Research Article

Convergence Theorems for a Common Point of Solutions of Equilibrium and Fixed Point of Relatively Nonexpansive Multivalued Mapping Problems

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We introduce an iterative process which converges strongly to a common point of set of solutions of equilibrium problem and set of fixed points of finite family of relatively nonexpansive multivalued mappings in Banach spaces.

1. Introduction

Let \(E\) be a real Banach space with dual \(E^*\). The function \(\phi : E \times E \to \mathbb{R^+}\), defined by

\[
\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \text{for } x, y \in E,
\]

is studied by Alber [1] and Reich [2], where \(J\) is the normalized duality mapping from \(E\) to \(2^{E^*}\) defined by \(Jx := \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}\), where \(\langle \cdot, \cdot \rangle\) denotes the generalized duality pairing. It is well known that \(E\) is smooth if and only if \(J\) is single valued and if \(E\) is uniformly smooth then \(J\) is uniformly continuous on bounded subsets of \(E\). We note that in a Hilbert space \(H\), \(J\) is the identity operator.

Let \(C\) be a nonempty closed convex subset of a Hilbert space \(H\). It is well known that the metric projection of \(H\) onto \(C\), \(P_C : H \to C\), is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In
Let $C$ be a nonempty closed and convex subset of a reflexive, strictly convex and smooth Banach space $E$. The \textit{generalized projection mapping}, introduced by Alber [1], is a mapping $\Pi_C : E \rightarrow C$, that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y,x)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$
\phi(\bar{x},x) = \min \{ \phi(y,x), y \in C \}. 
$$

(1.2)

Let $C$ be a nonempty closed convex subset of a Banach space $E$. Let $T : C \rightarrow C$ be a single-valued mapping. An element $p \in C$ is called a \textit{fixed point} of $T$ if $T(p) = p$. The set of fixed points of $T$ is denoted by $F(T)$. A point $p$ in $C$ is said to be an \textit{asymptotic fixed point} of $T$ (see [2]) if $C$ contains a sequence $\{x_n\}$ which converges weakly to $p$ such that $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of $T$ will be denoted by $\hat{F}(T)$. $T$ is said to be \textit{nonexpansive} if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$ and is called \textit{relatively nonexpansive} if (A1) $F(T) \neq \emptyset$; (A2) $\phi(p, Tx) \leq \phi(p, x)$ for $x \in C$ and $p \in F(T)$ and (A3) $F(T) = \hat{F}(T)$.

Let $C$ be a nonempty closed convex subset of a Banach space $E$ and let $N(C)$ and $CB(C)$ denote the family of nonempty subsets and nonempty closed bounded subsets of $C$, respectively. Let $H$ be the Hausdorff metric on $CB(C)$ defined by

$$
H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}, 
$$

(1.3)

for all $A, B \in CB(C)$, where $d(a, B) = \inf \{ \|a - b\| : b \in B \}$ is the distance from the point $a$ to the subset $B$.

Let $T : C \rightarrow CB(C)$ be a multivalued mapping, $T$ is said to be a \textit{nonexpansive} if $H(Tx, Ty) \leq \|x - y\|$, for $x, y \in C$. An element $p \in C$ is called a \textit{fixed point} of $T$, if $p \in F(T)$, where $F(T) := \{ p \in C : p \in T(p) \}$. A point $p \in C$ called an \textit{asymptotic fixed point} of $T$, if there exists a sequence $\{x_n\}$ in $C$ which converges weakly to $p$ such that $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$. $T$ is said to be \textit{relatively nonexpansive} if (B1) $F(T) \neq \emptyset$; (B2) $\phi(p, u) \leq \phi(p, x)$ for $x \in C, u \in Tx, p \in F(T)$ and (B3) $F(T) = \hat{F}(T)$, where $\hat{F}(T)$ is the set of asymptotic fixed points of $T$.

We remark that the class of relatively nonexpansive single-valued mappings is contained in a class of relatively nonexpansive multi-valued mappings. An example of relatively nonexpansive multi-valued mapping by Homaeipour and Razani [3] is given below.

\textit{Example 1.1.} Let $I = [0,1]$, $X = L^p(I), 1 < p < \infty$ and $C = \{ f \in X : f(x) \geq 0, \ \text{for all} \ x \in I \}$. Let $T : C \rightarrow CB(C)$ be defined by

$$
T(f) = \begin{cases} 
\left\{ g \in C : f(x) - \frac{3}{4} \leq g(x) \leq f(x) - \frac{1}{4}, \ \forall x \in I \right\}, & \text{if } f(x) > 1, \ x \in I, \\
\{0\}, & \text{otherwise}.
\end{cases}
$$

(1.4)

It is shown in [3] that $T$ is relatively nonexpansive multi-valued mapping which is not nonexpansive.
The study of fixed points for multi-valued nonexpansive mappings in relation to Hausdorff metric was introduced by Markin [4] (see also [5]). Since then a lot of activity in this area and fixed point theory for multi-valued nonexpansive mappings has been developed which has some nontrivial applications in pure and applied sciences including control theory, convex optimization, differential inclusion, and economics (see, e.g., [6] and references therein). Later, Lim [7] established the existence of fixed points for multi-valued nonexpansive mappings in uniformly convex Banach spaces.

It is well known that the normal Mann’s iterative [8] algorithm has only weak convergence in an infinite-dimensional Hilbert space even for nonexpansive single-valued mappings. Consequently, in order to obtain strong convergence, one has to modify the normal Mann’s iteration algorithm, the so called hybrid projection iteration method is such a modification. The hybrid projection iteration algorithm (HPIA) was introduced initially by Haugazeau [9] in 1968. For 40 years, (HPIA) has received rapid developments. For details, the readers are referred to papers [10–12] and the references therein.

In 2003, Nakajo and Takahashi [12] proposed the following modification of the Mann iteration method for a nonexpansive single-valued mapping $T$ in a Hilbert space $H$:

$$ x_0 \in C, \text{ chosen arbitrary,} $$

$$ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, $$

$$ C_n = \{ z \in C : \|y_n - z\| \leq \|x_n - z\| \}, $$

$$ Q_n = \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0 \}, $$

$$ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), \quad n \geq 0, $$

(1.5)

where $C$ is a closed convex subset of $H$, $P_C$ denotes the metric projection from $H$ onto $C$. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one then the sequence $\{x_n\}$ generated by (1.5) converges strongly to $P_{TF}(x_0)$.

In spaces more general than Hilbert spaces, Matsushita and Takahashi [11] proposed the following hybrid iteration method with generalized projection for relatively nonexpansive single-valued mapping $T$ in a Banach space $E$:

$$ x_0 \in C, \text{ chosen arbitrary,} $$

$$ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JTx_n), $$

$$ C_n = \{ z \in C : \phi(z, y_n) \leq \phi(z, x_n) \}, $$

$$ Q_n = \{ z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0 \}, $$

$$ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0), \quad n \geq 0. $$

(1.6)

They proved the following convergence theorem.

**Theorem MT.** Let $E$ be a uniformly convex and uniformly smooth Banach space, let $C$ be a nonempty closed convex subset of $E$, let $T$ be a relatively nonexpansive single-valued mapping from $C$ into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\lim \sup_{n \to \infty} \alpha_n < 1$. 
Suppose that \( \{x_n\} \) is given by (1.6), where \( J \) is the duality mapping on \( E \). If \( F(T) \) is nonempty, then \( \{x_n\} \) converges strongly to \( \Pi_{F(T)}x_0 \), where \( \Pi_{F(T)}(\cdot) \) is the generalized projection from \( E \) onto \( F(T) \).

Let \( f : C \times C \to \mathbb{R} \) be a bifunction, where \( \mathbb{R} \) is the set of real numbers. The equilibrium problem for \( f \) is

\[
\text{finding } x^* \in C \quad \text{such that } f(x^*, y) \geq 0, \quad \forall y \in C. \quad (1.7)
\]

The solution set of (1.7) is denoted by \( EP(f) \).

If \( f(x, y) = \langle Ax, y - x \rangle \), where \( A : C \to C \) is a monotone mapping, then the problem (1.7) reduces to the system of variational inequality problem

\[
\text{find an element } x^* \in C \quad \text{such that } \langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \quad (1.8)
\]

That is, the problem (1.8) is a special case of (1.7). The set of solutions of inequality (1.8) is denoted by \( VI(C, A) \).

For solving the equilibrium problem for a bifunction \( f : C \times C \to \mathbb{R} \), we assume that \( f \) satisfies the following conditions:

(A1) \( f(x, x) = 0 \), for all \( x \in C \),

(A2) \( f \) is monotone, that is, \( f(x, y) + f(y, x) \leq 0 \), for all \( x, y \in C \),

(A3) for each \( x, y, z \in C \), \( \lim_{t \to 0} f(tx + (1-t)x, y) \leq f(x, y) \),

(A4) for each \( x \in C \), \( y \to f(x, y) \) is convex and lower semicontinuous.

Recently, many authors studied the problem of finding a common element of the set of fixed points of nonexpansive or relatively nonexpansive single-valued mapping and the set of solutions of an equilibrium problems in the framework of Hilbert spaces and Banach spaces, respectively; see, for instance, [2, 13–21] and the references therein.

In [22], Kumam introduced the following iterative scheme in a Hilbert space:

\[
x_0 \in H,
\]

\[
u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n} (y - u_n, u_n - y_n) \geq 0, \quad \forall y \in C,
\]

\[
w_n = \alpha_n x_n + (1 - \alpha_n) Tu_n,
\]

\[
C_n = \{ z \in H : \| w_n - z \| \leq \| x_n - z \| \},
\]

\[
Q_n = \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0 \},
\]

\[
x_{n+1} = P_{C_n \cap Q_n}(x_0), \quad n \geq 0,
\]

for finding a common element of the set of fixed point of nonexpansive single-valued mapping \( T \) and set of solution of equilibrium problems.
In the case that \( E \) is a Banach space, Takahashi and Zembayashi [16] introduced the following iterative scheme which is called the shrinking projection method:

\[
x_0 \in C, \quad \text{chosen arbitrary,}
\]

\[
y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JTx_n),
\]

\[
u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \quad \forall y \in C,
\]

\[
C_{n+1} = \{ z \in C : \phi(z, u_n) \leq \phi(z, x_n) \},
\]

\[
x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad n \geq 0,
\]

where \( J \) is the duality mapping on \( E \), \( \Pi_C \) is the generalized projection from \( E \) onto \( C \) and \( T \) is relatively nonexpansive single-valued mapping. They proved that the sequence \( \{x_n\} \) converges strongly to a common element of the set of fixed point of relatively nonexpansive single-valued mapping and set of solution of equilibrium problem under appropriate conditions.

We remark that the computation of \( x_{n+1} \) in (1.9) and (1.10) is not simple because of the involvement of computation of \( C_{n+1} \) from \( C_n \) for each \( n \geq 0 \).

More recently, Homaeipour and Razani [3] studied the following iterative scheme for a fixed point of relatively nonexpansive multi-valued mapping in uniformly convex and uniformly smooth Banach space \( E \):

\[
x_0 \in C, \quad \text{chosen arbitrary,}
\]

\[
x_{n+1} = \Pi_C J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) Jz_n), \quad z_n \in Tx_n, \quad n \geq 0,
\]

where \( \{\alpha_n\} \subset (0, 1) \) for all \( n \geq 0 \) and \( \liminf_{n \to \infty} \alpha_n(1 - \alpha_n) > 0 \). They proved that if \( J \) is weakly sequentially continuous then the sequence \( \{x_n\} \) converges weakly to a fixed point of \( T \). Furthermore, it is shown that the scheme converges strongly to a fixed point of \( T \) if interior of \( F(T) \) is nonempty.

But it is worth mentioning that the convergence of the scheme is either weak or it requires that the interior of \( F(T) \) is nonempty.

In this paper, motivated by Kumam [22], Takahashi and Zembayashi [16], and Homaeipour and Razani [3], we construct an iterative scheme which converges strongly to a common point of set of solutions of equilibrium problem and set of fixed points of finite family of relatively nonexpansive multi-valued mappings in Banach spaces. Our scheme does not involve computation of \( C_n \) and \( Q_n \), for each \( n \geq 0 \), and the requirement that the interior of \( F \) is nonempty is dispensed with. Our theorems improve and unify most of the results that have been proved for this important class of nonlinear operators.
2. Preliminaries

Let $E$ be a normed linear space with $\dim E \geq 2$. The \textit{modulus of smoothness} of $E$ is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(\tau) := \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1; \|y\| = \tau \right\}. \quad (2.1)$$

The space $E$ is said to be \textit{smooth} if $\rho_E(\tau) > 0$, for all $\tau > 0$ and $E$ is called \textit{uniformly smooth} if and only if $\lim_{\tau \to 0} (\rho_E(\tau)/\tau) = 0$.

The \textit{modulus of convexity} of $E$ is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf \left\{ 1 - \frac{\|x + y\|^2}{2} : \|x\| = \|y\| = 1; \epsilon = \|x - y\| \right\}. \quad (2.2)$$

$E$ is called \textit{uniformly convex} if and only if $\delta_E(\epsilon) > 0$, for every $\epsilon \in (0, 2)$.

In the sequel, we will need the following results.

\textbf{Lemma 2.1} (see [1]). Let $K$ be a nonempty closed and convex subset of a real reflexive, strictly convex, and smooth Banach space $E$ and let $x \in E$. Then for all $y \in K$,

$$\phi(y, \Pi_K x) + \phi(\Pi_K x, x) \leq \phi(y, x). \quad (2.3)$$

We make use of the function $V : E \times E^* \to \mathbb{R}$, defined by

$$V(x, x^*) = \|x\|^2 - 2(x, x^*) + \|x^*\|^2, \quad \forall x \in E, \ x^* \in E^*, \quad (2.4)$$

studied by Alber [1]. That is, $V(x, x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$. We know the following lemma.

\textbf{Lemma 2.2} (see [1]). Let $E$ be reflexive strictly convex and smooth Banach space with $E^*$ as its dual. Then

$$V(x, x^*) + 2\left\langle J^{-1}x^* - x, y^* \right\rangle \leq V(x, x^* + y^*), \quad (2.5)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

\textbf{Lemma 2.3} (see [1]). Let $C$ be a convex subset of a real smooth Banach space $E$. Let $x \in E$. Then $x_0 = \Pi_C x$ if and only if

$$\langle z - x_0, Jx - Jx_0 \rangle \leq 0, \quad \forall z \in C. \quad (2.6)$$
Lemma 2.4 (see [23]). Let $E$ be a uniformly convex Banach space and $B_R(0)$ be a closed ball of $E$. Then, there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_N x_N\|^2 \leq \sum_{i=1}^{N} \alpha_i \|x_i\|^2 - \alpha_i \alpha_j g\left(\|x_i - x_j\|\right),$$

for $i, j \in \{1, \ldots, N\}$, $\alpha_i \in (0, 1)$ such that $\sum_{i=1}^{N} \alpha_i = 1$, and $x_i \in B_R(0) := \{x \in E : \|x\| \leq R\}$, for $i = 1, 2, \ldots, N$.

Lemma 2.5 (see [24]). Let $E$ be a real smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of $E$. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $x_n - y_n \to 0$, as $n \to \infty$.

Proposition 2.6 (see [3]). Let $E$ be a strictly convex and smooth Banach space and $C$ be a nonempty closed convex subset of $E$. Let $T : C \to N(C)$ be a relatively nonexpansive multi-valued mapping. Then $F(T)$ is closed and convex.

Lemma 2.7 (see [16]). Let $C$ be a nonempty, closed and convex subset of a uniformly smooth, strictly convex and reflexive real Banach space $E$. Let $f$ be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)–(A4). For $r > 0$ and $x \in E$, define the mapping $F_r : E \to C$ as follows:

$$F_r x := \left\{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}.$$

Then the following statements hold:

1. $F_r$ is single-valued,
2. $F(F_r) = EP(f),$
3. $\phi(q, F_r x) + \phi(F_r x, x) \leq \phi(q, x)$, for $q \in F(F_r),$
4. $EP(f)$ is closed and convex.

Lemma 2.8 (see [25]). Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_{i+1}}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:

$$a_{m_k} \leq a_{m_{k+1}}, \quad a_k \leq a_{m_{k+1}}. \quad (2.9)$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

Lemma 2.9 (see [26]). Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:

$$a_{n+1} \leq (1 - \beta_n) a_n + \beta_n \delta_n, \quad n \geq n_0, \text{ for some } n_0 \in \mathbb{N},$$

where $\{\beta_n\} \subset (0, 1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfying the following conditions: $\lim_{n \to \infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$, and $\limsup_{n \to \infty} \delta_n \leq 0$. Then, $\lim_{n \to \infty} a_n = 0$. 

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3. Main Result

Let $C$ be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive real Banach space $E$ with dual $E^*$. Let $f : C \times C \to \mathbb{R}$ be a bifunction. For the rest of this paper, $F_{r_n}x$ is a mapping defined as follows. For $x \in E$, let $F_{r_n} : E \to C$ be given by

$$F_{r_n}x := \left\{ z \in C : f(z, y) + \frac{1}{r_n}(y - z, Jz - Jx) \geq 0, \ \forall y \in C \right\},$$

(3.1)

where \( \{r_n\}_{n \in \mathbb{N}} \subset [c_1, \infty) \), for some $c_1 > 0$.

**Theorem 3.1.** Let $C$ be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space $E$. Let $f : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions (A1)–(A4). Let $T_i : C \to CB(C)$, for $i = 1, 2, \ldots, N$, be a finite family of relatively nonexpansive multi-valued mappings. Assume that $F := \cap_{i=1}^N F(T_i) \cap EP(f)$ is nonempty. Let \( \{x_n\} \) be a sequence generated by

$$x_0 = w \in C, \ \text{chosen arbitrarily,}$$

$$w_n = F_{r_n}x_n,$$

$$y_n = \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n) Jw_n),$$

$$x_{n+1} = J^{-1}\left( \frac{\beta_n}{2} Jw_n + \sum_{i=1}^{N} \beta_{n,i} Ju_{n,i} \right), \quad u_{n,i} \in Ti y_n, \quad n \geq 0,$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n \to \infty} \alpha_n = 0$. $\sum_{n=0}^{\infty} \alpha_n = 1$ and $\{\beta_{n,i}\} \subset [a, b] \subset (0, 1)$, for $i = 1, 2, \ldots, N$, satisfying $\beta_{n,0} + \beta_{n,1} + \cdots + \beta_{n,N} = 1$, for each $n \geq 0$. Then \( \{x_n\} \) converges strongly to an element of $F$.

**Proof.** Since $F$ is nonempty closed and convex, put $x^* := \Pi_F w$. Now from (3.2), Lemma 2.7(3) and property of $\phi$, we get that

$$\phi(x^*, y_n) = \phi\left(x^*, \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n) Jw_n)\right) $$

$$\leq \phi\left(x^*, J^{-1}(\alpha_n Jw + (1 - \alpha_n) Jw_n)\right) $$

$$= \|x^*\|^2 - 2\langle x^*, \alpha_n Jw + (1 - \alpha_n) Jw_n \rangle + \|\alpha_n Jw + (1 - \alpha_n) Jw_n\|^2 $$

$$\leq \|x^*\|^2 - 2\alpha_n \langle x^*, Jw \rangle - 2(1 - \alpha_n) \|x^*, Jw_n\|^2 $$

$$+ \alpha_n \|w\|^2 + (1 - \alpha_n) \|w_n\|^2 $$

$$\leq \alpha_n \phi(x^*, w) + (1 - \alpha_n) \phi(x^*, w_n) $$

$$= \alpha_n \phi(x^*, w) + (1 - \alpha_n) \phi(x^*, F_{r_n}x_n) $$

$$\leq \alpha_n \phi(x^*, w) + (1 - \alpha_n) \phi(x^*, x_n).$$

(3.3)
Now, from (3.2), Lemma 2.7(3), relatively nonexpansiveness of $T_i$, property of $\phi$ and (3.3), we have that

$$
\phi(x^*, x_{n+1}) = \phi \left( x^*, J^{-1}(\beta_{n,0} Jw_n + \sum_{i=1}^{N} \beta_{n,i} J u_{n,i}) \right)
\leq \beta_{n,0} \phi(x^*, w_n) + \sum_{i=1}^{N} \beta_{n,i} \phi(x^*, u_{n,i})
= \beta_{n,0} \phi(x^*, F_{r_n} x_n) + \sum_{i=1}^{N} \beta_{n,i} \phi(x^*, u_{n,i})
\leq \beta_{n,0} \phi(x^*, x_n) + (1 - \beta_{n,0}) \phi(x^*, y_n)
\leq \beta_{n,0} \phi(x^*, x_n) + (1 - \beta_{n,0}) [\alpha_n \phi(x^*, w) + (1 - \alpha_n) \phi(x^*, x_n)]
\leq \delta_n \phi(x^*, w) + (1 - \delta_n) \phi(x^*, x_n),
$$

where $\delta_n = (1 - \beta_{n,0}) \alpha_n$. Thus, by induction,

$$
\phi(x^*, x_{n+1}) \leq \max \{ \phi(x^*, x_0), \phi(x^*, w) \}, \quad \forall n \geq 0,
$$

which implies that $\{x_n\}$ is bounded and hence $\{y_n\}$ and $\{w_n\}$ are bounded. Now let $z_n = J^{-1}(\alpha_n Jw + (1 - \alpha_n) Jw_n)$. Then we have that $y_n = \Pi_C z_n$. Using Lemma 2.2 and property of $\phi$, we obtain that

$$
\phi(x^*, y_n) \leq \phi(x^*, z_n) = V(x^*, Jz_n)
\leq V(x^*, Jz_n - \alpha_n (Jw - Jx^*)) - 2(z_n - x^*, -\alpha_n (Jw - Jx^*))
= \phi \left( x^*, J^{-1}(\alpha_n Jx^* + (1 - \alpha_n) Jw_n) + 2\alpha_n (z_n - x^*, Jw - Jx^*) \right)
\leq \alpha_n \phi(x^*, x^*) + (1 - \alpha_n) \phi(x^*, w_n) + 2\alpha_n (z_n - x^*, Jw - Jx^*)
= (1 - \alpha_n) \phi(x^*, w_n) + 2\alpha_n (z_n - x^*, Jw - Jx^*)
\leq (1 - \alpha_n) \phi(x^*, x_n) + 2\alpha_n (z_n - x^*, Jw - Jx^*).
$$
Furthermore, from (3.2), Lemma 2.4, relatively nonexpansiveness of $T_i$, for each $i = 1, 2, \ldots, N$, Lemma 2.7(3), and (3.6) we have that

$$\phi(x^n, x_{n+1}) = \phi\left(x^n, J^{-1}\left(\beta_{n,0}\|w_n\| + \sum_{i=1}^{N}\beta_{n,i}\|u_{n,i}\|\right)\right)$$

$$\leq \beta_{n,0}\phi(x^n, w_n) + \sum_{i=1}^{N}\beta_{n,i}\phi(x^n, u_{n,i})$$

$$- \beta_{n,0}\beta_{n,i}g(\|w_n - J_{u_{n,i}}\|)$$

$$= \beta_{n,0}\phi(x^n, F_n x_n) + \sum_{i=1}^{N}\beta_{n,i}\phi(x^n, u_{n,i})$$

$$- \beta_{n,0}\beta_{n,i}g(\|w_n - J_{u_{n,i}}\|)$$

$$\leq \beta_{n,0}\phi(x^n, x_n) - \phi(x_n, w_n) + (1 - \beta_{n,0})\phi(x^n, y_n)$$

$$- \beta_{n,0}\beta_{n,i}g(\|w_n - J_{u_{n,i}}\|) \leq \beta_{n,0}\phi(x^n, x_n) - \beta_{n,0}\phi(x_n, w_n) + (1 - \beta_{n,0})$$

$$\times [(1 - \alpha_n)\phi(x^n, x_n) + 2\alpha_n(z_n - x^n, Jw - Jx^n)] \beta_{n,0}\beta_{n,i}g(\|w_n - J_{u_{n,i}}\|)$$

$$= (1 - \delta_n)\phi(x^n, x_n) + 2\delta_n(z_n - x^n, Jw - Jx^n)$$

$$- \beta_{n,0}\phi(x_n, w_n) - \beta_{n,0}\beta_{n,i}g(\|w_n - J_{u_{n,i}}\|),$$

and hence

$$\phi(x^n, x_{n+1}) \leq (1 - \delta_n)\phi(x^n, x_n) + 2\delta_n(z_n - x^n, Jw - Jx^n),$$

(3.8)

where $\delta_n := \alpha_n(1 - \beta_{n,0})$, for all $n \in \mathbb{N}$. Note that $\delta_n$ satisfies $\lim_n \delta_n = 0$ and $\sum_{n=1}^{\infty} \delta_n = \infty$. \hfill \Box

Now, we consider two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\phi(x^n, x_n)\}$ is nonincreasing for all $n \geq n_0$. In this situation, $\{\phi(x^n, x_n)\}$ is then convergent. Then from (3.7), we have that $\phi(x_n, w_n) \to 0$ and hence Lemma 2.5 implies that

$$x_n - w_n \to 0, \quad \text{as } n \to \infty.$$  (3.9)

Moreover, from (3.7), we have that $\beta_{n,0}\beta_{n,i}g(\|w_n - J_{u_{n,i}}\|) \to 0$, as $n \to \infty$, which implies by the property of $g$ that $Jw_n - J_{u_{n,i}} \to 0$, as $n \to \infty$, for each $i \in \{1, 2, \ldots, N\}$, and hence, since $J^{-1}$ uniformly continuous on bounded sets, we obtain that

$$w_n - u_{n,i} \to 0, \quad \text{as } n \to \infty.$$  (3.10)
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Furthermore, by Lemma 2.1, property of \( \phi \) and the fact that \( a_n \to 0 \), as \( n \to \infty \), imply that

\[
\phi(w_n, y_n) = \phi(w_n, \Pi_C z_n) \leq \phi(w_n, z_n)
\]

\[
= \phi \left( w_n, f^{-1}(a_n f w + (1 - a_n) f w_n) \right)
\]

\[
\leq a_n \phi(w_n, w) + (1 - a_n) \phi(w_n, w_n) \to 0, \quad \text{as } n \to \infty,
\]

and hence

\[
w_n - y_n \to 0, \quad w_n - z_n \to 0, \quad \text{as } n \to \infty.
\]  \hspace{1cm} (3.12)

Therefore, from (3.9), (3.10), and (3.12), we obtain that

\[
x_n - z_n \to 0, \quad y_n - x_n \to 0, \quad \text{as } n \to \infty,
\]  \hspace{1cm} (3.13)

\[
d(y_n, T_i y_n) \leq \|y_n - u_{n,i}\| \leq \|y_n - w_n\| + \|w_n - u_{n,i}\| \to 0,
\]  \hspace{1cm} (3.14)

as \( n \to \infty \), for each \( i \in \{1, 2, \ldots, N\} \).

Let \( \{z_{n_i}\} \) be a subsequence of \( \{z_n\} \) such that \( z_{n_i} \to z \) and \( \limsup_{n \to \infty} \langle z_n - x^*, f w - f x^* \rangle = \lim_{i \to \infty} \langle z_n - x^*, f w - f x^* \rangle \). Then, from (3.12), (3.13), and the uniform continuity of \( f \), we get that

\[
x_{n_i}, w_{n_i}, y_{n_i} \to z, \quad f x_n - f w_n \to 0, \quad \text{as } n \to \infty.
\]  \hspace{1cm} (3.15)

Now, we show that \( z \in EP(f) \). But, from the definition of \( w_n \) and (A2) we note that

\[
\frac{1}{r_{n_i}} \langle y - w_{n_i}, f w_n - f x_n \rangle \geq -f(w_n, y) \geq f(y, w_n), \quad \forall y \in C.
\]  \hspace{1cm} (3.16)

Letting \( i \to \infty \), we have from (3.15) and (A4) that \( f(y, z) \leq 0 \), for all \( y \in C \). Now, for \( 0 < t \leq 1 \) and \( y \in C \), let \( y_t = ty + (1 - t)z \). Since \( y \in C \) and \( z \in C \), we have \( y_t \in C \) and hence \( f(y_t, z) \leq 0 \). So, from the convexity of the equilibrium bifunction \( f(x, y) \) on the second variable \( y \), we have

\[
0 = f(y_t, y_t) \leq tf(y_t, y) + (1 - t)f(y_t, z) \leq tf(y_t, y),
\]  \hspace{1cm} (3.17)

and hence \( f(y_t, y) \geq 0 \). Now, letting \( t \to 0 \) and condition (A3), we obtain that \( f(z, y) \geq 0 \), for all \( y \in C \), and hence \( z \in EP(f) \).

Next, we show that \( z \in \cap_{i=1}^{N} F(T_i) \). But, since each \( T_i \) satisfies condition (B3) we obtain from (3.13) and (3.15) that \( z \in F(T_i) \), for each \( i = 1, 2, \ldots, N \), and hence \( z \in \cap_{i=1}^{N} F(T_i) \). Thus, from the above discussions we obtain that \( z \in F := \cap_{i=1}^{N} F(T_i) \cap EP(f) \). Therefore, by Lemma 2.3, we immediately obtain that \( \limsup_{n \to \infty} \langle z_n - x^*, f w - f x^* \rangle = \lim_{i \to \infty} \langle z_n - x^*, f w - f x^* \rangle = \langle z - x^*, f w - f x^* \rangle \leq 0 \). It follows from (3.8) and Lemma 2.9 that \( \phi(x^*, x_n) \to 0 \), as \( n \to \infty \). Consequently, \( x_n \to x^* \) by Lemma 2.5.
Case 2. Suppose that there exists a subsequence \(\{n_i\}\) of \(\{n\}\) such that
\[
\phi(x^*, x_{n_i}) < \phi(x^*, x_{n+1})
\] (3.18)
for all \(i \in \mathbb{N}\). Then, by Lemma 2.8, there exist a nondecreasing sequence \(\{m_k\} \subset \mathbb{N}\) such that \(m_k \to \infty\), \(\phi(x^*, x_{m_k}) \leq \phi(x^*, x_{m_{k+1}})\) and \(\phi(x^*, x_k) \leq \phi(x^*, x_{m_{k+1}})\), for all \(k \in \mathbb{N}\). Now, from (3.7) and the fact that \(\delta_n \to 0\), we have
\[
\beta_{m_k, i} \phi(x_k, w_{m_k}) + \beta_{m_k, i} \beta_{m_k, i} \tilde{g} (\|f w_{m_k} - J u_{m_k, i}\|)
\leq \left( \phi(x^*, x_{m_k}) - \phi(x^*, x_{m_{k+1}}) \right) - \delta_{m_k} \phi(x^*, x_{m_k}) + 2 \delta_{m_k} \langle z_{m_k} - x^*, f w - J x^* \rangle,
\] (3.19)
as \(k \to \infty\). Thus, using the same proof of Case 1, we obtain that \(x_{m_k} - w_{m_k} \to 0\) and \(w_{m_k} - u_{m_k, i} \to 0\), as \(k \to \infty\), for each \(i = 1, 2, \ldots, N\) and hence
\[
\limsup_{n \to \infty} \langle z_{m_k} - x^*, f w - J x^* \rangle \leq 0.
\] (3.20)
Then from (3.8), we have that
\[
\phi(x^*, x_{m_{k+1}}) \leq (1 - \delta_{m_k}) \phi(x^*, x_{m_k}) + 2 \delta_{m_k} \langle z_{m_k} - x^*, f w - J x^* \rangle.
\] (3.21)
Since \(\phi(x^*, x_{m_k}) \leq \phi(x^*, x_{m_{k+1}})\), (3.21) implies that
\[
\delta_{m_k} \phi(x^*, x_{m_k}) \leq \phi(x^*, x_{m_k}) - \phi(x^*, x_{m_{k+1}}) + 2 \delta_{m_k} \langle z_{m_k} - x^*, f w - J x^* \rangle
\leq 2 \delta_{m_k} \langle z_{m_k} - x^*, f w - J x^* \rangle.
\] (3.22)
In particular, since \(\delta_{m_k} > 0\), we get
\[
\phi(x^*, x_{m_k}) \leq 2 \langle z_{m_k} - x^*, f w - J x^* \rangle.
\] (3.23)
Then, from (3.20), we obtain that \(\phi(x^*, x_{m_k}) \to 0\), as \(k \to \infty\). This together with (3.21) gives \(\phi(x^*, x_{m_{k+1}}) \to 0\), as \(k \to \infty\). But \(\phi(x^*, x_k) \leq \phi(x^*, x_{m_{k+1}})\) for all \(k \in \mathbb{N}\), thus we obtain that \(x_k \to x^*\). Therefore, from the above two cases, we can conclude that \(\{x_n\}\) converges strongly to \(x^*\) and the proof is complete.

If in Theorem 3.1, we assume that \(f(x, y) = \langle Ax, y - x \rangle\), for \(A\) continuous monotone mapping, then we obtain the following corollary.

**Corollary 3.2.** Let \(C\) be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space \(E\). Let \(A : C \to E^*\) be a continuous monotone mapping.
Let $T_i : C \rightarrow CB(C)$, for $i = 1, 2, \ldots, N$, be a finite family of relatively nonexpansive multi-valued mappings. Assume that $F := \cap_{i=1}^{N} F(T_i) \cap VI(C, A)$ is nonempty. Let $\{x_n\}$ be a sequence generated by

$x_0 = w \in C$, chosen arbitrarily,

$$w_n \in C \text{ such that } \langle Aw_n, y - w_n \rangle + \frac{1}{r_n} \langle y - w_n, Jw_n - Jx_n \rangle \geq 0, \quad \forall y \in C,$$

$$y_n = \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n) Jw_n), \quad x_{n+1} = J^{-1}(\beta_{n,0} Jw_n + \sum_{i=1}^{N} \beta_{n,i} Jw_n), \quad u_{n,i} \in T_i y_n, \; n \geq 0,$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_{n,i}\} \subset [a, b] \subset (0, 1)$, for $i = 1, 2, \ldots, N$, satisfying $\beta_{n,0} + \beta_{n,1} + \cdots + \beta_{n,N} = 1$, for each $n \geq 0$. Then $\{x_n\}$ converges strongly to an element of $F$.

Proof. Let $f(x, y) = \langle Ax, y - x \rangle$. Since $A$ is monotone and continuous, we get that a bifunction $f$ satisfies conditions (A1)–(A4). Thus, the conclusion follows from Theorem 3.1. \qed

If in Theorem 3.1, we assume that $N = 1$, then we get the following theorem.

Corollary 3.3. Let $C$ be a nonempty, closed, and convex subset of a uniformly smooth and uniformly convex real Banach space $E$. Let $f : C \times C \rightarrow \mathbb{R}$, be a bifunction which satisfies conditions (A1)–(A4). Let $T : C \rightarrow CB(C)$ be a relatively nonexpansive multi-valued mapping. Assume that $F := F(T) \cap EP(f)$ is nonempty. Let $\{x_n\}$ be a sequence generated by

$$x_0 = w \in C$, chosen arbitrarily, $$

$$w_n = F_r x_n, \quad y_n = \Pi_C J^{-1}(\alpha_n Jw + (1 - \alpha_n) Jw_n), \quad x_{n+1} = J^{-1}(\beta_n Jw_n + (1 - \beta_n) Jw_n), \quad u_n \in Ty_n, \; n \geq 0,$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\} \subset [a, b] \subset (0, 1)$, for each $n \geq 0$. Then $\{x_n\}$ converges strongly to an element of $F$.

Proof. The proof follows from Theorem 3.1 with $N = 1$. \qed

If in Theorem 3.1, we assume that $f \equiv 0$, we get the following corollary.

Corollary 3.4. Let $C$ be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space $E$. Let $T_i : C \rightarrow CB(C)$, for $i = 1, 2, \ldots, N$, be a finite family of relatively
nonexpansive multi-valued mappings. Assume that \( F := \cap_{i=1}^{N} F(T_i) \) is nonempty. Let \( \{x_n\} \) be a se-
quency generated by

\[
x_0 = w \in C, \quad \text{chosen arbitrarily},
\]
\[
y_n = \Pi_C f^{-1}(a_n Jw + (1 - a_n) Jx_n),
\]
\[
x_{n+1} = f^{-1}\left( \beta_{n,0} Jx_n + \sum_{i=1}^{N} \beta_{n,i} Jy_{n,i} \right), \quad u_{n,i} \in T_i y_n, \quad n \geq 0,
\]

where \( a_n \in (0,1) \) such that \( \lim_{n \to \infty} a_n = 0 \), \( \sum_{n=1}^{\infty} a_n = \infty \), \( \{\beta_{n,i}\} \subset [a, b] \subset (0,1) \), for \( i = 1, 2, \ldots, N \) satisfying \( \beta_{n,0} + \beta_{n,1} + \cdots + \beta_{n,N} = 1 \), for each \( n \geq 0 \). Then \( \{x_n\} \) converges strongly to an element of \( F \).

If in Theorem 3.1, we assume that each \( T_i \), \( i = 1, 2, \ldots, N \) is single valued, we get the following corollary.

**Corollary 3.5.** Let \( C \) be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex real Banach space \( E \). Let \( f : C \times C \to \mathbb{R} \), be a bifunction which satisfies conditions (A1)–(A4). Let \( T_i : C \to C \), for \( i = 1, 2, \ldots, N \), be a finite family of relatively nonexpansive single-valued mappings. Assume that \( F := \cap_{i=1}^{N} F(T_i) \cap EP(f) \) is nonempty. Let \( \{x_n\} \) be a sequence generated by

\[
x_0 = w \in C, \quad \text{chosen arbitrarily},
\]
\[
w_n = F_n x_n,
\]
\[
y_n = \Pi_C f^{-1}(a_n Jw + (1 - a_n) Jw_n),
\]
\[
x_{n+1} = f^{-1}\left( \beta_{n,0} Jw_n + \sum_{i=1}^{N} \beta_{n,i} JT_i y_n \right), \quad n \geq 0,
\]

where \( a_n \in (0,1) \) such that \( \lim_{n \to \infty} a_n = 0 \), \( \sum_{n=1}^{\infty} a_n = \infty \), \( \{\beta_{n,i}\} \subset [a, b] \subset (0,1) \), for \( i = 1, 2, \ldots, N \), satisfying \( \beta_{n,0} + \beta_{n,1} + \cdots + \beta_{n,N} = 1 \), for each \( n \geq 0 \). Then \( \{x_n\} \) converges strongly to an element of \( F \).

If \( E = H \), a real Hilbert space, then \( E \) is uniformly convex and uniformly smooth real Banach space. In this case, \( J = I \), identity map on \( H \) and \( \Pi_C = P_C \), projection mapping from \( H \) onto \( C \). Thus, the following corollary holds.

**Corollary 3.6.** Let \( C \) be a nonempty, closed, and convex subset of a Hilbert space \( H \). Let \( f : C \times C \to \mathbb{R} \), be a bifunction which satisfies conditions (A1)–(A4). Let \( T_i : C \to CB(C) \), for \( i = 1, 2, \ldots, N \), be
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a finite family of relatively nonexpansive multi-valued mappings. Assume that \( F := \bigcap_{i=1}^{N} F(T_i) \cap \text{EP}(f) \) is nonempty. Let \( \{x_n\} \) be a sequence generated by

\[
x_0 = w \in C, \quad \text{chosen arbitrarily,}
\]

\[
w_n = F_{r_n}x_n,
\]

\[
y_n = P_C(\alpha_n w + (1 - \alpha_n) w_n),
\]

\[
x_{n+1} = \beta_n,0 w_n + \sum_{i=1}^{N} \beta_{n,i} u_{n,i}, \quad u_{n,i} \in T_i y_n, \quad n \geq 0,
\]

where \( \alpha_n \in (0, 1) \) such that \( \lim_{n \to \infty} \alpha_n = 0 \), \( \sum_{n=1}^{\infty} \alpha_n = \infty \), \( \{\beta_{n,i}\} \subset [a, b] \subset (0, 1) \), for \( i = 1, 2, \ldots, N \), satisfying \( \beta_{n,0} + \beta_{n,1} + \cdots + \beta_{n,N} = 1 \), for each \( n \geq 0 \). Then \( \{x_n\} \) converges strongly to an element of \( F \).

Remark 3.7. (1) Theorem 3.1 improves and extends the corresponding results of Kumanm [22] and Takahashi and Zembayashi [16] in the sense that either our scheme does not require computation of \( C_{n+1} \), for each \( n \geq 1 \), or the space considered is more general.

(2) Theorem 3.1 improves the corresponding results of Homaeipour and Razani [3] in the sense that our convergence is strong and the requirement that the interior of \( F \) is nonempty is dispensed with.

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References


