Review Article

Some Properties and Identities of Bernoulli and Euler Polynomials Associated with $p$-adic Integral on $\mathbb{Z}_p$


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We investigate some properties and identities of Bernoulli and Euler polynomials. Further, we give some formulae on Bernoulli and Euler polynomials by using $p$-adic integral on $\mathbb{Z}_p$.

1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$. Let $v_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$.

For $f \in UD(\mathbb{Z}_p)$, the $p$-adic invariant integral on $\mathbb{Z}_p$ in the bosonic sense is defined by

$$I(f) = \int_{\mathbb{Z}_p} f(x) d\mu(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) d\mu(x)$$

(1.1)

(see [1, 2]). The fermionic $p$-adic integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x$$

(1.2)
(see [3]). As is well known, Bernoulli polynomials are defined by

$$\frac{t}{e^t-1}e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (1.3)$$

with the usual convention about replacing $B^n(x)$ by $B_n(x)$, symbolically (see [1–19]). In the special case $x = 0$, $B_n(0) = B_n$ is called the $n$th Bernoulli number.

The Euler polynomials are also defined by the generating function as follows:

$$\frac{2}{e^t+1}e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (1.4)$$

with the usual convention about replacing $E^n(x)$ by $E_n(x)$, symbolically (see [1–19]). In the special case $x = 0$, $E_n(0) = E_n$ is called the $n$th Euler number.

By (1.3) and (1.4), we easily see that

$$B_n(x) = (B + x)^n = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} B_l, \quad (1.5)$$

$$E_n(x) = (E + x)^n = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} E_l,$$

where $\binom{n}{l} = n!/(n-l)!! = n(n-1)(n-2) \cdots (n-l+1)/l!$ (see [14, 16, 19]).

The following properties of Bernoulli numbers and polynomials are well known (see [10, 11]).

For $n \in \mathbb{Z}_+$, we have

$$\sum_{j=0}^{n} \binom{n}{j} y^{n-j} B_{j+1}(x) \frac{x^j}{j+1} = \frac{B_{n+1}(x+y) - y^{n+1}}{n+1}, \quad (1.6)$$

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} y^{n-2j} B_{2j+1}(x) \frac{x^j}{2j+1} \frac{B_{n+1}(x+y) + (-1)^n B_{n+1}(x-y)}{2n+2}, \quad (1.7)$$

$$\sum_{j=1}^{\lfloor (n+1)/2 \rfloor} \binom{n}{2j-1} y^{n+1-2j} B_{2j}(x) \frac{x^{j-1}}{2j} \frac{B_{n+1}(x+y) + (-1)^{n-1} B_{n+1}(x-y) - 2y^{n+1}}{2n+2}, \quad (1.8)$$

where $\lfloor \cdot \rfloor$ is Gauss’ symbol.

First, we investigate some identities of Euler polynomials corresponding to (1.6), (1.7) and (1.8). From those identities, we derive some interesting identities and properties by using $p$-adic integral on $\mathbb{Z}_p$. 

2. Some Identities of Bernoulli and Euler Polynomials

By (1.4), we get

\[ E_k(x + y) = \sum_{j=0}^{k} \binom{k}{j} y^{k-j} E_j(x), \quad \text{for } \in \mathbb{Z}. \]  

(2.1)

From (2.1), we note that

\[ E_k(x + y) = \sum_{j=0}^{k} \binom{k}{j} y^{k-j} E_j(x) \]

\[ = y^k + \sum_{j=1}^{k} \binom{k}{j} \binom{k-1}{j-1} y^{k-j} E_j(x). \]

Thus, we have

\[ \sum_{j=0}^{k-1} \binom{k-1}{j} y^{k-1-j} E_{j+1}(x) \frac{E_{j+1}(x)}{j+1} = \frac{E_k(x + y) - y^k}{k}. \]

(2.3)

Replacing \( k \) by \( k + 1 \) in (2.3), we obtain the following proposition.

Proposition 2.1. For \( k \in \mathbb{Z}_+ \), one has

\[ \sum_{j=0}^{k} \binom{k}{j} y^{k-j} E_{j+1}(x) \frac{E_{j+1}(x)}{j+1} = \frac{E_{k+1}(x + y) - y^{k+1}}{k + 1}. \]

(2.4)

Let us replace \( y \) by \( -y \) in Proposition 2.1. Then we have

\[ \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} y^{k-j} E_{j+1}(x) \frac{E_{j+1}(x)}{j+1} = \frac{E_{k+1}(x) - y^{k+1}}{k + 1}. \]

(2.5)

Thus, we see that

\[ \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} y^{k-j} E_{j+1}(x) \frac{E_{j+1}(x)}{j+1} = \frac{(-1)^k E_{k+1}(x - y) + y^{k+1}}{k + 1}. \]

(2.6)

Therefore, adding (2.4) and (2.6), we obtain the following proposition.

Proposition 2.2. For \( k \in \mathbb{Z}_+ \), one has

\[ \sum_{j=0}^{[k/2]} \binom{k}{2j} y^{k-2j} E_{2j+1}(x) \frac{E_{2j+1}(x)}{2j+1} = \frac{E_{k+1}(x + y) + (-1)^k E_{k+1}(x - y)}{2k + 2}. \]

(2.7)
From (2.2), we note that

\[
\sum_{j=1}^{k} \binom{k-1}{j-1} y^{k-j} (-1)^j E_j(x) = \frac{(-1)^k E_k(x - y) - y^k}{k}.
\] (2.8)

By (2.3) and (2.8), we get

\[
\left[\frac{k}{2}\right] \binom{k-1}{2j-1} \frac{k^{2j} E_{2j}(x)}{2j} = \frac{E_{k+1}(x + y) + (-1)^{k+1} E_{k+1}(x - y) - 2y^{k+1}}{2k+2}.
\] (2.9)

Therefore, replacing \( k \) by \( k + 1 \), we obtain the following proposition.

**Proposition 2.3.** For \( k \in \mathbb{N} \), one has

\[
\sum_{j=1}^{\left[\frac{k+1}{2}\right]} \binom{k+1-2j}{2j-1} E_{2j}(x) = \frac{E_{k+1}(x + y) + (-1)^{k+1} E_{k+1}(x - y) - 2y^{k+1}}{2k+2}.
\] (2.10)

Letting \( y = 1 \) in Proposition 2.1, we have

\[
\sum_{j=0}^{k} \binom{k}{j} E_{j+1}(x) = \frac{E_{k+1}(x + 1) - 1}{k + 1},
\] (2.11)

\[
E_{k+1}(x + 1) = \sum_{l=0}^{k+1} \binom{k+1}{l} (E + 1)^l x^{k+1-l}
\]

\[
= (2 - E_0) x^{k+1} - \sum_{l=1}^{k+1} \binom{k+1}{l} E_l x^{k+1-l}
\] (2.12)

\[
= 2x^{k+1} - \sum_{l=0}^{k+1} \binom{k+1}{l} E_l x^{k+1-l} = 2x^{k+1} - E_{k+1}(x).
\]

Therefore, by (2.11) and (2.12), we obtain the following corollary.

**Corollary 2.4.** For \( k \in \mathbb{Z}_+ \), one has

\[
\sum_{j=0}^{k} \binom{k}{j} \frac{E_{j+1}(x)}{j + 1} = -\frac{E_{k+1}(x)}{k + 1} + \frac{2x^{k+1} - 1}{k + 1}.
\] (2.13)
Replacing $y$ by 1 and $k$ by $2k$ in Proposition 2.2, we have

\[ \sum_{j=0}^{k} \binom{2k}{2j} \frac{E_{2j+1}(x)}{2j+1} = \frac{E_{2k+1}(x+1) + E_{2k+1}(x-1)}{4k+2} \]

\[ = \frac{E_{2k+1}(x+1) + E_{2k+1}(x) + E_{2k+1}(x) + E_{2k+1}(x-1)}{4k+2} - \frac{2E_{2k+1}(x)}{4k+2} \quad (2.14) \]

\[ = \frac{2x^{2k+1} + 2(x-1)^{2k+1}}{4k+2} - \frac{E_{2k+1}(x)}{2k+1}. \]

Therefore, by (2.14), we obtain the following corollary.

**Corollary 2.5.** For $k \in \mathbb{Z}_+$, one has

\[ \sum_{j=0}^{k} \binom{2k}{2j} \frac{E_{2j+1}(x)}{2j+1} = -\frac{E_{2k+1}(x)}{2k+1} + \frac{x^{2k+1} + (x-1)^{2k+1}}{2k+1}. \quad (2.15) \]

Replacing $y$ by 1 and $k$ by $2k$ in Proposition 2.3, we have

\[ \sum_{j=1}^{k} \binom{2k}{2j-1} \frac{E_{2j}(x)}{2j} = \frac{E_{2k+1}(x+1) - E_{2k+1}(x-1) - 2}{4k+2} \]

\[ = \frac{(E_{2k+1}(x+1) + E_{2k+1}(x)) - (E_{2k+1}(x) + E_{2k+1}(x-1))}{4k+2} - \frac{1}{2k+1} \quad (2.16) \]

\[ = \frac{2x^{2k+1} - 2(x-1)^{2k+1}}{4k+2} - \frac{1}{2k+1} \]

\[ = \frac{x^{2k+1} - (x-1)^{2k+1}}{2k+1} - \frac{1}{2k+1}. \]

Therefore, by (2.16), we obtain the following corollary.

**Corollary 2.6.** For $k \in \mathbb{N}$, one has

\[ \sum_{j=1}^{k} \binom{2k}{2j-1} \frac{E_{2j}(x)}{2j} = \frac{x^{2k+1} - (x-1)^{2k+1}}{2k+1} - \frac{1}{2k+1}. \quad (2.17) \]

Replacing $y$ by 1/2 and $k$ by $2k$ in Proposition 2.3, we get

\[ \sum_{j=1}^{k} \binom{2k}{2j-1} \left( \frac{1}{2} \right)^{2k+1-2j} \frac{E_{2j}(x)}{2j} = \frac{E_{2k+1}(x+1/2) - E_{2k+1}(x-1/2) - 2^{-2k}}{4k+2}. \quad (2.18) \]
Thus, we have

\[ \sum_{j=1}^{k} \binom{2k}{2j-1} 2^{j} \frac{E_{2j}(x)}{2j} = \frac{2^{2k} (E_{2k+1}(x + 1/2) - E_{2k+1}(x - 1/2)) - 1}{2k + 1}, \tag{2.19} \]

\[ E_{2k+1}(x + 1/2) = E_{2k+1}(x - 1/2 + 1) = \sum_{l=0}^{2k+1} \binom{2k+1}{l} \left( x - \frac{1}{2} \right)^{2k+1-l} (E+1)^l \]

\[ = 2 \left( x - \frac{1}{2} \right)^{2k+1} - \sum_{l=0}^{2k+1} \binom{2k+1}{l} \left( x - \frac{1}{2} \right)^{2k+1-l} E_{l} \tag{2.20} \]

\[ = 2 \left( x - \frac{1}{2} \right)^{2k+1} - E_{2k+1}(x - \frac{1}{2}). \]

Therefore, by (2.19) and (2.20), we obtain the following corollary.

**Corollary 2.7.** For \( k \in \mathbb{N} \), we have

\[ \sum_{j=1}^{k} \binom{2k}{2j-1} 2^{j} \frac{E_{2j}(x)}{2j} = -\frac{2^{2k+1} E_{2k+1}(x - 1/2)}{2k + 1} + \frac{2^{2k+1} (x - 1/2)^{2k+1}}{2k + 1} - \frac{1}{2k + 1}. \tag{2.21} \]

Replacing \( y \) by 1 and \( k \) by \( 2k + 1 \) in Proposition 2.2, we get

\[ \sum_{j=0}^{k} \binom{2k+1}{2j} \frac{E_{2j+1}(x)}{2j + 1} = \frac{E_{2k+2}(x + 1) - E_{2k+2}(x - 1)}{4k + 4} \]

\[ = \frac{(E_{2k+2}(x + 1) + E_{2k+2}(x)) - (E_{2k+2}(x) + E_{2k+2}(x - 1))}{4k + 4} \tag{2.22} \]

\[ = \frac{2x^{2k+2} - 2(x - 1)^{2k+2}}{4k + 4} = \frac{x^{2k+2} - (x - 1)^{2k+2}}{2k + 2}. \]

Therefore, by (2.22), we obtain the following corollary.

**Corollary 2.8.** For \( k \in \mathbb{Z}_+ \), one has

\[ \sum_{j=0}^{k} \binom{2k+1}{2j} \frac{E_{2j+1}(x)}{2j + 1} = \frac{x^{2k+2} - (x - 1)^{2k+2}}{2k + 2}. \tag{2.23} \]
Replacing $k$ by $2k + 1$ and $y$ by 1 in Proposition 2.3, we get

\[ \sum_{j=1}^{k+1} \left( \frac{2k + 1}{2j - 1} \right) \frac{E_{2j}(x)}{2j} = \frac{E_{2k+2}(x + 1) + E_{2k+2}(x - 1) - 2}{4k + 4} \]

\[ = \frac{(E_{2k+2}(x + 1) + E_{2k+2}(x)) + (E_{2k+2}(x) + E_{2k+2}(x - 1))}{4k + 4} - \frac{E_{2k+2}(x) + 1}{2k + 2} \]

\[ = \frac{x^{2k+2} + (x - 1)^{2k+2}}{2k + 2} - \frac{E_{2k+2}(x) + 1}{2k + 2}. \]  \hfill (2.24)

Therefore, by (2.24), we obtain the following corollary.

**Corollary 2.9.** For $k \in \mathbb{Z}_+$, we have

\[ \sum_{j=1}^{k+1} \left( \frac{2k + 1}{2j - 1} \right) \frac{E_{2j}(x)}{2j} = \frac{x^{2k+2} + (x - 1)^{2k+2}}{2k + 2} - \frac{E_{2k+2}(x) + 1}{2k + 2}. \]  \hfill (2.25)

Replacing $k$ by $2k + 1$ and $y$ by $1/2$ in Proposition 2.2, we have

\[ \sum_{j=0}^{k} \left( \frac{2k + 1}{2j} \right) \left( \frac{1}{2} \right)^{2k+1-2j} \frac{E_{2j+1}(x)}{2j + 1} = \frac{E_{2k+2}(x + 1/2) - E_{2k+2}(x - 1/2)}{4k + 4}. \]  \hfill (2.26)

Thus, by multiplying $2^{2k+1}$ on both sides, we get

\[ \sum_{j=0}^{k} \left( \frac{2k + 1}{2j} \right) \frac{E_{2j+1}(x)}{2j + 1} = \frac{2^{2k+1} \left( E_{2k+2}(x + 1/2) - E_{2k+2}(x - 1/2) \right)}{4k + 4}. \]  \hfill (2.27)

By (2.20) and (2.27), we see that

\[ \sum_{j=0}^{k} \left( \frac{2k + 1}{2j} \right) \frac{E_{2j+1}(x)}{2j + 1} = \frac{2^{2k} \left( 2(x - 1/2)^{2k+2} - 2E_{2k+2}(x - 1/2) \right)}{2k + 2} \]

\[ = \frac{2^{2k} \left( x - 1/2 \right)^{2k+2} - 2^{2k}E_{2k+2}(x - 1/2)}{k + 1}. \]  \hfill (2.28)

Therefore, by (2.28), we obtain the following corollary.

**Corollary 2.10.** For $k \in \mathbb{Z}_+$, we have

\[ \sum_{j=0}^{k} \left( \frac{2k + 1}{2j} \right) \frac{E_{2j}(x)}{2j + 1} = \frac{2^{2k} E_{2k+2}(x - 1/2)}{k + 1} + \frac{2^{2k} (x - 1/2)^{2k+2}}{k + 1}. \]  \hfill (2.29)
Let us take the $p$-adic integral on both sides in (2.30) as follows: for $k \in \mathbb{N}$,

$$I_1 = \sum_{j=0}^{k-1} \sum_{l=0}^{j+1} \frac{1}{j+1} \binom{k}{j} \binom{j+1}{l} B_{j+1-l} B_l = B_k - \frac{1}{k+1}. \tag{2.32}$$

Therefore, by (2.31) and (2.32), we obtain the following theorem.

**Theorem 2.11.** For $k \in \mathbb{N}$, one has

$$\sum_{j=0}^{k-1} \sum_{l=0}^{j+1} \frac{1}{j+1} \binom{k}{j} \binom{j+1}{l} B_{j+1-l} B_l = B_k - \frac{1}{k+1}. \tag{2.33}$$

In (2.30), let us take the fermionic $p$-adic integral on both sides as follows:

$$I_2 = \sum_{j=0}^{k-1} \frac{1}{j+1} \int_{\mathbb{Z}_p} B_{j+1}(x) d\mu_1(x)$$

$$= \sum_{j=0}^{k-1} \sum_{l=0}^{j+1} \binom{k}{j} \binom{j+1}{l} B_{j+1-l} \int_{\mathbb{Z}_p} x^l d\mu_1(x) \tag{2.34}$$

On the other hand

$$I_2 = \int_{\mathbb{Z}_p} x^k d\mu_1(x) - \frac{1}{k+1} \int_{\mathbb{Z}_p} d\mu_1(x) = E_k - \frac{1}{k+1}. \tag{2.35}$$

Therefore, by (2.34) and (2.35), we obtain the following theorem.
Theorem 2.12. For $k \in \mathbb{N}$, one has

\[
\sum_{j=0}^{k-1} \sum_{l=0}^{j+1} \frac{1}{j+1} \binom{k}{j} \binom{j+1}{l} B_{j+1-l} = E_k - \frac{1}{k+1}.
\] (2.36)

From (1.7), we can easily derive the following equation:

\[
\sum_{j=0}^{k-1} \binom{2k}{2j+1} B_{2j+1}(x) = \frac{x^{2k} - (x - 1)^{2k}}{2}.
\] (2.37)

Let us take $\int_{Z_p} d\mu(x)$ on both sides in (2.37). Then we have

\[
I_3 = \sum_{j=0}^{k-1} \binom{2k}{2j} \frac{1}{2j+1} \int_{Z_p} B_{2j+1}(x) d\mu(x)
\]

\[
= \sum_{j=0}^{k-1} \binom{2k}{2j+1} \sum_{l=0}^{2j+1} \binom{2j+1}{l} \int_{Z_p} x^l d\mu(x)
\] (2.38)

\[= \sum_{j=0}^{k-1} \sum_{l=0}^{2j+1} \frac{1}{2j+1} \binom{2k}{2j} \binom{2j+1}{l} B_{2j+1-l} B_l.
\]

On the other hand,

\[
I_3 = \frac{1}{2} \left( \int_{Z_p} x^{2k} d\mu(x) - \int_{Z_p} (x - 1)^{2k} d\mu(x) \right)
\]

\[= \frac{1}{2} (B_{2k} - B_{2k}(-1)) = \frac{1}{2} (B_{2k} - B_{2k}(2))
\] (2.39)

\[= \frac{1}{2} (B_{2k} - (2k + \delta_{1,2k} + B_{2k})),
\]

where $\delta_{n,k}$ is a Kronecker symbol.

Therefore, by (2.38) and (2.39), we obtain the following theorem.

Theorem 2.13. For $k \in \mathbb{N}$, one has

\[
\sum_{j=0}^{k-1} \sum_{l=0}^{2j+1} \frac{1}{2j+1} \binom{2k}{2j} \binom{2j+1}{l} B_{2j+1-l} B_l = -k.
\] (2.40)
Taking \( \int_{z_p} d\mu_{-1}(x) \) on both sides in (2.37), we get

\[
I_4 = \sum_{j=0}^{k-1} \sum_{l=0}^{2j+1} \frac{1}{2j+1} \left( \begin{array}{c} 2k \\ 2j \\ \end{array} \right) \left( \begin{array}{c} 2j+1 \\ l \\ \end{array} \right) B_{2j+1-l} \int_{z_p} x^l d\mu_{-1}(x)
= \sum_{j=0}^{k-1} \sum_{l=0}^{2j+1} \frac{1}{2j+1} \left( \begin{array}{c} 2k \\ 2j \\ \end{array} \right) \left( \begin{array}{c} 2j+1 \\ l \\ \end{array} \right) B_{2j+1-l} E_i.
\] (2.41)

On the other hand

\[
I_4 = \frac{1}{2} \left( \int_{z_p} x^k \, d\mu_{-1}(x) - \int_{z_p} (x-1)^{2k} \, d\mu_{-1}(x) \right)
= \frac{1}{2} \left( \int_{z_p} x^{2k} \, d\mu_{-1}(x) - \int_{z_p} (x+2)^{2k} \, d\mu_{-1}(x) \right)
= \frac{1}{2} (E_{2k} - (x+2)^{2k} - (x-1)^{2k})
= -1 + \delta_{0,k}.
\] (2.42)

Therefore, by (2.41) and (2.42), we obtain the following theorem.

**Theorem 2.14.** For \( k \in \mathbb{N} \), one has

\[
\sum_{j=0}^{k-1} \sum_{l=0}^{2j+1} \frac{1}{2j+1} \left( \begin{array}{c} 2k \\ 2j \\ \end{array} \right) \left( \begin{array}{c} 2j+1 \\ l \\ \end{array} \right) B_{2j+1-l} E_i = -1.
\] (2.43)

From (1.8), we can also derive the following equation:

\[
\sum_{j=1}^{k} \left( \begin{array}{c} 2k \\ 2j-1 \end{array} \right) B_{2j}(x) \frac{x^{2k} + (x-1)^{2k}}{2} = \frac{1}{2k+1}.
\] (2.44)

Let us take the bosonic \( p \)-adic integral on both sides in (2.44). Then we get

\[
I_5 = \sum_{j=1}^{k} \left( \begin{array}{c} 2k \\ 2j-1 \end{array} \right) \frac{1}{2j} \int_{z_p} B_{2j}(x) d\mu(x)
= \sum_{j=1}^{k} \left( \begin{array}{c} 2k \\ 2j-1 \end{array} \right) \frac{1}{2j} \sum_{l=0}^{2j} \left( \begin{array}{c} 2j \\ l \end{array} \right) B_{2j-l} \int_{z_p} x^l d\mu(x)
= \sum_{j=1}^{k} \sum_{l=0}^{2j} \frac{1}{2j} \left( \begin{array}{c} 2k \\ 2j-1 \end{array} \right) \left( \begin{array}{c} 2j \\ l \end{array} \right) B_{2j-l} B_l.
\] (2.45)
On the other hand,

\[ I_5 = \frac{1}{2} \int_{\mathbb{Z}_p} (x^{2k} + (-1 + x)^{2k}) d\mu(x) - \frac{1}{2k + 1} \int_{\mathbb{Z}_p} d\mu(x) \]

\[ = \frac{1}{2} \int_{\mathbb{Z}_p} (x^{2k} + (x + 2)^{2k}) d\mu(x) - \frac{1}{2k + 1} \]

\[ = \frac{1}{2} (B_{2k} + B_{2k}(2)) - \frac{1}{2k + 1} \]

\[ = \frac{1}{2} (B_{2k} + 2k + B_{2k} + \delta_{1,2k}) - \frac{1}{2k + 1}. \tag{2.46} \]

Therefore, by (2.45) and (2.46), we obtain the following theorem.

**Theorem 2.15.** For \( k \in \mathbb{N} \), one has

\[ \sum_{j=1}^{k} \sum_{l=0}^{2j} \frac{1}{2j} \binom{2k}{2j-1} \binom{2j}{l} B_{2j-l} = B_{2k} + k - \frac{1}{2k + 1}. \tag{2.47} \]

Now, let us consider the fermionic \( p \)-adic integral on both sides in (2.44):

\[ I_6 = \sum_{j=1}^{k} \left( \binom{2k}{2j-1} \right) \left( \frac{1}{2j} \sum_{l=0}^{2j} \binom{2j}{l} \right) B_{2j-l} \int_{\mathbb{Z}_p} x^l d\mu_{-1}(x) \]

\[ = \sum_{j=1}^{k} \sum_{l=0}^{2j} \frac{1}{2j} \binom{2k}{2j-1} \binom{2j}{l} B_{2j-l} E_l. \tag{2.48} \]

On the other hand,

\[ I_6 = \frac{1}{2} \int_{\mathbb{Z}_p} (x^{2k} + (x - 1)^{2k}) d\mu_{-1}(x) - \frac{1}{2k + 1} \int_{\mathbb{Z}_p} d\mu_{-1}(x) \]

\[ = \frac{1}{2} \int_{\mathbb{Z}_p} (x^{2k} + (x + 2)^{2k}) d\mu_{-1}(x) - \frac{1}{2k + 1} \]

\[ = \frac{1}{2} (E_{2k} + E_{2k}(2)) - \frac{1}{2k + 1} \]

\[ = \frac{1}{2} (E_{2k} + (2 + E_{2k} - 2\delta_{0,2k})) - \frac{1}{2k + 1} \]

\[ = E_{2k} + 1 - \delta_{0,2k} - \frac{1}{2k + 1} = \frac{2k}{2k + 1}. \tag{2.49} \]

Therefore, by (2.48) and (2.49), we obtain the following theorem.
Theorem 2.16. For $k \in \mathbb{N}$, one has
\[
\sum_{j=1}^{k} \sum_{i=0}^{2j} \frac{1}{2j} \binom{2j}{2j-1} \binom{2j}{i} B_{2j-i} E_i = \frac{2k}{2k+1}.
\] (2.50)

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References


