Research Article

Minimum-Norm Fixed Point of Pseudocontractive Mappings

Habtu Zegeye, Naseer Shahzad, and Mohammad Ali Alghamdi

1 Department of Mathematics, University of Botswana, Private Bag 00704, Gaborone, Botswana
2 Department of Mathematics, King Abdulaziz University, P.O. Box. 80203, Jeddah 21589, Saudi Arabia

Correspondence should be addressed to Naseer Shahzad, nshahzad@kau.edu.sa

Received 7 May 2012; Accepted 14 June 2012

Abstract

Let $K$ be a closed convex subset of a real Hilbert space $H$ and let $T : K \to K$ be a continuous pseudocontractive mapping. Then for $\beta \in (0, 1)$ and each $t \in (0, 1)$, there exists a sequence $\{y_t\} \subset K$ satisfying $y_t = \beta P_K[(1 - t)y_t + (1 - \beta)T(y_t)]$ which converges strongly, as $t \to 0^+$, to the minimum-norm fixed point of $T$. Moreover, we provide an explicit iteration process which converges strongly to a minimum-norm fixed point of $T$ provided that $T$ is Lipschitz. Applications are also included. Our theorems improve several results in this direction.

1. Introduction

Let $K$ be a nonempty subset of a real Hilbert space $H$. A mapping $T : K \to H$ is called Lipschitz if there exists $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in K. \quad (1.1)$$

If $L \in [0, 1)$, then $T$ is called a contraction; if $L = 1$ then $T$ is called a nonexpansive. It is easy to see from (1.1) that every contraction mapping is nonexpansive, and every nonexpansive mapping is Lipschitz.

A mapping $T$ is called strongly pseudocontractive if there exists $\alpha \in (0, 1)$ such that inequality

$$\langle Tx - Ty, x - y \rangle \leq \alpha \|x - y\|^2, \quad (1.2)$$
holds for all \( x, y \in K \). \( T \) is called pseudocontractive if the inequality

\[
\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2,
\]

holds for all \( x, y \in K \). Note that inequality (1.3) can be equivalently written as

\[
\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in K.
\]

It is easy to see that nonexpansive and strongly pseudocontractive mappings are pseudocontractive mappings. However, the converse may not be true (see [1, 2] for details).

Interest in pseudocontractive mappings stems mainly from their firm connection with the important class of nonlinear monotone mappings, where a mapping \( A \) with domain \( D(A) \) and range \( R(A) \) in \( H \) is called monotone if the inequality

\[
\langle Ax - Ay, x - y \rangle \geq 0,
\]

holds for every \( x, y \in D(A) \). We note that \( A \) is monotone if and only if \( T := I - A \) is pseudocontractive, and hence a zero of \( A, N(A) := \{ x \in D(A) : Ax = 0 \} \) is a fixed point of \( T, F(T) := \{ x \in D(T) : Tx = x \} \).

Let \( K \) be a nonempty closed convex subset of a real Hilbert space \( H \) and \( T : K \to K \) a pseudocontractive mapping. Assume that the set of fixed points of \( T \) is nonempty. It is known from [3] that \( F(T) \) is closed and convex.

Let the variational inequality (VI) be given as finding a point \( x^* \) with the property that

\[
x^* \in F(T) \text{ such that } \langle x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T).
\]

Then, \( x^* \) is the minimum-norm fixed point of \( T \) which exists uniquely and is exactly the (nearest point or metric) projection of the origin onto \( F(T) \), that is, \( x^* = P_{F(T)}(0) \). We also observe that the minimum-norm fixed point of pseudocontractive \( T \) is the minimum-norm solution of a monotone operator equation \( Ax = 0 \), where \( A = (I - T) \).

It is quite often to seek the minimum-norm solution of a given nonlinear problem. In an abstract way, we may formulate such problems as finding a point \( x^* \) with the property

\[
x^* \in K, \quad \|x^*\| = \min_{x \in K} \|x\|.
\]

In other words, \( x^* \) is the projection of the origin onto \( K \), that is,

\[
x^* = P_K(0).
\]

A typical example is the split feasibility problem (SFP), formulated as finding a point \( x^* \) with the property that

\[
x^* \in K, \quad Ax^* \in Q,
\]
where $K$ and $Q$ are nonempty closed convex subsets of the infinite-dimension real Hilbert spaces $H_1$ and $H_2$, respectively, and $A$ is bounded linear mapping from $H_1$ to $H_2$. Equation (1.9) models many applied problems arising from image reconstructions and learning theory (see, e.g., [4]). Some works on the finite dimensional setting with relevant projection methods for solving image recovery problems can be found in [5–7]. Defining the proximity function $f$ by

$$ f(x) := \frac{1}{2} \|Ax - P_QAx\|^2, \quad (1.10) $$

we consider the convex optimization problem:

$$ \min_{x \in K} f(x) := \min_{x \in K} \frac{1}{2} \|Ax - P_QAx\|^2. \quad (1.11) $$

It is clear that $x^*$ is a solution to the split feasibility problem (1.9) if and only if $x^* \in K$ and $Ax^* - P_QAx^* = 0$ which is the minimum-norm solution of the minimization problem (1.11).

Motivated by the above split feasibility problem, we study the general case of finding the minimum-norm fixed point of a pseudocontractive mapping $T : K \to K$, that is, we find minimum norm fixed point of $T$ which satisfies

$$ x^* \in F(T) \quad \text{such that } \|x^*\| = \min\{\|x\| : x \in F(T)\}. \quad (1.12) $$

Let $T : K \to K$ be a nonexpansive self-mapping on closed convex subset $K$ of a Banach space $E$. For a given $u \in K$ and for a given $t \in (0,1)$ define a contraction $T_t : K \to K$ by

$$ T_t x = (1-t)u + tTx, \quad x \in K. \quad (1.13) $$

By Banach contraction principle, it yields a fixed point $z_t \in K$ of $T_t$, that is, $z_t$ is the unique solution of the equation:

$$ z_t = (1-t)u + tTz_t. \quad (1.14) $$

Browder [8] proved that as $t \to 1$, $z_t$ converges strongly to a fixed point of $T$ which is closer to $u$, that is, the nearest point projection of $u$ onto $F(T)$. In 1980, Reich [9] extended the result of Browder to a more general Banach spaces. Furthermore, Takahashi and Ueda [10] and Morales and Jung [11] improved results of Reich [9] to the class of continuous pseudocontractive mappings. For other results on pseudocontractive mappings, we refer to [12–15].

We note that the above methods can be used to find the minimum-norm fixed point $x^*$ of $T$ if $0 \in K$. However, if $0 \notin K$ neither Browder’s, Reich’s, Takahashi and Ueda’s, nor Morales and Jung’s method works to find minimum-norm fixed point of $T$.

Our concern is now the following: is it possible to construct a scheme, implicit or explicit, which converges strongly to the minimum-norm fixed point of $T$ for any closed convex domain $K$ of $T$?
In this direction, Yang et al. [4] introduced an implicit and explicit iteration processes which converge strongly to the minimum-norm fixed point of nonexpansive self-mapping $T$, in real Hilbert spaces. In fact, they proved the following theorems.

**Theorem YLY1**(see [4]). Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$ and $T : K \to K$ a nonexpansive mapping with $F(T) \neq \emptyset$. For $\beta \in (0,1)$ and each $t \in (0,1)$, let $y_t$ be defined as the unique solution of fixed point equation:

$$y_t = \beta Ty_t + (1 - \beta) P_K [(1 - t)y_t], \quad t \in (0,1). \quad (1.15)$$

Then the net $\{y_t\}$ converges strongly, as $t \to 0$, to the minimum-norm fixed point of $T$.

**Theorem YLY2**(see [4]). Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$, and let $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For a given $x_0 \in K$, define a sequence $\{x_n\}$ iteratively by

$$x_{n+1} = \beta Tx_n + (1 - \beta) P_K [(1 - \alpha_n)x_n], \quad n \geq 1, \quad (1.16)$$

where $\beta \in (0,1)$ and $\alpha_n \in (0,1)$, satisfying certain conditions. Then the sequence $\{x_n\}$ converges strongly to the minimum-norm fixed point of $T$.

A natural question arises whether the above theorems can be extended to a more general class of pseudocontractive mappings or not.

Let $K$ be a closed convex subset a real Hilbert space $H$ and let $T : K \to K$ be continuous pseudocontractive mapping.

It is our purpose in this paper to prove that for $\beta \in (0,1)$ and each $t \in (0,1)$, there exists a sequence $\{y_t\} \subset K$ satisfying $y_t = \beta P_K [(1 - t)y_t] + (1 - \beta)T(y_t)$ which converges strongly, as $t \to 0^+$, to the minimum-norm fixed point of $T$. Moreover, we provide an explicit iteration process which converges strongly to the minimum-norm fixed point of $T$ provided that $T$ is Lipschitz. Our theorems improve Theorem YLY1 and Theorem YLY2 of Yang et al. [4] and Theorems 3.1, and 3.2 of Cai et al. [16].

2. Preliminaries

In what follows, we shall make use of the following lemmas.

**Lemma 2.1** (see [11]). Let $H$ be a real Hilbert space. Then, for any given $x, y \in H$, the following inequality holds:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle. \quad (2.1)$$

**Lemma 2.2** (see [17]). Let $K$ be a closed and convex subset of a real Hilbert space $H$. Let $x \in H$. Then $x_0 = P_K x$ if and only if

$$\langle z - x_0, x - x_0 \rangle \leq 0, \quad \forall z \in K. \quad (2.2)$$
Lemma 2.3 (see [18]). Let \( \{\lambda_n\}, \{\alpha_n\}, \) and \( \{\gamma_n\} \) be sequences of nonnegative numbers satisfying the conditions: \( \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \) and \( \gamma_n/\alpha_n \to 0, \) as \( n \to \infty. \) Let the recursive inequality:

\[
\lambda_{n+1} \leq \lambda_n - \alpha_n \varphi(\lambda_{n+1}) + \gamma_n, \quad n = 1, 2, \ldots,
\]

be given where \( \varphi : [0, \infty) \to [0, \infty) \) is a strictly increasing function such that it is positive on \( (0, \infty) \) and \( \varphi(0) = 0. \) Then \( \lambda_n \to 0, \) as \( n \to \infty. \)

Lemma 2.4 (see [3]). Let \( H \) be a real Hilbert space, \( K \) be a closed convex subset of \( H \) and \( T : K \to K \) be a continuous pseudocontractive mapping, then

(i) \( F(T) \) is closed convex subset of \( K; \)

(ii) \((I - T)\) is demiclosed at zero, that is, if \( \{x_n\} \) is a sequence in \( K \) such that \( x_n \to x \) and \( Tx_n - x_n \to 0, \) as \( n \to \infty, \) then \( x = T(x). \)

Lemma 2.5 (see [19]). Let \( H \) be a real Hilbert space. Then for all \( x, y \in H \) and \( \alpha \in [0, 1], \) the following equality holds:

\[
\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2.
\]

3. Main Results

Theorem 3.1. Let \( K \) be a nonempty closed and convex subset of a real Hilbert space \( H. \) Let \( T : K \to K \) be a continuous pseudocontractive mapping with \( F(T) \neq \emptyset. \) Then for \( \beta \in (0, 1) \) and each \( t \in (0, 1), \) there exists a sequence \( \{y_t\} \subset K \) satisfying the following condition:

\[
y_t = \beta P_K[(1 - t)y_t] + (1 - \beta)T(y_t)
\]

and the net \( \{y_t\} \) converges strongly, as \( t \to 0^{+}, \) to the minimum-norm fixed point of \( T. \)

Proof. For \( \beta \in (0, 1) \) and each \( t \in (0, 1) \) let \( T_t(y) := \beta P_K[(1 - t)y] + (1 - \beta)T(y). \) Then using nonexpansiveness of \( P_K \) and pseudocontractivity of \( T, \) for \( x, y \in K, \) we have that

\[
\langle T_t x - T_t y, x - y \rangle = \beta \langle P_K[(1 - t)x] - P_K[(1 - t)y], x - y \rangle
\]

\[
+ (1 - \beta)\langle T(x) - T(y), x - y \rangle
\]

\[
\leq \beta(1 - t)\|x - y\|^2 + (1 - \beta)\|x - y\|^2
\]

\[
\leq (1 - t\beta)\|x - y\|^2.
\]

This implies that \( T_t \) is strongly pseudocontractive on \( K. \) Thus, by Corollary 1 of [20] \( T_t \) has a unique fixed point, \( y_t, \) in \( K. \) This means that the equation:

\[
y_t = \beta P_K[(1 - t)y_t] + (1 - \beta)T(y_t)
\]
has a unique solution for each \( t \in (0, 1) \). Furthermore, since \( F(T) \neq \emptyset \), for \( y^* \in F(T) \), we have that

\[
\|y_t - y^*\|^2 = \langle \beta P_k [(1-t)y_t] + (1-\beta)Ty_t, y_t - y^* \rangle \\
= \beta \langle P_k [(1-t)y_t] - P_k y^*, y_t - y^* \rangle + (1-\beta) \langle Ty_t - Ty^*, y_t - y^* \rangle \\
\leq \beta \| (1-t)y_t - y^* \| \cdot \| y_t - y^* \| + (1-\beta) \| y_t - y^* \|^2 \\
\leq \beta [(1-t)\| y_t - y^* \| + t\| y^* \|] \| y_t - y^* \| + (1-\beta) \| y_t - y^* \|^2,
\]

which implies that

\[
\|y_t - y^*\| \leq \beta (1-t)\| y_t - y^* \| + \beta t\| y^* \| + (1-\beta) \| y_t - y^* \|,
\]

and hence \( \|y_t - y^*\| \leq \| y^* \|. \) Therefore, \( \{y_t\} \) and hence \( \{Ty_t\} \) is bounded.

Furthermore, from (3.3) and using nonexpansiveness of \( P_k \) we get that

\[
\|y_t - Ty_t\| = \| \beta P_k [(1-t)y_t] + (1-\beta)T(y_t) - Ty_t \| \\
= \beta \| P_k [(1-t)y_t] - P_k Ty_t \| \\
\leq \beta \| (1-t)y_t - Ty_t \| \\
\leq \beta \| y_t - Ty_t \| + \beta t\| y_t \|,
\]

which implies that

\[
\|y_t - Ty_t\| \leq \frac{\beta}{(1-\beta)} t\| y_t \| \longrightarrow 0, \quad \text{as} \quad t \longrightarrow 0.
\]

Furthermore, from (3.3), convexity of \( \| \cdot \|^2 \), (1.4), and (3.7), we get that

\[
\|y_t - y^*\|^2 = \| (1-\beta) (Ty_t - y^*) + \beta \langle P_k [(1-t)y_t] - P_k y^* \| \|^2 \\
= (1-\beta) \| Ty_t - y^* \|^2 + \beta \| P_k [(1-t)y_t] - P_k y^* \|^2 \\
\leq (1-\beta) \| y_t - y^* \|^2 + (1-\beta) \| Ty_t - y_t \|^2 + \beta \| (1-t)y_t - y^* \|^2 \\
\leq (1-\beta) \| y_t - y^* \|^2 + (1-\beta) \| Ty_t - y_t \|^2 + \beta \| (1-t)y_t - y^* \|^2 \\
\leq (1-\beta) \| y_t - y^* \|^2 + \frac{\beta^2}{(1-\beta)} t^2\| y_t \|^2 \\
+ \beta \big( \| y_t - y^* \|^2 - 2t\| y_t - y^* \|^2 - 2t\langle y^*, y_t - y^* \rangle + t^2\| y_t \|^2 \big).
\]
This implies that
\[ \|y_t - y^*\|^2 \leq \langle y^*, y^* - y_t \rangle + tM, \]
for some \( M > 0 \).

Now, for \( t_n \to 0 \), as \( n \to \infty \), let \( \{y_n\} \) be a subsequence of \( \{y_t\} \) such that \( y_n \to y' \). Then, we have from (3.7) and Lemma 2.4 that \( y' \in F(T) \). Furthermore, replacing \( y^* \) by \( y' \) in (3.9) and the fact that \( y_n \to y' \) imply that
\[ \|y_n - y'\|^2 \leq \langle y', y' - y_n \rangle \to 0 \quad \text{as } n \to \infty, \]
which implies that
\[ y_n \to y', \quad \text{as } n \to \infty. \]

Thus, from (3.9) and (3.11), we have that
\[ \|y' - y^*\|^2 \leq \langle y^*, y^* - y' \rangle, \quad \text{as } n \to \infty, \]
which is equivalent to the inequality:
\[ \langle y', y^* - y' \rangle \geq 0 \quad \text{and hence } y' = P_T0. \]

If there is another subsequence \( \{y_m\} \) of \( \{y_t\} \) such that \( y_m \to y'' \), similar argument gives that \( y'' = P_T0 \), which implies, by uniqueness of \( P_T0 \), that \( y'' = y' \). Therefore, the net \( y_t \to y' = P_T0 \) which is the minimum-norm zero of fixed point of \( T \). The proof is complete.

We now state and prove a convergence theorem for the minimum-norm zero of a monotone mapping \( A \).

**Theorem 3.2.** Let \( H \) be a real Hilbert space. Let \( A : H \to H \) be a continuous monotone mapping with \( N(A) \neq \emptyset \). Then for \( \beta \in (0, 1) \) and each \( t \in (0, 1) \), there exists a sequence \( \{y_t\} \subset H \) satisfying the following condition:
\[ y_t = \beta(1-t)y_t + (1-\beta)(I-A)y_t, \]
and the net \( \{y_t\} \) converges strongly, as \( t \to 0^+ \), to the minimum-norm zero of \( A \).

**Proof.** Let \( Tx := (I-A)x \). Then, we get that \( T \) is continuous pseudocontractive mapping with \( F(T) = N(A) \neq \emptyset \). Moreover, since \( P_H \) is an identity mapping on \( H \), when \( A \) is replaced with \( (I-T) \) scheme (3.14) reduces to scheme (3.1), and hence the conclusion follows from Theorem 3.1.

If in Theorem 3.1, we consider \( \{t_n\}, \{\beta_n\} \subset (0, 1) \) such that \( t_n \to 0, \beta_n \to 0 \) and \( y_n := y_{t_n} \), the method of proof of Theorem 3.1 provides the following corollary.
**Corollary 3.3.** Let $K$ be a nonempty closed and convex subset of a real Hilbert space $H$. Let $T : K \rightarrow K$ be continuous pseudocontractive mapping with $F(T) \neq \emptyset$. Then the sequence $\{y_n\} \subset K$ defined by

$$y_n = \beta_n P_K[(1-t_n)y_n] + (1-\beta_n)T(y_n),$$

(3.15)

where $\{t_n\}, \{\beta_n\} \subset (0,1)$ such that $t_n \rightarrow 0$, $\beta_n \rightarrow 0$, as $n \rightarrow \infty$, converges strongly, as $n \rightarrow \infty$, to the minimum-norm fixed point of $T$.

The following proposition and lemma play an important role in proving the next theorem.

**Proposition 3.4.** Let $K$ be a nonempty closed and convex subset of a real Hilbert space $H$. Let $T : K \rightarrow K$ be continuous pseudocontractive mapping. Then the sequence $\{y_n\}$ in (3.15) satisfies the following inequality:

$$\|y_n - y_{n-1}\| \leq \frac{\theta_{n-1} - \theta_n}{\theta_n t_n} \left[\|y_n\| + \|P_K[(1-t_n)y_{n-1}]\|\right] + \frac{\theta_{n-1}}{\theta_n} \frac{t_n - t_{n-1}}{t_n} \|y_{n-1}\|,$$

(3.16)

where $\theta_n := \beta_n/(1-\beta_n)$ for $\{\beta_n\}$ decreasing sequence.

**Proof.** If we put $\theta_n := \beta_n/(1-\beta_n)$, (3.15) reduces to

$$y_n = Ty_n + \theta_n (P_K[(1-t_n)y_n] - y_n).$$

(3.17)

Thus, using pseudocontractivity of $T$ and nonexpansiveness of $P_K$ we get that

$$\|y_n - y_{n-1}\|^2 = \|Ty_n + \theta_n (P_K[(1-t_n)y_n] - y_n) - Ty_{n-1} - \theta_{n-1} (P_K[(1-t_{n-1})y_{n-1}] - y_{n-1})\|^2$$

$$= \|Ty_n - Ty_{n-1} + \theta_{n-1}y_{n-1} - \theta_n y_n + \theta_{n-1}y_n - \theta_n y_{n-1}\|^2$$

$$+ \theta_n P_K[(1-t_n)y_n] - \theta_{n-1} P_K[(1-t_{n-1})y_{n-1}]\|^2$$

$$= \langle Ty_n - Ty_{n-1} + \theta_{n-1}(y_{n-1} - y_n) + (\theta_{n-1} - \theta_n) y_n, y_n - y_{n-1} \rangle$$

$$+ \langle \theta_n P_K[(1-t_n)y_n] - \theta_{n-1} P_K[(1-t_{n-1})y_{n-1}], y_n - y_{n-1} \rangle$$

$$+ \langle \theta_{n-1} P_K[(1-t_n)y_{n-1}] - \theta_n P_K[(1-t_{n-1})y_{n-1}], y_n - y_{n-1} \rangle$$

$$\leq \|y_n - y_{n-1}\|^2 - \theta_{n-1} \|y_n - y_{n-1}\|^2 + (\theta_{n-1} - \theta_n) \|y_n\|$$

$$\times \|y_n - y_{n-1}\| + \theta_n \|1-t_n\| \|y_n - y_{n-1}\|^2$$

$$+ (\theta_n - \theta_{n-1}) \|P_K[(1-t_n)y_{n-1}]\| \|y_{n-1} - y_n\|$$

$$+ \theta_{n-1} |t_n - t_{n-1}| \|y_{n-1}\| \|y_n - y_{n-1}\|,$$

(3.18)
which implies, using the fact that $\theta_n$ is decreasing, that

\[
\|y_n - y_{n-1}\| \leq (1 - \theta_n) \|y_n - y_{n-1}\| + |\theta_n - \theta_{n-1}| \|y_n\| + \|P_K[(1 - t_n)y_{n-1}]\| + (1 - t_n) \|y_n - y_{n-1}\| + |\theta_n - \theta_{n-1}| \|y_n\| + \|P_K[(1 - t_n)y_{n-1}]\| + \frac{\theta_n - \theta_{n-1}}{\theta_n t_n} \|y_n - y_{n-1}\|. \tag{3.19}
\]

and hence

\[
\|y_n - y_{n-1}\| \leq \frac{\|y_n\| + \|P_K[(1 - t_n)y_{n-1}]\| + \frac{\|y_n - y_{n-1}\| + \|P_K[(1 - t_n)y_{n-1}]\|}{\theta_n t_n}}{\theta_n t_n} \|y_n - y_{n-1}\|. \tag{3.20}
\]

The proof is complete. \hfill \Box

For the rest of this paper, let $\{\lambda_n\}$, $\{\theta_n\}$ (decreasing) and $\{t_n\}$ be real sequences in $(0, 1]$ satisfying the following conditions: (i) $\lim_{n \to \infty} \lambda_n = 0 = \lim_{n \to \infty} t_n$; (ii) $\lambda_n(1 + \theta_n) \leq 1$; (iii) $\sum \lambda_n \theta_n t_n = \infty$, $\lim_{n \to \infty} \lambda_n / \theta_n = 0$; (iv) $\lim_{n \to \infty} (\lambda_n \theta_n t_n^2) = 0$ and $\lim_{n \to \infty} (t_n - 1) = 0$. Examples of real sequences which satisfy these conditions are $\lambda_n = 1/(n + 1)^{1/2}$, $\theta_n = 1/(n + 1)^{1/4}$ and $t_n = 1/(n + 1)^{1/14}$.

Lemma 3.5. Let $K$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : K \to K$ be a Lipschitz pseudocontractive mapping with Lipschitz constant $L \geq 0$ and $F(T) \neq \emptyset$. Let a sequence 

\[ x_n := (1 - \lambda_n)x_n + \lambda_n T x_n - \lambda_n \theta_n (x_n - P_K[(1 - t_n)x_n]), \tag{3.21} \]

for all positive integers $n \geq 1$. Then \{x_n\} is bounded.

Proof. We follow the method of proof of Chidume and Zegeye [21]. Since $\lambda_n/(\theta_n t_n) \to 0$, there exists $N_0 > 0$ such that $\lambda_n/(\theta_n t_n) \leq d := 1/(2(3 + L)^2)$, for all $n \geq N_0$. Let $x^* \in F(T)$ and $r > 0$ be sufficiently large such that $B_r(x^*) \subset B_r(x^*)$ and $\|x^*\| \leq r/(2(4 + L))$. Now, we show by induction that \{x_n\} belongs to $B := B_r(x^*)$ for all integers $n \geq N_0$. By construction, we have $x_{N_0} \in B$. Assume that $x_n \in B$ for any $n > N_0$. Then, we prove that $x_{n+1} \in B$. Suppose $x_{n+1}$ is
Theorem 3.2: Let $T : H \rightarrow H$ be a bounded mapping, $K$ a nonempty closed convex subset of $H$, and $x^* \in K$. Let $\{t_n\}_{n=1}^{\infty}$ be a sequence in $(0, 1)$, $\{\lambda_n\}_{n=1}^{\infty}$ a sequence in $(0, 1)$, and $\{\theta_n\}_{n=1}^{\infty}$ a sequence in $(0, 1)$ such that $\sum_{n=1}^{\infty} \lambda_n = 1$. Suppose that $\|x_n - x^*\| \leq r$, $\sum_{n=1}^{\infty} \lambda_n = 1$, and $\sum_{n=1}^{\infty} \theta_n = 1$. Then the sequence $\{x_n\}_{n=1}^{\infty}$ generated by the following recursion formula

$$
\begin{align*}
x_{n+1} &= (1 - \lambda_n) (x_n - \theta_n (x_n - P_K (1 - t_n)x_n)) + \lambda_n (x_n - TX_n)
\end{align*}
$$

satisfies the relation

$$
\|x_{n+1} - x^*\| \leq \|x_n - x^*\| - \lambda_n \|x_n - x^*\| - \theta_n \|x_n - P_K (1 - t_n)x_n\| + \lambda_n \|x_n - TX_n\|
$$

for all $n \in \mathbb{N}$. Furthermore, if $T$ is a nonexpansive map, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $x^*$. 

Proof: We shall prove the theorem by induction on $n$. For $n = 1$, since $x_1 = x_0$, the inequality holds trivially. Assume that the inequality holds for some $n \geq 1$. We consider two cases.

1. If $\lambda_n = 0$, then $x_{n+1} = x_n$, and the inequality trivially holds.

2. If $\lambda_n > 0$, then

$$
\begin{align*}
x_{n+1} &= (1 - \lambda_n) (x_n - \theta_n (x_n - P_K (1 - t_n)x_n)) + \lambda_n (x_n - TX_n) \\
      &= x_n - \theta_n (x_n - P_K (1 - t_n)x_n) + \lambda_n (x_n - TX_n) \\
      &= x_n - \theta_n (x_n - P_K (1 - t_n)x_n) - \lambda_n (x_n - TX_n)
\end{align*}
$$

Then

$$
\begin{align*}
\|x_{n+1} - x^*\| &= \|x_n - x^* - \lambda_n \|T(1 - t_n)x_n\| + \theta_n \|x_n - P_K (1 - t_n)x_n\| + \lambda_n \|x_n - TX_n\|
\end{align*}
$$

By the induction hypothesis, we have

$$
\|x_{n+1} - x^*\| \leq \|x_n - x^*\| - \lambda_n \|x_n - x^*\| - \theta_n \|x_n - P_K (1 - t_n)x_n\| + \lambda_n \|x_n - TX_n\|
$$

for all $n \in \mathbb{N}$.
Since \( \|x_{n+1} - x^*\| > \|x_n - x^*\| \), from (3.24) we get that

\[
\|x_{n+1} - x^*\| \leq \frac{\lambda_n}{\theta_n t_n} (3 + L)^2 \|x_n - x^*\| + (4 + L) \|x^*\|,
\]

and hence \( \|x_{n+1} - x^*\| \leq r \), since \( x_n \in B, \|x^*\| \leq r/(2(4 + L)) \) and \( \lambda_n/\theta_n t_n \leq 1/2(3 + L)^2 \) for all \( n \geq N_0 \). But this is a contradiction. Therefore, \( x_n \in B \) for all positive integers \( n \geq N_0 \), and hence the sequence \( \{x_n\} \) is bounded.

For the next theorem, let \( \{y_n\} \) denotes the sequence defined by \( y_n := y_{sn} = s_n Ty_{sn} + (1 - s_n)P_K[(1 - t_n)y_n], s_n = 1/(1 + \theta_n) \), for all \( n \geq 1 \), guaranteed by Corollary 3.3 (which reduces to \( \theta_n(P_K[(1 - t_n)y_n] - y_n) - (y_n - Ty_n) = 0 \).

**Theorem 3.6.** Let \( K \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( T : K \to K \) be a Lipschitz pseudocontractive mapping with Lipschitz constant \( L \geq 0 \) and \( F(T) \neq \emptyset \). Let a sequence \( \{x_n\} \) be generated from arbitrary \( x_1 \in K \) by

\[
x_{n+1} := (1 - \lambda_n)x_n + \lambda_n Tx_n - \lambda_n\theta_n(x_n - P_K[(1 - t_n)x_n]),
\]

for all positive integers \( n \geq 1 \). Then \( \{x_n\} \) converges strongly to the minimum-norm fixed point of \( T \), as \( n \to \infty \).

**Proof.** By Lemma 3.5, we have that the sequence \( \{x_n\} \) is bounded. Now, we show that it converges strongly to a minimum-norm fixed point of \( T \). But from (3.26) and Lemma 2.1, we have that

\[
\begin{align*}
\|x_{n+1} - y_n\|^2 &\leq \|x_n - y_n\|^2 - 2\lambda_n\theta_n \langle (x_{n+1} - y_n), j(x_{n+1} - y_n) \rangle \\
&\quad + 2\lambda_n \theta_n \langle x_{n+1} - y_n, (x_n - Tx_n) \rangle \\
&\quad - \theta_n (x_n - P_K[(1 - t_n)x_n]), j(x_{n+1} - y_n) \\
&= \|x_n - y_n\|^2 - 2\lambda_n \theta_n \|x_{n+1} - y_n\|^2 + 2\lambda_n \theta_n (x_{n+1} - x_n) \\
&\quad + \|x_{n+1} - y_n\|^2 - \theta_n(P_K[(1 - t_n)x_n] - y_n) - (y_n - Ty_n) \\
&\quad - (y_n - Ty_n)] + \theta_n(P_K[(1 - t_n)x_n] - P_K[(1 - t_n)y_n]) \\
&\quad + ((x_{n+1} - Tx_{n+1}) - (x_n - Tx_n)), j(x_{n+1} - y_n)).
\end{align*}
\]

Observe that by the property of \( y_n \) and pseudocontractivity of \( T \) we have \( \theta_n(P_K[(1 - t_n)x_n] - y_n) - (y_n - Ty_n) = 0 \) (see (3.17)) and \((x_{n+1} - Tx_{n+1}) - (y_n - Ty_n), j(x_{n+1} - y_n) \geq 0 \) for all \( n \geq 1 \). Thus, we have from (3.27) that

\[
\begin{align*}
\|x_{n+1} - y_n\|^2 &\leq \|x_n - y_n\|^2 - 2\lambda_n \theta_n \|x_{n+1} - y_n\|^2 + 2\lambda_n \theta_n (x_{n+1} - x_n) \\
&\quad + \|x_{n+1} - y_n\|^2 - \theta_n(P_K[(1 - t_n)x_n] - P_K[(1 - t_n)x_{n+1}]) \\
&\quad + P_K[(1 - t_n)x_{n+1}] - P_K[(1 - t_n)y_n]) \\
&\quad + (x_{n+1} - Tx_{n+1}) - (x_n - Tx_n), j(x_{n+1} - y_n)).
\end{align*}
\]
Corollary 3.7. Let \( H \) be a real Hilbert space. Let \( A : H \to H \) be a Lipschitz monotone mapping with Lipschitz constant \( L \geq 0 \) and \( N(A) \neq \emptyset \). Let a sequence \( \{x_n\} \) be generated from arbitrary \( x_1 \in H \) by

\[
x_{n+1} = x_n - \lambda_n Ax_n + \lambda_n \theta_n t_n x_n,
\]

for all positive integers \( n \). Then \( \{x_n\} \) converges strongly to the minimum-norm solution of the equation \( Ax = 0 \).

\[\text{Proof.}\] Let \( T := (I - A) \). Then \( T \) is a Lipschitz pseudocontractive mapping with Lipschitz constant \( L' := (L + 1) \), and the minimum-norm solution of the equation \( Ax = 0 \) is the minimum-norm fixed point of \( T \). Moreover, if we replace \( T \) by \( (I - A) \) in (3.26), then the equation reduces to (3.32). Thus, the conclusion follows from Theorem 3.6. \( \square \)
4. Applications

For the rest of this paper, let $H$ be a Hilbert space and $A : H \to H$ a bounded linear operator. Consider the convexly constrained linear inverse problem, which has extensively been discussed in the literature (see, e.g., [22]), given by:

$$x \in K, \quad Ax = b,$$

where $K$ is closed and convex subset of $H$ and $b \in H$, which is a special case of the SFP problem (1.9). Set

$$\varphi(x) := \frac{1}{2} \|Ax - b\|^2.$$  \hfill (4.2)

The least-square solution of (4.1) is the least-norm minimizer of the minimization problem (4.2). Let $\Omega$ denote the solution set of (4.2). It is known that $\Omega$ is nonempty if and only if $P_{\partial \Omega}(b) \in A(K)$. In this case, $\Omega$ has a unique element with minimum norm which is a least-square solution of (4.1), that is, there exists a unique point $x^* \in H$ such that

$$\|x^*\| = \min \{\|x\| : x \in \Omega\}. \hfill (4.3)$$

We note that $\varphi(x)$ is a quadratic function with gradient:

$$\nabla \varphi(x) = A^*(Ax - b), \hfill (4.4)$$

where $A^*$ is adjoint of $A$. Let $\gamma > 0$ and $x^* \in \Omega$. Thus, $x^*$ is the minimum-norm solution of the minimization problem (4.2) if and only if $x^*$ a solution of

$$\gamma \nabla \varphi(x) = \gamma A^*(Ax - b) = 0. \hfill (4.5)$$

Now, we state applications of our theorems.

**Theorem 4.1.** Assume that the solution set of convexly constrained linear inverse problem (4.1) with $K := H$, a real Hilbert space, is nonempty and that $\nabla \varphi$ is monotone. Then for $\beta \in (0, 1)$ and each $t \in (0, 1)$, there exists a sequence $\{y_t\} \subset H$ satisfying the following condition:

$$y_t = \beta (1 - t)y_t + (1 - \beta) (y_t - \gamma A^*(Ay_t - b)), \hfill (4.6)$$

where $A^*$ is adjoint of $A$, and the net $\{y_t\}$ converges strongly, as $t \to 0^+$, to the minimum-norm solution of the split feasibility problem (4.1).

**Proof.** We note that $\varphi(x)$ is continuously differentiable function with gradient:

$$\nabla \varphi(x) = A^*(Ax - b), \hfill (4.7)$$
where $A^*$ is adjoint of $A$, which is Lipschitz (see Lemma 8.1 of [5]) and monotone (by hypothesis). Thus, the conclusion follows from Theorem 3.2.

**Theorem 4.2.** Assume that the solution set of split feasibility problem (4.1) is nonempty and that $\nabla_\rho$ with $K := H$, a real Hilbert space, is monotone. Let a sequence $\{x_n\}$ be generated from arbitrary $x_1 \in E$ by

$$x_{n+1} = x_n - \lambda_n \gamma A^*(Ax_n - b) + \lambda_n \theta_n t_n x_n, \quad (4.8)$$

for all positive integers $n$, where $\gamma > 0$ and $A^*$ is adjoint of $A$. Then, $\{x_n\}$ converges strongly to the minimum-norm solution of the split feasibility problem (4.1).

**Remark 4.3.** Theorem 3.1 improves Theorem YLY1 and Theorem 3.1 of Cai et al. [16] to a more general class of pseudocontractive mappings. Moreover, Theorem 3.6 improves Theorem YLY1 and Theorem 3.2 of Cai et al. [16] in the sense that our scheme provides a minimum-norm fixed point of pseudocontractive mapping $T$.

**References**


