1)x, u_2, \ldots, u_n)
\]
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\[
\begin{align*}
&+ \frac{1}{|k-1|} \varphi(0, (k-1)x, u_2, \ldots, u_n) + \frac{k^2 + |k|-1}{|k-1|} \varphi(0, 2kx, u_2, \ldots, u_n) \\
&+ \left| \frac{k}{k-1} \right| \varphi(0, (3k-1)x, u_2, \ldots, u_n) + \frac{k^2}{|k-1|} \varphi(0, 2(k-1)x, u_2, \ldots, u_n)
\end{align*}
\]

(2.27)

for all \( x, u_2, \ldots, u_n \in X \).

On the other hand, it follows from (2.21) and (2.27) that

\[
\|8f(2kx) - 16f(kx) + (k - k^3)f(4x) + (2k^3 - 10k)f(2x) + 16kf(x), u_2, \ldots, u_n\|_Y
\]

\[
\leq (|k| + 1) \left[ \varphi(x, (2k + 1)x, u_2, \ldots, u_n) + \varphi(x, (2k - 1)x, u_2, \ldots, u_n) \right] + \varphi(3x, x, u_2, \ldots, u_n) \\
+ k \varphi(2x, x, u_2, \ldots, u_n) + \varphi(2x, 2kx, u_2, \ldots, u_n) + 2 \varphi(x, (k + 1)x, u_2, \ldots, u_n) \\
+ 2 \varphi(x, (k - 1)x, u_2, \ldots, u_n) + 2 \varphi(2x, x, u_2, \ldots, u_n) + 2 \varphi(2x, kx, u_2, \ldots, u_n)
\]

\[
+ \frac{2}{|k-1|} \varphi(0, x, u_2, \ldots, u_n) + \frac{2|k|}{|k-1|} \varphi(0, kx, u_2, \ldots, u_n) + \frac{|k| + 1}{|k-1|} \varphi(0, (k+1)x, u_2, \ldots, u_n)
\]

\[
+ \frac{1}{|k-1|} \varphi(0, (k-1)x, u_2, \ldots, u_n) + \frac{k^2 + |k|-1}{|k-1|} \varphi(0, 2kx, u_2, \ldots, u_n)
\]

\[
+ \left| \frac{k}{k-1} \right| \varphi(0, (3k-1)x, u_2, \ldots, u_n) + \frac{k^2}{|k-1|} \varphi(0, 2(k-1)x, u_2, \ldots, u_n)
\]

(2.28)

for all \( x, u_2, \ldots, u_n \in X \). Therefore by (2.24) and (2.28), we get

\[
\|f(4x) - 10f(2x) + 16f(x), u_2, \ldots, u_n\|_Y
\]

\[
\leq \frac{1}{|k^3 - k|}
\]

\[
\times \left\{ (|k| + 1) \left[ \varphi(x, (2k + 1)x, u_2, \ldots, u_n) + \varphi(x, (2k - 1)x, u_2, \ldots, u_n) \right] \\
+ \varphi(3x, x, u_2, \ldots, u_n) + \left( 8k^2 + 1 \right) \varphi(x, x, u_2, \ldots, u_n) + \varphi(x, 3kx, u_2, \ldots, u_n) \\
+ \varphi(x, kx, u_2, \ldots, u_n) + k^2 \varphi(2x, 2x, u_2, \ldots, u_n) + \varphi(2x, 2kx, u_2, \ldots, u_n) \\
+ 2 \varphi(x, (k + 1)x, u_2, \ldots, u_n) + 2 \varphi(x, (k - 1)x, u_2, \ldots, u_n) + 2 \varphi(2x, x, u_2, \ldots, u_n) \\
+ 2 \varphi(2x, kx, u_2, \ldots, u_n) + 8 \varphi \left( \frac{x}{2}, \frac{kx}{2}, u_2, \ldots, u_n \right) + 8 |k| \varphi \left( \frac{x}{2}, \frac{(2k - 1)x}{2}, u_2, \ldots, u_n \right) \\
+ 8 |k| \varphi \left( \frac{x}{2}, \frac{(2k + 1)x}{2}, u_2, \ldots, u_n \right) + 8 \varphi \left( \frac{x}{2}, \frac{3kx}{2}, u_2, \ldots, u_n \right)
\right\}
\]
\begin{align*}
+ |k| + 1 & \varphi(0, (k + 1)x, u_2, \ldots, u_n) + \frac{8k^2 + 1}{|k - 1|} \varphi(0, (k - 1)x, u_2, \ldots, u_n) \\
+ \frac{2}{|k - 1|} & \varphi(0, x, u_2, \ldots, u_n) + \left| \frac{k}{|k - 1|} \right| \varphi(0, (3k - 1)x, u_2, \ldots, u_n) \\
+ \frac{k^2}{|k - 1|} & \varphi(0, 2(k - 1)x, u_2, \ldots, u_n) + \frac{k^2 + |k| - 1}{|k - 1|} \varphi(0, 2kx, u_2, \ldots, u_n) \\
+ \frac{8|k|}{|k - 1|} & \varphi(0, \frac{(3k - 1)x}{2}, u_2, \ldots, u_n) \\
+ \frac{8|k|}{|k - 1|} & \varphi(0, \frac{(k + 1)x}{2}, u_2, \ldots, u_n) + \frac{8k^2 + 2|k| - 8}{|k - 1|} \varphi(0, kx, u_2, \ldots, u_n) \right) \\
:= \tilde{\varphi}(x, u_2, \ldots, u_n)
\end{align*}

(2.29)

for all \( x, u_2, \ldots, u_n \in X \).

Now, let \( g : X \to Y \) be the mapping defined by \( g(x) := f(2x) - 8f(x) \) for all \( x, u_2, \ldots, u_n \in X \). Then, (2.29) means that

\[
\|f(4x) - 10f(2x) + 16f(x), u_2, \ldots, u_n\|_Y \leq \tilde{\varphi}(x, u_2, \ldots, u_n)
\]

(2.30)

for all \( x, u_2, \ldots, u_n \in X \). Also, we get

\[
\|g(2x) - 2g(x), u_2, \ldots, u_n\|_Y \leq \tilde{\varphi}(x, u_2, \ldots, u_n)
\]

(2.31)

for all \( x \in X \). Replacing \( x \) by \( 2^jx \) in (2.31) and dividing both sides of (2.31) by \( 2^{j+1} \), we get

\[
\left\| \frac{1}{2^j} g(2^jx) - \frac{1}{2^{j+1}} g(2^{j+1}x), u_2, \ldots, u_n \right\|_Y \leq \frac{1}{2^{j+1}} \tilde{\varphi}(2^jx, u_2, \ldots, u_n)
\]

(2.32)

for all \( x, u_2, \ldots, u_n \in X \) and all integers \( j \geq 0 \). For all integers \( l, m \) with \( 0 \leq l < m \), we have

\[
\left\| \frac{1}{2^l} g(2^lx) - \frac{1}{2^m} g(2^mx), u_2, \ldots, u_n \right\|_Y \leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} g(2^jx) - \frac{1}{2^{j+1}} g(2^{j+1}x), u_2, \ldots, u_n \right\|_Y
\]

(2.33)

\[
\leq \sum_{j=l}^{m-1} \frac{1}{2^{j+1}} \tilde{\varphi}(2^jx, u_2, \ldots, u_n)
\]

for all \( x, u_2, \ldots, u_n \in X \). So, we get

\[
\lim_{l,m \to \infty} \left\| \frac{1}{2^l} g(2^lx) - \frac{1}{2^m} g(2^mx), u_2, \ldots, u_n \right\|_Y = 0
\]

(2.34)
for all \( x, u_2, \ldots, u_n \in X \). This shows that the sequence \( \{(1/2^i)g(2^i x)\} \) is a Cauchy sequence in \( Y \). Since \( Y \) is an \( n \)-Banach space, the sequence \( \{(1/2^i)g(2^i x)\} \) converges. So, we can define a mapping \( A : X \to Y \) by

\[
A(x) := \lim_{j \to \infty} \frac{1}{2^j} g\left(\frac{2^j x}{2^j}\right) \tag{2.35}
\]

for all \( x \in X \). Putting \( l = 0 \), then passing the limit \( m \to \infty \) in (2.33), and using Lemma 1.6(4), we get

\[
\|g(x) - A(x), u_2, \ldots, u_n\|_Y \leq \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \|\varphi(2^j x, u_2, \ldots, u_n)\|_Y \tag{2.36}
\]

for all \( x, u_2, \ldots, u_n \in X \).

Now we show that \( A \) is additive. By Lemma 1.6, (2.2), (2.32), and (2.35), we have

\[
\|A(2x) - 2A(x), u_2, \ldots, u_n\|_Y = \lim_{j \to \infty} \left\| \frac{1}{2^j} g\left(\frac{2^j x}{2^j}\right) - \frac{1}{2^j} g\left(\frac{2^j x}{2^j}\right), u_2, \ldots, u_n \right\|_Y
= 2 \lim_{j \to \infty} \left\| \frac{1}{2^{j+1}} g\left(\frac{2^{j+1} x}{2^{j+1}}\right) - \frac{1}{2^j} g\left(\frac{2^j x}{2^j}\right), u_2, \ldots, u_n \right\|_Y
\leq \lim_{j \to \infty} \frac{1}{2^j} \|\varphi(2^j x, u_2, \ldots, u_n)\|_Y = 0
\]

for all \( x, u_2, \ldots, u_n \in X \). By Lemma 1.6(3), \( A(2x) = 2A(x) \) for all \( x \in X \). Also, by Lemma 1.6(4), (2.2), (2.3), and (2.35), we get

\[
\|DA(x, y), u_2, \ldots, u_n\|_Y
= \lim_{j \to \infty} \frac{1}{2^j} \left\| Dg\left(\frac{2^j x, 2^j y}{2^j}\right), u_2, \ldots, u_n \right\|_Y
= \lim_{j \to \infty} \frac{1}{2^j} \left\| Df\left(\frac{2^{j+1} x, 2^{j+1} y}{2^{j+1}}\right) - 8Df\left(\frac{2^j x, 2^j y}{2^j}\right), u_2, \ldots, u_n \right\|_Y
\leq \lim_{j \to \infty} \frac{1}{2^j} \left[ \left\| Df\left(\frac{2^{j+1} x, 2^{j+1} y}{2^{j+1}}\right), u_2, \ldots, u_n \right\|_Y + 8 \left\| Df\left(\frac{2^j x, 2^j y}{2^j}\right), u_2, \ldots, u_n \right\|_Y \right]
\leq \lim_{j \to \infty} \frac{1}{2^j} \left[ \varphi\left(\frac{2^{j+1} x, 2^{j+1} y, u_2, \ldots, u_n}{2^{j+1}}\right) + 8 \varphi\left(\frac{2^j x, 2^j y, u_2, \ldots, u_n}{2^j}\right) \right] = 0
\]

for all \( x, y, u_2, \ldots, u_n \in X \). By Lemma 1.6(3), \( DA(x, y) = 0 \) for all \( x, y \in X \). Hence, the mapping \( A \) satisfies (1.1). By [11, Lemma 2.3], the mapping \( x \to A(2x) - 8A(x) \) is additive. Therefore, \( A(2x) = 2A(x) \) implies that the mapping \( A \) is additive.
To prove the uniqueness of \(A\), let \(B : X \to Y\) be another additive mapping satisfying (2.4). Fix \(x \in X\). Clearly, \(A(2^l x) = 2^l A(x)\) and \(B(2^l x) = 2^l B(x)\) for all \(l \in \mathbb{N}\). It follows from (2.4) that

\[
\|A(x) - B(x), u_2, \ldots, u_n\|_Y = \left\| \frac{A(2^l x)}{2^l} - \frac{B(2^l x)}{2^l}, u_2, \ldots, u_n \right\|_Y
\]

\[
\leq \frac{1}{2^l} \left[ \left\| f \left( 2^{l+1} x \right) - 8f \left( 2^l x \right) - A \left( 2^l x \right), u_2, \ldots, u_n \right\|_Y 
+ \left\| B \left( 2^l x \right) - f \left( 2^{l+1} x \right) + 8f \left( 2^l x \right), u_2, \ldots, u_n \right\|_Y \right]
\]

\[
\leq \frac{1}{2^l} \sum_{j=0}^{\infty} \frac{1}{2^j} \tilde{q} \left( 2^{l+j} x, u_2, \ldots, u_n \right)
\]

\[
\leq \sum_{j=0}^{\infty} \frac{1}{2^{l+j}} \tilde{q} \left( 2^{l+j} x, u_2, \ldots, u_n \right) = \sum_{j=0}^{\infty} \frac{1}{2^j} \tilde{q} \left( 2^j x, u_2, \ldots, u_n \right)
\]

for all \(x, u_2, \ldots, u_n \in X\), and \(l \in \mathbb{N}\). By (2.2), we see that the right-hand side of the above inequality tends to 0 as \(l \to \infty\). Therefore, \(\|A(x) - B(x), u_2, \ldots, u_n\|_Y = 0\) for all \(u_2, \ldots, u_n \in X\). By Lemma 1.6, we can conclude that \(A(x) = B(x)\) for all \(x \in X\). So, \(A = B\). This proves the uniqueness of \(A\).

**Theorem 2.2.** Let \(X\) be a linear space and \(Y\) an \(n\)-Banach space. Let \(f : X \to Y\) be a mapping with \(f(0) = 0\) for which there is a function \(\varphi : X^{n+1} \to [0, \infty)\) such that

\[
\sum_{j=1}^{\infty} 2^j \varphi \left( \frac{x}{2^j}, u_2, \ldots, u_n \right) < \infty,
\]

\[
\|Df(x, y), u_2, \ldots, u_n\|_Y \leq \varphi(x, y, u_2, \ldots, u_n)
\]

for all \(x, y, u_2, \ldots, u_n \in X\). Then, there is a unique additive mapping \(A : X \to Y\) such that

\[
\|f(2x) - 8f(x) - A(x), u_2, \ldots, u_n\|_Y \leq \sum_{j=1}^{\infty} 2^{j-1} \tilde{q} \left( \frac{x}{2^j}, u_2, \ldots, u_n \right)
\]

(2.41)

for all \(x, u_2, \ldots, u_n \in X\), where \(\tilde{q}(x, u_2, \ldots, u_n)\) is defined as in Theorem 2.1.

**Proof.** The proof is similar to the proof of Theorem 2.1. \(\square\)

**Corollary 2.3.** Let \(X\) be a normed space and \(Y\) an \(n\)-Banach space. Let \(\theta \in [0, \infty), p_2, \ldots, p_n \in (0, \infty)\) such that \(p \neq 1\), and let \(f : X \to Y\) be a mapping with \(f(0) = 0\) such that

\[
\|Df(x, y), u_2, \ldots, u_n\|_Y \leq \theta \left( \|x\|^p_X + \|y\|^p_X \right) \|u_2\|^2_X \cdots \|u_n\|^p_X
\]

(2.42)
for all \( x, y, u_2, \ldots, u_n \in X \). Then, there exists a unique additive mapping \( A : X \to Y \) such that

\[
\| f(2x) - 8f(x) - A(x), u_2, \ldots, u_n \|_Y \leq \frac{\theta\|x\|^p_X\|u_2\|^p_X \cdots \|u_n\|^p_X}{(2 - 2^p)(k^3 - k)}
\]  

(2.43)

for all \( x, u_2, \ldots, u_n \in X \), where

\[
e = \left(1 + |k| + 2^{3-p}|k|\right)[(2k + 1)^p + (2k - 1)^p] + 2|k| + 13 + 3^p + 3|k|^p + 16k^2 + 3^p|k|^p + 2^{p+1}k^3
\]

\[+ 2^p\left(5 + |k|^p\right) + 2|k + 1|^p + 2|k - 1|^p + 2^{3-p}\left(2 + |k| + |k|^p + 3|k|^p\right) + \frac{(|k| + 1)|k + 1|^p}{|k - 1|}
\]

\[+ \frac{2}{|k - 1|} + \frac{|k|(2^{3-p} + 1)}{|k - 1|}3k - 1|^p + \frac{8k^2 + 2k - 8}{|k - 1|}|k|^p.
\]

(2.44)

Proof. Define \( \phi(x, y) = \theta\|x\|^p_X + \|y\|^p_X\|u_2\|^p_X \cdots \|u_n\|^p_X \) for all \( x, y, u_2, \ldots, u_n \in X \), and apply Theorems 2.1 and 2.2.

The following example shows that the assumption \( p \neq 1 \) cannot be omitted in Corollary 2.3.

Example 2.4. Let \( X = \mathbb{C} \) be a linear space over \( \mathbb{R} \). Define \( \| \cdot \| : X \times X \to \mathbb{R} \) by \( \|x_1, x_2\| = |a_1b_2 - a_2b_1| \), where \( x_i = a_i + b_i \in \mathbb{C}, a_j, b_j \in \mathbb{R}, j = 1, 2 \) (i.e., \( i = \sqrt{-1} \) is the imaginary unit). Then, \( (X, \| \cdot \|) \) is a 2-normed linear space.

Let \( \phi : \mathbb{C} \to \mathbb{C} \) defined by

\[
\phi(x) = \begin{cases} 
x, & \text{for } |x| < 1, \\
1, & \text{for } |x| \geq 1.
\end{cases}
\]

(2.45)

Consider the function \( f : \mathbb{C} \to \mathbb{C} \) defined by

\[
f(x) = \sum_{m=0}^{\infty} \alpha^{-m}\phi(\alpha^m x)
\]

(2.46)

for all \( x \in \mathbb{C} \), where \( \alpha > |k| \). Then, \( f \) satisfies the functional inequality

\[
\|Df(x, y), u\| \leq \frac{4\alpha^2(|k| + 1)}{\alpha - 1}(|x| + |y|)|u|
\]

(2.47)

for all \( x, y, u \in \mathbb{C} \), but there do not exist an additive mapping \( A : \mathbb{C} \to \mathbb{C} \) and a constant \( d > 0 \) such that \( \|f(x) - A(x), u\| \leq d \|x\||u| \) for all \( x, u \in \mathbb{C} \).
It is clear that \(|f(x)| \leq \alpha/ (\alpha - 1)\) for all \(x \in \mathbb{C}\). If \(|x| + |y| = 0\) or \(|x| + |y| \geq 1/\alpha\) for all \(x, y \in \mathbb{C}\), then the inequality (2.47) holds. Now suppose that \(0 < |x| + |y| < 1/\alpha\). Then, there exists an integer \(n \geq 1\) such that

\[
\frac{1}{\alpha^{n+1}} \leq |x| + |y| < \frac{1}{\alpha^n}. \tag{2.48}
\]

Hence, \(\alpha^m |kx \pm y| < 1, \alpha^m |x \pm y| < 1, \alpha^m |x| < 1\) for all \(m = 0, 1, \ldots, n - 1\). From the definition of \(f\) and (2.48), we obtain that

\[
\|Df(x, y), u\| = \left\| \sum_{m=0}^{\infty} \alpha^{-m}\phi(\alpha^m (kx + y)) + \sum_{m=0}^{\infty} \alpha^{-m}\phi(\alpha^m (kx - y)) - k \sum_{m=0}^{\infty} \alpha^{-m}\phi(\alpha^m (x + y)) 
\right.
\]

\[
- k \sum_{m=0}^{\infty} \alpha^{-m}\phi(\alpha^m (x - y)) - 2 \sum_{m=0}^{\infty} \alpha^{-m}\phi(\alpha^m kx) + 2k \sum_{m=0}^{\infty} \alpha^{-m}\phi(\alpha^m x), u \right\| \leq \frac{4\alpha^2 (|k| + 1)}{\alpha - 1} (|x| + |y|) |u|.
\]

Therefore, \(f\) satisfies (2.47). Now, we claim that the functional equation (1.1) is not stable for \(p = 1\) in Corollary 2.3. Suppose on the contrary that there exist an additive mapping \(A : \mathbb{C} \to \mathbb{C}\) and a constant \(d > 0\) such that \(\|f(x) - A(x), u\| \leq d |x| |u|\) for all \(x, u \in \mathbb{C}\). Then, there exists a constant \(c \in \mathbb{C}\) such that \(A(x) = cx\) for all rational numbers \(x\). So, we obtain that

\[
\|f(x), u\| \leq (d + |c|) |x| |u| \tag{2.50}
\]

for all rational numbers \(x\) and all \(u \in \mathbb{C}\). Let \(s \in \mathbb{N}\) with \(s + 1 > d + |c|\). If \(x\) is a rational number in \((0, \alpha^{-s})\) and \(u = bi (b \in \mathbb{R})\), then \(\alpha^m x \in (0, 1)\) for all \(m = 0, 1, \ldots, s\), and we get

\[
\|f(x), u\| = \left\| \sum_{m=0}^{s} \frac{\phi(\alpha^m x)}{\alpha^m}, u \right\| \geq \sum_{m=0}^{s} \frac{\phi(\alpha^m x)}{\alpha^m} |b| = (s + 1) x |b| > (d + |c|) x |b| = (d + |c|) |x| |u|,
\]

which contradicts (2.50).

**Theorem 2.5.** Let \(X\) be a linear space and \(Y\) an \(n\)-Banach space. Let \(f : X \to Y\) be a mapping with \(f(0) = 0\) for which there is a function \(\varphi : X^{n+1} \to [0, \infty)\) such that

\[
\sum_{j=0}^{\infty} \frac{1}{\alpha^j} \varphi(2^j x, 2^j y, u_2, \ldots, u_n) < \infty, \tag{2.52}
\]

\[
\|Df(x, y, u_2, \ldots, u_n)\|_Y \leq \varphi(x, y, u_2, \ldots, u_n) \tag{2.53}
\]
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for all \( x, y, u_2, \ldots, u_n \in X \). Then, there is a unique cubic mapping \( C : X \to Y \) such that

\[
\| f(2x) - 2f(x) - C(x), u_2, \ldots, u_n \|_Y \leq \sum_{j=0}^{\infty} \frac{1}{8^{i+1}} \tilde{\phi}(2^i x, u_2, \ldots, u_n)
\]  

(2.54)

for all \( x, u_2, \ldots, u_n \in X \), where \( \tilde{\phi}(x, u_2, \ldots, u_n) \) is defined as in Theorem 2.1.

Proof. As in the proof of Theorem 2.1, we have

\[
\| f(4x) - 10f(2x) + 16f(x), u_2, \ldots, u_n \|_Y \leq \tilde{\phi}(x, u_2, \ldots, u_n)
\]  

(2.55)

for all \( x \in X \), where \( \tilde{\phi}(x, u_2, \ldots, u_n) \) is defined as in Theorem 2.1.

Now, let \( h : X \to Y \) be the mapping defined by \( h(x) := f(2x) - 2f(x) \). By (2.55), we have

\[
\| h(2x) - 8h(x), u_2, \ldots, u_n \|_Y \leq \tilde{\phi}(x, u_2, \ldots, u_n)
\]  

(2.56)

for all \( x \in X \). Replacing \( x \) by \( 2^i x \) in (2.56) and dividing both sides of (2.56) by \( 8^{i+1} \), we get

\[
\left\| \frac{1}{8^i} h(2^i x) - \frac{1}{8^{i+1}} h(2^{i+1} x), u_2, \ldots, u_n \right\|_Y \leq \frac{1}{8^{i+1}} \tilde{\phi}(2^i x, u_2, \ldots, u_n)
\]  

(2.57)

for all \( x, u_2, \ldots, u_n \in X \) and all integers \( j \geq 0 \). For all integers \( l, m \) with \( 0 \leq l < m \), we have

\[
\left\| \frac{1}{8^l} h(2^l x) - \frac{1}{8^m} h(2^m x), u_2, \ldots, u_n \right\|_Y \leq \sum_{j=0}^{m-1} \left\| \frac{1}{8^j} h(2^j x) - \frac{1}{8^{j+1}} h(2^{j+1} x), u_2, \ldots, u_n \right\|_Y
\]  

\[
\leq \sum_{j=0}^{m-1} \frac{1}{8^{j+1}} \tilde{\phi}(2^j x, u_2, \ldots, u_n)
\]  

(2.58)

for all \( x, u_2, \ldots, u_n \in X \). So, we get

\[
\lim_{l,m \to \infty} \left\| \frac{1}{8^l} h(2^l x) - \frac{1}{8^m} h(2^m x), u_2, \ldots, u_n \right\|_Y = 0
\]  

(2.59)

for all \( x, u_2, \ldots, u_n \in X \). This shows that the sequence \( \{(1/8^l)h(2^l x)\} \) is a Cauchy sequence in \( Y \). Since \( Y \) is an \( n \)-Banach space, the sequence \( \{(1/8^l)h(2^l x)\} \) converges. So, we can define a mapping \( C : X \to Y \) by

\[
C(x) := \lim_{j \to \infty} \frac{1}{8^j} h\left( 2^j x \right)
\]  

(2.60)
for all \( x \in X \). Putting \( l = 0 \), then passing the limit \( m \to \infty \) in (2.58), and using Lemma 1.6(4), we get

\[
\|h(x) - C(x), u_2, \ldots, u_n\|_Y \leq \sum_{j=0}^{\infty} \frac{1}{8^{j+1}} \varphi \left( 2^{j} x, u_2, \ldots, u_n \right) \tag{2.61}
\]

for all \( x, u_2, \ldots, u_n \in X \).

Now we show that \( C \) is cubic. By Lemma 1.6, (2.52), (2.58), and (2.60), we have

\[
\|C(2x) - 8C(x), u_2, \ldots, u_n\|_Y = \lim_{j \to \infty} \left\| \frac{1}{8^j} h(2^{j+1} x) - \frac{1}{8^j} h(2^j x), u_2, \ldots, u_n \right\|_Y
\]

\[
= 8 \lim_{j \to \infty} \left\| \frac{1}{8^j} h(2^{j+1} x) - \frac{1}{8^j} h(2^j x), u_2, \ldots, u_n \right\|_Y \tag{2.62}
\]

\[
\leq \lim_{j \to \infty} \frac{1}{8^j} \varphi \left( 2^j x, u_2, \ldots, u_n \right) = 0
\]

for all \( x, u_2, \ldots, u_n \in X \). By Lemma 1.6(3), \( C(2x) = 8C(x) \) for all \( x \in X \). Also, by Lemma 1.6(4), (2.52), (2.53), and (2.60), we get

\[
\|DC(x, y), u_2, \ldots, u_n\|_Y
\]

\[
= \lim_{j \to \infty} \frac{1}{8^j} \left\| Dh(2^j x, 2^j y), u_2, \ldots, u_n \right\|_Y
\]

\[
= \lim_{j \to \infty} \frac{1}{8^j} \left\| Df(2^{j+1} x, 2^{j+1} y) - 2Df(2^j x, 2^j y), u_2, \ldots, u_n \right\|_Y \tag{2.63}
\]

\[
\leq \lim_{j \to \infty} \frac{1}{8^j} \left[ \left\| Df(2^{j+1} x, 2^{j+1} y), u_2, \ldots, u_n \right\|_Y + 2 \left\| Df(2^j x, 2^j y), u_2, \ldots, u_n \right\|_Y \right]
\]

\[
\leq \lim_{j \to \infty} \frac{1}{8^j} \left[ \varphi \left( 2^{j+1} x, 2^{j+1} y, u_2, \ldots, u_n \right) + 2 \varphi \left( 2^j x, 2^j y, u_2, \ldots, u_n \right) \right] = 0
\]

for all \( x, y, u_2, \ldots, u_n \in X \). By Lemma 1.6(3), \( DC(x, y) = 0 \) for all \( x, y \in X \). Hence the mapping \( C \) satisfies (1.1). By [11, Lemma 2.3], the mapping \( x \to C(2x) - 2C(x) \) is cubic. Therefore, \( C(2x) = 8C(x) \) implies that the mapping \( C \) is cubic.
proof. The proof is similar to the proof of Theorem 2.5. Let $x \in X$. Clearly, $C(2^l x) = 8^l A(x)$ and $S(2^l x) = 8^l S(x)$ for all $l \in \mathbb{N}$. It follows from (2.54) that

$$
\|C(x) - S(x), u_2, \ldots, u_n\|_Y = \left\| \frac{C(2^l x)}{8^l} - S(2^l x), u_2, \ldots, u_n \right\|_Y
$$

$$
\leq \frac{1}{8^l} \left[ \left\| f(2^{l+1} x) - 2 f(2^l x) - C(2^l x), u_2, \ldots, u_n \right\|_Y 
+ \left\| S(2^l x) - f(2^{l+1} x) + 2 f(2^l x), u_2, \ldots, u_n \right\|_Y \right]
$$

$$
\leq \frac{1}{8^l} \sum_{j=0}^\infty \frac{1}{8^j} \varphi(2^{l+1} x, u_2, \ldots, u_n)
$$

$$
\leq \sum_{j=0}^\infty \frac{1}{8^{j+1}} \varphi(2^{l+1} x, u_2, \ldots, u_n) = \sum_{j=0}^\infty \frac{1}{8^j} \varphi(2^l x, u_2, \ldots, u_n)
$$

for all $x, u_2, \ldots, u_n \in X$, and $l \in \mathbb{N}$. By (2.52), we see that the right-hand side of the above inequality tends to 0 as $l \to \infty$. Therefore, $\|C(x) - S(x), u_2, \ldots, u_n\|_Y = 0$ for all $u_2, \ldots, u_n \in X$. By Lemma 1.6, we can conclude that $C(x) = S(x)$ for all $x \in X$. So $C = S$. This proves the uniqueness of $C$. 

Theorem 2.6. Let $X$ be a linear space and $Y$ an n-Banach space. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X^{n+1} \to [0, \infty)$ such that

$$
\sum_{j=1}^\infty 8^j \varphi \left( \frac{x}{2^j}, \frac{y}{2^j}, u_2, \ldots, u_n \right) < \infty,
$$

$$
\|Df(x, y), u_2, \ldots, u_n\|_Y \leq \varphi(x, y, u_2, \ldots, u_n)
$$

for all $x, y, u_2, \ldots, u_n \in X$. Then, there is a unique cubic mapping $C : X \to Y$ such that

$$
\|f(2^l x) - f(x) - C(x), u_2, \ldots, u_n\|_Y \leq \sum_{j=1}^\infty 8^j \varphi \left( \frac{x}{2^j}, u_2, \ldots, u_n \right)
$$

for all $x, u_2, \ldots, u_n \in X$, where $\varphi(x, u_2, \ldots, u_n)$ is defined as in Theorem 2.1.

Proof. The proof is similar to the proof of Theorem 2.5. 

Corollary 2.7. Let $X$ be a normed space and $Y$ an n-Banach space. Let $\theta \in [0, \infty)$, $p_2, \ldots, r_n \in (0, \infty)$ such that $p \neq 3$, and let $f : X \to Y$ be a mapping with $f(0) = 0$ such that

$$
\|Df(x, y), u_2, \ldots, u_n\|_Y \leq \theta \left( \|x\|^p_X + \|y\|^p_X \right) \|u_2\|^r_2 \cdots \|u_n\|^r_n
$$

(2.67)
for all \( x, y, u_2, \ldots, u_n \in X \). Then, there exists a unique cubic mapping \( C : X \to Y \) such that

\[
\| f(2x) - 2f(x) - C(x), u_2, \ldots, u_n \|_Y \leq \frac{\theta \epsilon \| x \|_X^p \| u_2 \|_X^{p_2} \cdots \| u_n \|_X^{p_n}}{(8 - 2^p)(k^3 - k)}
\]

(2.68)

for all \( x, u_2, \ldots, u_n \in X \), where \( \epsilon \) is defined as in Corollary 2.3.

Proof. Define \( \varphi(x, y) = \theta(\| x \|_X^p + \| y \|_X^p)\| u_2 \|_X^{p_2} \cdots \| u_n \|_X^{p_n} \) for all \( x, y, u_2, \ldots, u_n \in X \), and apply Theorems 2.5 and 2.6.

The following example shows that the the generalized Hyers-Ulam stability problem for the case of \( p = 3 \) was excluded in Corollary 2.7.

Example 2.8. Let \( X = \mathbb{C} \) be a linear space over \( \mathbb{R} \), and let \( \| \cdot , \cdot \| : X \times X \to \mathbb{R} \) be defined as in Example 2.4. Then, \( (X, \| \cdot , \cdot \|) \) is a 2-normed linear space.

Let \( \phi : \mathbb{C} \to \mathbb{C} \) be defined by

\[
\phi(x) = \begin{cases} 
  x^3, & \text{for } |x| < 1, \\
  1, & \text{for } |x| \geq 1.
\end{cases}
\]

(2.69)

Consider the function \( f : \mathbb{C} \to \mathbb{C} \) defined by

\[
f(x) = \sum_{m=0}^{\infty} a^{-3m} \phi(a^m x)
\]

(2.70)

for all \( x \in \mathbb{C} \), where \( a > |k| \). Then, \( f \) satisfies the functional inequality

\[
\| Df(x, y), u \| \leq \frac{4a^6(|k| + 1)}{a^3 - 1} \left( |x|^3 + |y|^3 \right) |u|
\]

(2.71)

for all \( x, y, u \in \mathbb{C} \), but there do not exist a cubic mapping \( C : \mathbb{C} \to \mathbb{C} \) and a constant \( d > 0 \) such that \( \| f(x) - C(x), u \| \leq d |x|^3 |u| \) for all \( x, u \in \mathbb{C} \).

It is clear that \( |f(x)| \leq a^3/(a^3 - 1) \) for all \( x \in \mathbb{C} \). If \( |x|^3 + |y|^3 = 0 \) or \( |x|^3 + |y|^3 \geq 1/a^3 \) for all \( x, y \in \mathbb{C} \), then inequality (2.71) holds. Now suppose that \( 0 < |x|^3 + |y|^3 < 1/a^3 \). Then, there exists an integer \( n \geq 1 \) such that

\[
\frac{1}{a^{3(n+1)}} \leq |x|^3 + |y|^3 < \frac{1}{a^{3n}}.
\]

(2.72)
Hence, \(a^m|kx ± y| < 1, a^m|x ± y| < 1, a^m|x| < 1\) for all \(m = 0, 1, \ldots, n - 1\). From the definition of \(f\) and (2.72), we obtain that

\[
\|Df(x, y), u\| = \left\| \sum_{m=n}^{\infty} x^{-3m} \phi(a^m(kx + y)) + \sum_{m=n}^{\infty} y^{-3m} \phi(a^m(ky - x)) - k \sum_{m=n}^{\infty} x^{-3m} \phi(a^m(x + y)) \right\|
\]

\[
- k \sum_{m=n}^{\infty} x^{-3m} \phi(a^m(x - y)) - 2 \sum_{m=n}^{\infty} y^{-3m} \phi(a^m kx) + 2k \sum_{m=n}^{\infty} -3m \phi(a^m x), u \right\|
\]

\[
\leq \frac{4a^3(|k| + 1)}{\alpha^3 - 1} \left( |x|^3 + |y|^3 \right) |u|.
\]

(2.73)

Therefore, \(f\) satisfies (2.71). Now, we claim that the functional equation (1.1) is not stable for \(p = 3\) in Corollary 2.7. Suppose on the contrary that there exist a cubic mapping \(C: \mathbb{C} \to \mathbb{C}\) and a constant \(d > 0\) such that \(\|f(x) - C(x), u\| \leq d |x|^3 |u|\) for all \(x, u \in \mathbb{C}\). Then, there exists a constant \(\beta \in \mathbb{C}\) such that \(C(x) = \beta x^3\) for all rational numbers \(x\). So, we obtain that

\[
\|f(x), u\| \leq (d + |\beta|) |x|^3 |u|
\]

(2.74)

for all rational numbers \(x\) and all \(u \in \mathbb{C}\). Let \(s \in \mathbb{N}\) with \(s + 1 > d + |\beta|\). If \(x\) is a rational number in \((0, a^s)\) and \(u = bi (b \in \mathbb{R})\), then \(a^m x \in (0, 1)\) for all \(m = 0, 1, \ldots, s\), and we get

\[
\|f(x), u\| = \left\| \sum_{m=0}^{\infty} \frac{\phi(a^m x)}{a^{3m}}, u \right\| \geq \sum_{m=0}^{s} \frac{\phi(a^m x)}{a^{3m}} |b|
\]

\[
= (s + 1)x^3 |b| > (d + |\beta|) x^3 |b| = (d + |\beta|) |x|^3 |u|,
\]

which contradicts (2.74).

**Theorem 2.9.** Let \(X\) be a linear space and \(Y\) an \(n\)-Banach space. Let \(f : X \to Y\) be a mapping with \(f(0) = 0\) for which there is a function \(\phi : X^{n+1} \to [0, \infty)\) such that

\[
\sum_{j=0}^{\infty} \frac{1}{2^j} \phi \left( 2^j x, 2^j y, u_j, \ldots, u_n \right) < \infty,
\]

(2.76)

\[
\|Df(x, y), u_2, \ldots, u_n\|_Y \leq \phi(x, y, u_2, \ldots, u_n)
\]

(2.77)

for all \(x, y, u_2, \ldots, u_n \in X\). Then, there exist a unique additive mapping \(A : X \to Y\) and a unique cubic mapping \(C : X \to Y\) such that

\[
\|f(x) - A(x) - C(x), u_2, \ldots, u_n\|_Y \leq \frac{1}{6} \sum_{j=0}^{\infty} \left( \frac{1}{2^j} + \frac{1}{8^j} \right) \phi \left( 2^j x, u_2, \ldots, u_n \right)
\]

(2.78)

for all \(x, u_2, \ldots, u_n \in X\), where \(\tilde{\phi}(x, u_2, \ldots, u_n)\) is defined as in Theorem 2.1.
Proof. By Theorems 2.1 and 2.5, there exist an additive mapping \( A' : X \to Y \) and a cubic mapping \( C' : X \to Y \) such that

\[
\| f(2x) - 8f(x) - A'(x), u_2, \ldots, u_n \|_Y \leq \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \phi \left( 2^j x, u_2, \ldots, u_n \right),
\]

\[
\| f(2x) - 2f(x) - C'(x), u_2, \ldots, u_n \|_Y \leq \sum_{j=0}^{\infty} \frac{1}{8^{j+1}} \phi \left( 2^j x, u_2, \ldots, u_n \right)
\]

(2.79)

for all \( x, u_2, \ldots, u_n \in X \). Hence,

\[
\left\| f(x) + \frac{1}{6} A'(x) - \frac{1}{6} C'(x), u_2, \ldots, u_n \right\|_Y \leq \frac{1}{3} \sum_{j=0}^{\infty} \left( \frac{1}{2^{j+1}} + \frac{1}{8^{j+1}} \right) \phi \left( 2^j x, u_2, \ldots, u_n \right)
\]

(2.80)

for all \( x \in X \). So, we obtain (2.78) by letting \( A(x) = -(1/6) A'(x) \) and \( C(x) = (1/6) C'(x) \) for all \( x \in X \).

To prove the uniqueness of \( A \) and \( C \), let \( A'', C'' : X \to Y \) be another additive and cubic mapping satisfying (2.78). Fix \( x \in X \). Let \( A_1 = A - A'' \) and \( C_1 = C - C'' \). So,

\[
\| A_1(x) + C_1(x), u_2, \ldots, u_n \|_Y \\
\leq \| f(x) - A(x) - C(x), u_2, \ldots, u_n \|_Y + \| f(x) - A''(x) - C''(x), u_2, \ldots, u_n \|_Y \\
\leq \frac{1}{3} \sum_{j=0}^{\infty} \left( \frac{1}{2^{j+1}} + \frac{1}{8^{j+1}} \right) \phi \left( 2^j x, u_2, \ldots, u_n \right)
\]

(2.81)

for all \( x, u_2, \ldots, u_n \in X \). Then (2.76) implies that

\[
\lim_{n \to \infty} \frac{1}{8^n} \| A_1(2^n x) + C_1(2^n x), u_2, \ldots, u_n \|_Y = 0
\]

(2.82)

for all \( x, u_2, \ldots, u_n \in X \). Thus, \( C_1 = 0 \). So, it follows from (2.81) that

\[
\| A_1(x), u_2, \ldots, u_n \|_Y \leq \frac{1}{3} \sum_{j=0}^{\infty} \left( \frac{1}{2^{j+1}} + \frac{1}{8^{j+1}} \right) \phi \left( 2^j x, u_2, \ldots, u_n \right)
\]

(2.83)

for all \( u_2, \ldots, u_n \in X \). Therefore, \( A_1 = 0 \). \( \square \)

Similarly to Theorem 2.9, one can prove the following result.
Theorem 2.10. Let $X$ be a linear space and $Y$ an $n$-Banach space. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X^{n+1} \to [0, \infty)$ such that

$$\sum_{j=0}^{\infty} 2^j \varphi\left(\frac{x}{2^j} , \frac{y}{2^j} , u_2 , \ldots , u_n \right) < \infty,$$

for all $x, y, u_2, \ldots, u_n \in X$. Then, there exist a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$\| f(x) - A(x) - C(x) , u_2 , \ldots , u_n \|_Y \leq \frac{1}{6} \sum_{j=1}^{\infty} \left(2^{j-1} + 8^{j-1}\right) \tilde{\varphi}\left(\frac{x}{2^j} , u_2 , \ldots , u_n \right)$$

for all $x, u_2, \ldots, u_n \in X$, where $\tilde{\varphi}(x, u_2, \ldots, u_n)$ is defined as in Theorem 2.1.

Proof. The proof is similar to the proof of Theorem 2.9 and the result follows from Theorems 2.2 and 2.6. \(\square\)

Theorem 2.11. Let $X$ be a linear space and $Y$ an $n$-Banach space. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is a function $\varphi : X^{n+1} \to [0, \infty)$ such that

$$\sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{2^j} , \frac{y}{2^j} , u_2 , \ldots , u_n \right) < \infty, \quad \sum_{j=0}^{\infty} \frac{1}{8^j} \varphi\left(2^{j-1} x , 2^{j-1} y , u_2 , \ldots , u_n \right) < \infty,$$

for all $x, y, u_2, \ldots, u_n \in X$. Then, there exist a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$\| f(x) - A(x) - C(x) , u_2 , \ldots , u_n \|_Y \leq \varphi(x, y, u_2, \ldots, u_n)$$

for all $x, u_2, \ldots, u_n \in X$, where $\tilde{\varphi}(x, u_2, \ldots, u_n)$ is defined as in Theorem 2.1.

Proof. The proof is similar to the proof of Theorem 2.9 and the result follows from Theorems 2.2 and 2.5. \(\square\)

Corollary 2.12. Let $X$ be a normed space and $Y$ an $n$-Banach space. Let $\theta \in [0, \infty), r_2, \ldots, r_n \in (0, \infty), p \in (0, 1) \cup (1, 3) \cup (3, \infty)$, and let $f : X \to Y$ be a mapping with $f(0) = 0$ such that

$$\| Df(x, y) , u_2 , \ldots , u_n \|_Y \leq \theta\left(\|x\|_X^p + \|y\|_X^p\right) \|u_2\|_X^p \cdots \|u_n\|_X^p$$

(2.88)
for all \( x, y, u_2, \ldots, u_n \in X \). Then, there exist a unique additive mapping \( A : X \to Y \) and a unique cubic mapping \( C : X \to Y \) such that

\[
\| f(x) - A(x) - C(x), u_2, \ldots, u_n \|_Y \leq \frac{1}{6|k^3 - k|} \left( \frac{1}{|2 - 2^p|} + \frac{1}{|8 - 2^p|} \right) \theta \epsilon \| x \|_X^p \| u_2 \|_X^2 \cdots \| u_n \|_X^n
\]

(2.89)

for all \( x, u_2, \ldots, u_n \in X \), where \( \epsilon \) is defined as in Corollary 2.3.

**Proof.** Define \( \varphi(x, y) = \theta(|x|_X^p + |y|_X^p)\| u_2 \|_X^2 \cdots \| u_n \|_X^n \) for all \( x, y, u_2, \ldots, u_n \in X \), and apply Theorems 2.9–2.11.

**Remark 2.13.** The generalized Hyers-Ulam stability problem for the cases of \( p = 1 \) and \( p = 3 \) was excluded in Corollary 2.12 (see Examples 2.4 and 2.8).

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**References**


