Research Article

Strong Convergence Theorems for Maximal Monotone Operators with Nonspraying Mappings in a Hilbert Space

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We prove the strong convergence theorems for finding a common element of the set of fixed points of a nonspraying mapping $T$ and the solution sets of zero of a maximal monotone mapping and an $\alpha$-inverse strongly monotone mapping in a Hilbert space. Manaka and Takahashi (2011) proved weak convergence theorems for maximal monotone operators with nonspraying mappings in a Hilbert space; there we introduced new iterative algorithms and got some strong convergence theorems for maximal monotone operators with nonspraying mappings in a Hilbert space.

1. Introduction

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $C$ be a nonempty closed convex subset of $H$. We denote by $F(T)$ the set of fixed point of $T$. Then, a mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The mapping $T : C \to C$ is said to be firmly nonexpansive if $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle$ for all $x, y \in C$; see, for instance, Browder [1] and Goebel and Kirk [2]. The mapping $T : C \to C$ is said to be firmly nonspraying [3] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|x - Ty\|^2, \quad (1.1)$$

for all $x, y \in C$. Iemoto and Takahashi [4] proved that $T : C \to C$ is nonspraying if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty \rangle, \quad (1.2)$$
for all $x, y \in C$. It is not hard to know that a nonspreading mapping is deduced from a firmly nonexpansive mapping; see [5, 6], and a firmly nonexpansive mapping is a nonexpansive mapping.

Many studies have been done for structuring the fixed point of nonexpansive mapping $T$. In 1953, Mann [7] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n,$$  \hspace{1cm} (1.3)

where the initial guess $x_1 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0,1]$. It is known that under appropriate settings, the sequence $\{x_n\}$ converges weakly to a fixed point of $T$. However, even in a Hilbert space, Mann iteration may fail to converge strongly, for example see [8].

Some attempts to construct iteration method guaranteeing the strong convergence have been made. For example, Halpern [9] proposed the following so-called Halpern iteration:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n,$$  \hspace{1cm} (1.4)

where $u, x_1 \in C$ are arbitrary and $\{\alpha_n\}$ is a real sequence in $[0,1]$ which satisfies $\alpha_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$. Then, $\{x_n\}$ converges strongly to a fixed point of $T$; see [9, 10].

In 1975, Baillon [11] first introduced the nonlinear ergodic theorem in Hilbert space as follows:

$$S_n x = \sum_{k=0}^{n-1} T^k x$$  \hspace{1cm} (1.5)

converges weakly to a fixed point of $T$ for some $x \in C$.

Recently, in the case when $T : C \to C$ is a nonexpansive mapping, $A : C \to H$ is an $\alpha$-inverse strongly monotone mapping, and $B \in H \times H$ is a maximal monotone operator, Takahashi et al. [12] proved a strong convergence theorem for finding a point of $F(T) \cap (A + B)^{-1}(0)$, where $F(T)$ is the set of fixed points of $T$ and $(A + B)^{-1}(0)$ is the set of zero points of $A + B$.

In 2011, Manaka and Takahashi [13] for finding a point of the set of fixed points of $T$ and the set of zero points of $A + B$ in a Hilbert space, they introduced an iterative scheme as follows:

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)T(f_{I_n}(I - \lambda_n A)x_n),$$  \hspace{1cm} (1.6)

where $T$ is a nonspreading mapping, $A$ is an $\alpha$-inverse strongly monotone mapping, and $B$ is a maximal monotone operator such that $f_{I_n} = (I - \lambda B)^{-1}$; $\{\beta_n\}$ and $\{\lambda_n\}$ are sequences which satisfy $0 < c \leq \beta_n \leq d < 1$ and $0 < a \leq \lambda_n \leq b < 2a$. Then they proved that $\{x_n\}$ converges weakly to a point $p = \lim_{n \to \infty} P_{F(T) \cap (A + B)^{-1}(0)}x_n$.

Motivated by above authors, we generalize and modify the iterative algorithms (1.5) and (1.6) for finding a common element of the set of fixed points of a nonspreading mapping $T$ and the set of zero points of monotone operator $A + B$ ($A$ is an $\alpha$-inverse strongly monotone
mapping, and $B$ is a maximal monotone operator. First, we prove that the sequence generated by our iterative method is weak convergence under the property conditions. Then, we prove that the strong convergence in a Hilbert space. As expected, we get some weak and strong convergence theorems about the common element of the set of fixed points of a nonspecifying mapping and the set of zero points of an $\alpha$-inverse strongly monotone mapping and a maximal monotone operator in a Hilbert space.

2. Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $C$ be a nonempty closed convex subset of $H$. A set-valued mapping $B : D(B) \subseteq H \rightarrow H$ is said to be monotone if for any $x, y \in D(B)$ and $x^* \in Bx$ and $y^* \in By$, it holds that

$$\langle x - y, x^* - y^* \rangle \geq 0.$$ (2.1)

A monotone operator $B$ on $H$ is said to be maximal if $B$ has no monotone extension, that is, its graph is not properly contained in the graph of any other monotone operator on $H$. For a maximal monotone operator $B$ on $H$ and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1} : 2^H \rightarrow D(B)$, which is called the resolvent of $B$ for $r > 0$. Let $B$ be a maximal monotone operator on $H$, and let $B^{-1}(0) = \{ x \in H : 0 \in Bx \}$. For a constant $\alpha > 0$, the mapping $A : C \rightarrow H$ is said to be an $\alpha$-inverse strongly monotone if for any for all $x, y \in C$,

$$\langle x - y, Ax - Ay \rangle \geq \alpha \| Ax - Ay \|^2.$$ (2.2)

Remark 2.1. It is not hard to know that if $A$ is an $\alpha$-inverse strongly monotone mapping, then it is $1/\alpha$-Lipschitzian and hence uniformly continuous. Clearly, the class of monotone mappings include the class of an $\alpha$-inverse strongly monotone mappings.

Remark 2.2. It is well known that if $T : C \rightarrow C$ is a nonexpansive mapping, then $I - T$ is 1/2-inverse strongly monotone, where $I$ is the identity mapping on $H$; see, for instance, [14]. It is known that the resolvent $J_r$ is firmly nonexpansive and $B^{-1}(0) = F(J_r)$ for all $r > 0$.

For a single-valued mapping $T$, a point $p$ is called a fixed point of $T$ if $p =Tp$. For a multivalued mapping $T$, a point $p$ is called a fixed point of $T$ if $p \in Tp$. The set of fixed points of $T$ is denoted by $F(T)$.

Let $E$ be a uniformly convex real Banach space, $K$ be a nonempty closed convex subset of $E$. A multivalued mapping $T : K \rightarrow CB(K)$ is said to be as follows.

(i) Contraction if there exists a constant $k \in [0,1)$ such that

$$H(Tx,Ty) \leq k \| x - y \|, \quad \forall x,y \in K.$$ (2.3)

(ii) Nonexpansive if

$$H(Tx,Ty) \leq \| x - y \|, \quad \forall x,y \in K.$$ (2.4)
(iii) Quasinonexpansive if $F(T) \neq \emptyset$ and

$$H(Tx, Tp) \leq \|x - p\|, \quad \forall x \in K, \forall p \in F(T). \quad (2.5)$$

It is well known that every nonexpansive multivalued mapping $T$ with $F(T) \neq \emptyset$ is multivalued quasinonexpansive. But there exist multivalued quasi-nonexpansive mappings that are not multivalued nonexpansive. It is clear that if $T$ is a quasi-nonexpansive multivalued mapping, then $F(T)$ is closed.

A Banach space $E$ is said to satisfy Opial’s condition if whenever $\{x_n\}$ is a sequence in $E$ which converges weakly to $x$, then

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|, \quad \forall y \in E, \; x \neq y. \quad (2.6)$$

**Lemma 2.3** (Manaka and Takahashi [13]). Let $H$ be a real Hilbert space, and let $C$ be a nonempty closed convex subset of $H$. Let $\alpha > 0$. Let $A$ be an $\alpha$-inverse strongly monotone mapping of $C$ into $H$, and let $B$ be a maximal monotone operator on $H$ such that the domain of $B$ is included in $C$. Let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of $B$ for any $\lambda > 0$. Then, the following hold

(i) if $u, v \in (A + B)^{-1}(0)$, then $Au = Av$;

(ii) for any $\lambda > 0$, $u \in (A + B)^{-1}(0)$ if and only if $u = J_\lambda(I - \lambda A)u$.

**Lemma 2.4** (Schu [15]). Suppose that $E$ is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers $n$. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of $E$ such that $\limsup_{n \to \infty} \|x_n\| \leq r$, $\limsup_{n \to \infty} \|y_n\| \leq r$, and $\lim_{n \to \infty} \|t_n x_n + (1 - t_n) y_n\| = r$ hold for some $r \geq 0$. Then, $\liminf_{n \to \infty} \|x_n - y_n\| = 0$.

**Lemma 2.5** (Liu [16] and Xu [17]). Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property as follows

$$a_{n+1} \leq (1 - t_n)a_n + b_n + t_n c_n, \quad (2.7)$$

where $\{t_n\}$, $\{b_n\}$, and $\{c_n\}$ satisfy the restrictions as follows

(i) $\sum_{n=0}^{\infty} t_n = \infty$,

(ii) $\sum_{n=0}^{\infty} b_n < \infty$,

(iii) $\limsup_{n \to \infty} c_n \leq 0$.

Then, $\{a_n\}$ converges to zero as $n \to \infty$.

### 3. Strong Convergence Theorem

In this section, we prove the strong convergence theorems for finding a common element in common set of the fixed sets of a nonspreading mapping and the solution sets of zero of a maximal monotone operator and an $\alpha$-inverse strongly monotone operator and in a Hilbert space.
Theorem 3.1. Let $C$ be a nonempty convex closed subset of a real Hilbert space $H$, let $A : C \to H$ be an $\alpha$-inverse strongly monotone, let $B : D(B) \subseteq C \to 2^H$ be maximal monotone, let $J_1 = (I + \lambda B)^{-1}$ be the resolvent of $B$ for any $\lambda > 0$, and let $T : C \to C$ be a nonspreading mapping. Assume that $F := F(T) \cap (A + B)^{-1}(0) \neq \emptyset$. We define

$$x_1 = x \in C, \text{ arbitrarily,}$$
$$z_n = J_{\lambda_n}(I - \lambda_n A)x_n,$$
$$y_n = \frac{1}{n} \sum_{k=1}^n T^k z_n,$$
$$x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n,$$  \hspace{1cm} (3.1)

where $\{\alpha_n\}$ is sequences in $[0, 1]$ such that $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$. There exists $a, b$ such that $0 < a \leq \lambda_n \leq b < 2a$ for each $n \in N$. Then, $\{x_n\}$ converges strongly to $Pu$, and $P$ is the metric projection of $H$ onto $F$.

Proof. First, we prove that $\{x_n\}$ is bounded and $\lim_{n \to \infty} \|x_n - p\|$ exists for each $p \in F(T)$. In fact, from Lemma 2.3, we have $p = J_{\lambda_n}(I - \lambda_n A)p$, together with (3.1) and $A$ is an $\alpha$-inverse strongly monotone, we get that

$$\|z_n - p\|^2 = \|J_{\lambda_n}(I - \lambda_n A)x_n - J_{\lambda_n}(I - \lambda_n A)p\|^2 \leq \|(I - \lambda_n A)x_n - (I - \lambda_n A)p\|^2 \leq \|x_n - p\|^2 - 2\lambda_n (x_n - p, Ax_n - Ap) + \lambda_n^2 \|Ax_n - Ap\|^2 \leq \|x_n - p\|^2 - 2\lambda_n \alpha \|Ax_n - Ap\|^2 + \lambda_n^2 \|Ax_n - Ap\|^2 \leq \|x_n - p\|^2 - \lambda_n (2\alpha - \lambda_n) \|Ax_n - Ap\|^2 \leq \|x_n - p\|^2.$$  \hspace{1cm} (3.2)

From the definition of $y_n$ and $T$ is nonspreading mapping, we obtain that

$$\|y_n - p\| = \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k z_n - p \right\| = \frac{1}{n} \sum_{k=1}^{n-1} \|T^k z_n - p\| \leq \frac{1}{n} \sum_{k=1}^{n-1} \|z_n - p\| = \|z_n - p\| \leq \|x_n - p\|.$$  \hspace{1cm} (3.3)

Together with (3.1), we have that

$$\|x_{n+1} - p\| = \|\alpha_n u + (1 - \alpha_n)y_n - p\| \leq \alpha_n \|u - p\| + (1 - \alpha_n) \|y_n - p\| \leq \alpha_n \|u - p\| + (1 - \alpha_n) \|x_n - p\|.$$  \hspace{1cm} (3.4)
Hence, we get that
\[
\|x_{n+1} - p\| \leq \max\{\|u - p\|, \|x_n - p\|\},
\]
for all \(n \in N\). This means that \(\{x_n - p\}\) is bounded, so \(\{x_n\}\) is bounded. From \(T\) is nonspreading, (3.3), and (3.2), we get that \(\{y_n\}, \{z_n\},\) and \(\{T^n z_n\}\) are all bounded.

Since \(\{x_n\}\) is bounded, there exists a subsequence \(\{x_{n_i}\}\) of \(\{x_n\}\) such that \(\lim_{i \to \infty} \|x_{n_i} - p\|\) exists. Since \(\{x_{n_i}\}\) is bounded, there exists a subsequence \(\{x_{n_{i_k}}\}\) of \(\{x_{n_i}\}\) such that \(x_{n_{i_k}} \to w\) \(\in C\) as \(i \to \infty\). Now, we prove that \(w \in F(T)\). Since \(\|x_{n+1} - y_n\| = a_n\|u - y_n\|\), replacing \(n\) by \(n_{i_k}\), we have \(\|x_{n_{i_k+1}} - y_{n_{i_k}}\| = a_{n_{i_k}}\|u - y_{n_{i_k}}\|\). Together with \(a_n \to 0\) and \(\{y_n\}\) is bounded, we obtain that \(\lim_{i \to \infty} \|x_{n_{i_k+1}} - y_{n_{i_k}}\| = 0\), so we have \(y_{n_{i_k}} \to w\).

Let \(n \in N\). Since \(T\) is nonspreading, we have that for all \(y \in C\) and \(k = 0, 1, 2, \ldots, n - 1\),
\[
\left\|T^{k+1} z_n - Ty\right\|^2 \leq \left\|T^k z_n - y\right\|^2 + 2\left\langle T^k z_n - T^{k+1} z_n, y - Ty\right\rangle
\]
\[
= \left\|T^k z_n - Ty\right\|^2 + \left\|Ty - y\right\|^2 + 2\left\langle T^k z_n - Ty, Ty - y\right\rangle + 2\left\langle T^k z_n - T^{k+1} z_n, y - Ty\right\rangle.
\]
Summing these inequalities from \(k = 0\) to \(n - 1\) and dividing by \(n\), we have
\[
\frac{1}{n} \left(\|T^n z_n - Ty\|^2 - \|z_n - Ty\|^2\right) \leq \|Ty - y\|^2 + 2\left\langle y_n - Ty, Ty - y\right\rangle + \frac{2}{n}\left\langle z_n - T^n z_n, y - Ty\right\rangle.
\]
Replacing \(n\) by \(n_{i_k}\), we have
\[
\frac{1}{n_{i_k}} \left(\|T^{n_{i_k}} z_{n_{i_k}} - Ty\|^2 - \|z_{n_{i_k}} - Ty\|^2\right)
\]
\[
\leq \|Ty - y\|^2 + 2\left\langle y_{n_{i_k}} - Ty, Ty - y\right\rangle + \frac{2}{n_{i_k}}\left\langle z_{n_{i_k}} - T^{n_{i_k}} z_{n_{i_k}}, y - Ty\right\rangle.
\]
Since \(\{z_n\}\) and \(\{T^n z_n\}\) are bounded, we have that
\[
0 \leq \|Ty - y\|^2 + 2\left\langle w - Ty, Ty - y\right\rangle
\]
as \(i \to \infty\). Putting \(y = w\), we have
\[
0 \leq \|Tw - w\|^2 + 2\langle w - Tw, Tw - w\rangle = -\|Tw - w\|^2.
\]
Hence, \(w \in F(T)\).
Next, we prove that $w \in (A + B)^{-1}(0)$. From (3.2) and (3.3) we have that

$$
\|x_{n+1} - p\|^2 \leq \alpha_n \|u - p\|^2 + (1 - \alpha_n) \|y_n - p\|^2
$$

$$
\leq \alpha_n \|u - p\|^2 + (1 - \alpha_n) \|z_n - p\|^2
$$

$$
\leq \alpha_n \|u - p\|^2 + (1 - \alpha_n) \left( \|x_n - p\|^2 - \lambda_n (2\alpha - \lambda_n) \|Ax_n - Ap\|^2 \right)
$$

$$
= \alpha_n \left( \|u - p\|^2 - \|x_n - p\|^2 \right) + \|x_n - p\|^2 - (1 - \alpha_n) \lambda_n (2\alpha - \lambda_n) \|Ax_n - Ap\|^2.
$$

(3.11)

We rewrite above inequality as follows:

$$
(1 - \alpha_n) \lambda_n (2\alpha - \lambda_n) \|Ax_n - Ap\|^2 \leq \alpha_n \left( \|u - p\|^2 - \|x_n - p\|^2 \right) + \|x_n - p\|^2 - \|x_{n+1} - p\|^2.
$$

(3.12)

Replacing $n$ by $n_k$, we have

$$
(1 - \alpha_{n_k}) \lambda_{n_k} (2\alpha - \lambda_{n_k}) \|Ax_{n_k} - Ap\|^2
$$

$$
\leq \alpha_{n_k} \left( \|u - p\|^2 - \|x_{n_k} - p\|^2 \right)
$$

$$
+ \|x_{n_k} - p\|^2 - \|x_{n_k+1} - p\|^2.
$$

(3.13)

Together with $\lim_{n \to \infty} \alpha_n = 0$, $0 < a \leq \lambda_n \leq b < 2\alpha$ and since $\lim_{k \to \infty} \|x_{n_k} - p\|$ exists, we obtain that

$$
\lim_{k \to \infty} \|Ax_{n_k} - Ap\| = 0.
$$

(3.14)

Since $J_{\lambda_n}$ is firmly nonexpansive, and from (3.2), we have that

$$
\|z_n - p\|^2 = \|J_{\lambda_n} (I - \lambda_n A)x_n - J_{\lambda_n} (I - \lambda_n A)p\|^2
$$

$$
\leq \langle z_n - p, (I - \lambda_n A)x_n - (I - \lambda_n A)p \rangle
$$

$$
= \frac{1}{2} \left( \|z_n - p\|^2 + \|(I - \lambda_n A)x_n - (I - \lambda_n A)p\|^2
$$

$$
- \|z_n - p - (I - \lambda_n A)x_n + (I - \lambda_n A)p\|^2 \right)
$$

$$
\leq \frac{1}{2} \left( \|z_n - p\|^2 + \|x_n - p\|^2 - \|z_n - p - (I - \lambda_n A)x_n + (I - \lambda_n A)p\|^2 \right)
$$

$$
= \frac{1}{2} \left( \|z_n - p\|^2 + \|x_n - p\|^2 - \|z_n - x_n\|^2 - 2\lambda_n \langle z_n - x_n, Ax_n - Ap \rangle - \lambda_n^2 \|Ax_n - Ap\|^2 \right).
$$

(3.15)
This means that

\[
\|z_n - p\|^2 \leq \|x_n - p\|^2 - \|z_n - x_n\|^2 - 2\lambda_n\langle z_n - x_n, Ax_n - Ap \rangle - \lambda_n^2\|Ax_n - Ap\|^2. \tag{3.16}
\]

Together with (3.1) and (3.3), we have

\[
\|x_{n+1} - p\|^2 \leq \alpha_n\|u - p\|^2 + (1 - \alpha_n)\|y_n - p\|^2 \\
\leq \alpha_n\|u - p\|^2 + (1 - \alpha_n)\|z_n - p\|^2 \\
\leq \alpha_n\|u - p\|^2 + (1 - \alpha_n) \\
\times \left\{ \|x_n - p\|^2 - \|z_n - x_n\|^2 - 2\lambda_n\langle z_n - x_n, Ax_n - Ap \rangle - \lambda_n^2\|Ax_n - Ap\|^2 \right\} \\
\leq \alpha_n\|u - p\|^2 + \|x_n - p\|^2 - \|z_n - x_n\|^2 \\
- 2\lambda_n\langle z_n - x_n, Ax_n - Ap \rangle - \lambda_n^2\|Ax_n - Ap\|^2. \tag{3.17}
\]

Therefore, we have

\[
\|z_n - x_n\|^2 \leq \alpha_n\|u - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
- 2\lambda_n\langle z_n - x_n, Ax_n - Ap \rangle - \lambda_n^2\|Ax_n - Ap\|^2. \tag{3.18}
\]

Replacing \(n\) by \(n_k\), we have

\[
\|z_{n_k} - x_{n_k}\|^2 \leq \alpha_{n_k}\|u - p\|^2 + \|x_{n_k} - p\|^2 - \|x_{n_{k+1}} - p\|^2 \\
- 2\lambda_{n_k}\langle z_{n_k} - x_{n_k}, Ax_{n_k} - Ap \rangle - \lambda_{n_k}^2\|Ax_{n_k} - Ap\|^2. \tag{3.19}
\]

Since \(\lim_{k \to \infty}\|x_{n_k} - p\|\) exists, from (3.14) and \(\lim_{n \to \infty}\alpha_n = 0\), we obtain

\[
\lim_{n \to \infty}\|z_{n_k} - x_{n_k}\| = 0. \tag{3.20}
\]

Since \(A\) is Lipschitz continuous, we also obtain

\[
\lim_{n \to \infty}\|Az_{n_k} - Ax_{n_k}\| = 0. \tag{3.21}
\]
By the definition of $J_{\lambda_n}$ and (3.1), we have that

$$z_n = (I - \lambda_n B)^{-1}(I - \lambda_n A)x_n$$

$$\iff (I - \lambda_n A)x_n \in (I - \lambda_n B)z_n = z_n + \lambda_n Bz_n$$

$$\iff x_n - z_n - \lambda_n Ax_n \in \lambda_n Bz_n$$

$$\iff \frac{1}{\lambda_n}(x_n - z_n - \lambda_n Ax_n) \in Bz_n.$$

Since $B$ is monotone, so for $(e, f) \in B$, we have that

$$\left\langle z_n - e, \frac{1}{\lambda_n}(x_n - z_n - \lambda_n Ax_n) - f \right\rangle \geq 0,$$

and hence

$$\left\langle z_n - e, x_n - z_n - \lambda_n(Ax_n + f) \right\rangle \geq 0.$$

Replacing $n$ by $n_k$, we have that

$$\left\langle z_{n_k} - e, x_{n_k} - z_{n_k} - \lambda_{n_k}(Ax_{n_k} + f) \right\rangle \geq 0.$$  

Since $A$ is an $\alpha$-inverse strongly monotone, we have

$$\left\langle x_{n_k} - w, Ax_{n_k} - Aw \right\rangle \geq \alpha \|Ax_{n_k} - Aw\|^2.$$

This means that $Ax_{n_k} \to Aw$ as $i \to \infty$. From (3.20) and $x_{n_k} \to w$, we get that $z_{n_k} \to w$, together with (3.25), we have that

$$\left\langle w - e, -Aw - f \right\rangle \geq 0.$$

Since $B$ is maximal monotone, so $(-Aw) \in Bw$. That is, $w \in (A + B)^{-1}(0)$.

Now, we prove that $x_n \to Pu$ as $n \to \infty$. Without loss of generality, we may assume that there exists a subsequence $\{x_{n_i+1}\}$ of $\{x_{n+1}\}$ such that

$$\limsup_{n \to \infty} \left\langle u - Pu, x_{n+1} - Pu \right\rangle = \lim_{i \to \infty} \left\langle u - Pu, x_{n_i+1} - Pu \right\rangle.$$

Since $P$ is the metric projection of $H$ onto $F$ and $x_{n+1} \to w \in F$, we have

$$\lim_{i \to \infty} \left\langle u - Pu, x_{n_i+1} - Pu \right\rangle = \left\langle u - Pu, w - Pu \right\rangle \leq 0.$$
This implies that
\[ \lim_{n \to \infty} \langle u - Pu, x_{n+1} - Pu \rangle \leq 0. \tag{3.30} \]

From (2.1), (3.1), and (3.3), we have
\begin{align*}
\|x_{n+1} - Pu\|^2 &= \|(1 - \alpha_n)(y_n - Pu) + \alpha_n(u - Pu)\|^2 \\
&\leq (1 - \alpha_n)^2\|y_n - Pu\|^2 + 2\alpha_n\langle u - Pu, x_{n+1} - Pu \rangle \\
&\leq (1 - \alpha_n)\|x_n - Pu\|^2 + 2\alpha_n\langle u - Pu, x_{n+1} - Pu \rangle. \tag{3.31}
\end{align*}

From Lemma 2.5 and (3.30), we have
\[ \lim_{n \to \infty} \|x_n - Pu\| = 0. \tag{3.32} \]

This means that \( x_n \to Pu \) as \( n \to \infty \).

\[ \square \]

References


