Inequalities between Arithmetic-Geometric, Gini, and Toader Means

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We find the greatest values $p_1, p_2$ and least values $q_1, q_2$ such that the double inequalities $S_{p_1}(a,b) < M(a,b) < S_{q_1}(a,b)$ and $S_{p_2}(a,b) < T(a,b) < S_{q_2}(a,b)$ hold for all $a, b > 0$ with $a \neq b$ and present some new bounds for the complete elliptic integrals. Here $M(a,b)$, $T(a,b)$, and $S_p(a,b)$ are the arithmetic-geometric, Toader, and $p$th Gini means of two positive numbers $a$ and $b$, respectively.

1. Introduction

For $p \in \mathbb{R}$ the $p$th Gini mean $S_p(a,b)$ and power mean $M_p(a,b)$ of two positive real numbers $a$ and $b$ are defined by

$$
S_p(a,b) = \begin{cases} 
\left( \frac{a^{p-1} + b^{p-1}}{a + b} \right)^{1/(p-2)}, & p \neq 2, \\
(a^{a}b^{b})^{1/(a+b)}, & p = 2,
\end{cases} \quad (1.1)
$$

$$
M_p(a,b) = \begin{cases} 
\left( \frac{a^p + b^p}{2} \right)^{1/p}, & p \neq 0, \\
\sqrt{ab}, & p = 0,
\end{cases} \quad (1.2)
$$

respectively.
It is well known that $S_p(a,b)$ and $M_p(a,b)$ are continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. Many means are special case of these means, for example,

\[
S_1(a,b) = M_1(a,b) = \frac{a + b}{2} = A(a,b) \text{ is the arithmetic mean,}
\]

\[
S_0(a,b) = M_0(a,b) = \sqrt{ab} = G(a,b) \text{ is the geometric mean,}
\]

\[
M_{-1}(a,b) = \frac{2ab}{a + b} = H(a,b) \text{ is the harmonic mean.}
\]

Recently, the Gini and power means have been the subject of intensive research. In particular, many remarkable inequalities for these means can be found in the literature [1–7].

In [8], Toader introduced the Toader mean $T(a,b)$ of two positive numbers $a$ and $b$ as follows:

\[
T(a,b) = \frac{2}{\pi} \int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \, d\theta
\]

\[
= \begin{cases} 
\frac{2a\mathcal{E}\left(\sqrt{1-(b/a)^2}\right)}{\pi}, & a > b, \\
\frac{2b\mathcal{E}\left(\sqrt{1-(a/b)^2}\right)}{\pi}, & a < b, \\
a, & a = b,
\end{cases}
\]

where $\mathcal{E}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 t)^{-1/2} \, dt$, $r \in [0,1]$, is the complete elliptic integrals of the second kind.

The classical arithmetic-geometric mean $M(a,b)$ of two positive number $a$ and $b$ is defined as the common limit of sequences $\{a_n\}$ and $\{b_n\}$, which are given by

\[
a_0 = a, \quad b_0 = b,
\]

\[
a_{n+1} = \frac{a_n + b_n}{2} = A(a_n, b_n), \quad b_{n+1} = \sqrt{a_nb_n} = G(a_n, b_n).
\]

The Gauss identity [9] shows that

\[
M(1,r)\mathcal{K}\left(\sqrt{1-r^2}\right) = \frac{\pi}{2}
\]

for $r \in (0,1)$, where $\mathcal{K}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 t)^{-1/2} \, dt$, $r \in [0,1]$, is the complete elliptic integrals of the first kind.
Vuorinen [10] conjectured that

\[ M_{3/2}(a, b) < T(a, b) \]  

(1.7)

for all \( a, b > 0 \) with \( a \neq b \). This conjecture was proved by Qiu and Shen in [11] and Barnard et al. in [12], respectively.

In [13], Alzer and Qiu presented a best possible upper power mean bound for the Toader mean as follows:

\[ T(a, b) < M_{\log 2/\log(\pi/2)}(a, b) \]  

(1.8)

for all \( a, b > 0 \) with \( a \neq b \).

In [14–17], the authors proved that

\[ M_0(a, b) = G(a, b) < M(a, b) < M_{1/2}(a, b), \]  

(1.9)

\[ L(a, b) < M(a, b) < \frac{\pi}{2} L(a, b) \]  

(1.10)

for all \( a, b > 0 \) with \( a \neq b \), where

\[ L(a, b) = \begin{cases} \frac{a - b}{\log a - \log b}, & a \neq b, \\ a, & a = b, \end{cases} \]  

(1.11)

denotes the classical logarithmic mean of two positive numbers \( a \) and \( b \).

Very recently, Chu and Wang [18] and Guo and Qi [19] proved that

\[ L_0(a, b) < T(a, b) < L_{1/4}(a, b) \]  

(1.12)

for all \( a, b > 0 \) with \( a \neq b \), and \( L_0(a, b) \) and \( L_{1/4}(a, b) \) are the best possible lower and upper Lehmer mean bounds for the Toader mean \( T(a, b) \), respectively. Here, the \( p \)th Lehmer mean \( L_p(a, b) \) of two positive numbers \( a \) and \( b \) is defined by \( L_p(a, b) = (a^{p+1} + b^{p+1})/(a^p + b^p) \).

The main purpose of this paper is to find the greatest values \( p_1, p_2 \) and least values \( q_1, q_2 \) such that the double inequalities \( S_{p_1}(a, b) < M(a, b) < S_{q_1}(a, b) \) and \( S_{p_2}(a, b) < T(a, b) < S_{q_2}(a, b) \) hold for all \( a, b > 0 \) with \( a \neq b \) and present some new bounds for the complete elliptic integrals.

2. Preliminary Knowledge

Throughout this paper, we denote \( r' = \sqrt{1-r^2} \) for \( r \in [0, 1] \).
For $0 < r < 1$, the following derivative formulas were presented in [9, Appendix E, pages 474–475]:

\[
\frac{d\mathcal{K}(r)}{dr} = \frac{\mathcal{E}(r) - r^2\mathcal{K}(r)}{r}, \quad \frac{d\mathcal{E}(r)}{dr} = \frac{\mathcal{E}(r) - \mathcal{K}(r)}{r}, \quad \frac{d}{dr}\left[\mathcal{E}(r) - r^2\mathcal{K}(r)\right] = r\mathcal{K}(r), \quad \frac{d\left[\mathcal{K}(r) - \mathcal{E}(r)\right]}{dr} = \frac{r\mathcal{E}(r)}{r^2}. \tag{2.1}
\]

\[
\mathcal{K}\left(\frac{2\sqrt{r}}{1 + r}\right) = (1 + r)\mathcal{K}(r), \tag{2.2}
\]

\[
\mathcal{E}\left(\frac{2\sqrt{r}}{1 + r}\right) = \frac{2\mathcal{E}(r) - r^2\mathcal{K}(r)}{1 + r}. \tag{2.3}
\]

Lemma 2.1 can be found in [9, Theorem 3.21(7), (8), and (10), and Exercise 3.43(13) and (46)].

**Lemma 2.1.** (1) $r^c\mathcal{K}(r)$ is strictly decreasing from $[0, 1)$ onto $(0, \pi/2]$ for $c \in [1/2, \infty)$;  
(2) $r^c\mathcal{E}(r)$ is strictly increasing on $(0, 1)$ if and only if $c \leq -1/2$ and strictly decreasing if and only if $c > 0$;  
(3) $\mathcal{K}(r)/\log(4/\pi)$ is strictly decreasing from $(0, 1)$ onto $(1, \pi/\log 16)$;  
(4) $2\mathcal{E}(r) - r^2\mathcal{K}(r)$ is strictly increasing from $(0, 1)$ onto $(\pi/2, 2)$;  
(5) $[\mathcal{E}(r) - r^2\mathcal{K}(r)]/[r^2\mathcal{K}(r)]$ is strictly decreasing from $(0, 1)$ onto $(0, 1/2)$.

### 3. Main Results

**Theorem 3.1.** Inequality $S_{1/2}(a, b) < M(a, b) < S_1(a, b)$ holds for all $a, b > 0$ with $a \neq b$, and $S_{1/2}(a, b)$ and $S_1(a, b)$ are the best possible lower and upper Gini mean bounds for the arithmetic-geometric mean $M(a, b)$.

**Proof.** From (1.1) and (1.5) we clearly see that both $S_r(a, b)$ and $M(a, b)$ are symmetric and homogenous of degree 1. Without loss of generality, we assume that $a = 1 > b$. Let $t = b$ and $r = (1 - t)/(1 + t)$. Then from (1.1) and (1.6) together with (2.2) we clearly see that

\[
M(a, b) - S_{1/2}(a, b) = \frac{\pi}{2\mathcal{K}\left(\sqrt{1 - t^2}/2\right)} - \left[\frac{1 + t}{\mathcal{K}\left(\sqrt{1-t^2}/2\right)}\right]^{2/3} \tag{3.1}
\]

\[
= \frac{\pi}{2(1 + r)\mathcal{K}(r)} - \left[\frac{2\sqrt{1 - r}}{(1 + r)(\sqrt{1 + r + \sqrt{1 - r}})}\right]^{2/3}.
\]

\[
= \frac{1}{1 + r}\left[\frac{\pi}{2\mathcal{K}(r)} - \left(\frac{2r'}{\sqrt{1 + r + \sqrt{1 - r}}}\right)^{2/3}\right].
\]
Let

\[ F(r) = \left( \frac{\pi}{2K(r)} \right)^3 - \left( \frac{2r'}{\sqrt{1+r} + \sqrt{1-r}} \right)^2. \]  

(3.2)

Then \( F(r) \) can be rewritten as

\[ F(r) = \left( \frac{\pi}{2K(r)} \right)^3 - \frac{2r'^2}{1+r^2} = \frac{2r'^2}{1+r^2} F_1(r), \]

(3.3)

where

\[ F_1(r) = \left( \frac{\pi}{2} \right)^3 \frac{1+r'}{2r'^2K(r)^3} - 1. \]  

(3.4)

It is well known that the function \( r \to \sqrt{r} + 1/\sqrt{r} \) is positive and strictly decreasing in \((0, 1)\). Then (3.4) and Lemma 2.1(1) lead to the conclusion that \( F_1(r) \) is strictly increasing in \((0, 1)\), so that \( F_1(r) > F_1(0) = 0 \) for \( r \in (0, 1)\).

Therefore, \( M(a, b) > S_{1/2}(a, b) \) follows from (3.1)–(3.3).

On the other hand, \( M(a, b) < S_1(a, b) = A(a, b) \) follows directly from (1.9).

Next, we prove that \( S_{1/2}(a, b) \) and \( S_1(a, b) \) are the best possible lower and upper Gini mean bounds for the arithmetic-geometric mean \( M(a, b) \).

For any \( 0 < \varepsilon < 1/2 \) and \( 0 < x < 1 \), from (1.1), (1.6), and Lemma 2.1(3) we have

\[
[M(1, 1-x)]^{3-2\varepsilon} - [S_{1/2+\varepsilon}(1, 1-x)]^{3-2\varepsilon} = \left[ \frac{\pi}{2 \int_0^{\pi/2} \left( 1 - (2x - x^2 \sin^2 t) \right)^{-1/2} dt} \right]^{3-2\varepsilon} - \left[ \frac{(2-x)(1-x)^{1/2-\varepsilon}}{1 + (1-x)^{1/2-\varepsilon}} \right]^2,
\]

(3.5)

\[
\lim_{x \to 0} \frac{M(1, x)}{S_{1-\varepsilon}(1, x)} = \lim_{x \to 0} \left[ \frac{2}{\pi x^{\varepsilon/(1+\varepsilon)} K\left( \sqrt{1-x^2} \right)} \left( \frac{1+x^\varepsilon}{1+x} \right)^{1/(1+\varepsilon)} \right]
\]

\[
= \lim_{x \to 0} \frac{2}{\pi x^{\varepsilon/(1+\varepsilon)} K\left( \sqrt{1-x^2} \right)} = \lim_{x \to 0} \frac{2}{\pi x^{\varepsilon/(1+\varepsilon)} \log(4/x) K\left( \sqrt{1-x^2} \right)}
\]

(3.6)

\[
= \lim_{x \to 0} \frac{2}{\pi x^{\varepsilon/(1+\varepsilon)} \log(4/x)} = +\infty.
\]
Letting \( x \to 0 \) and making use of the Taylor expansion, one has

\[
\begin{align*}
\left[ \frac{\pi}{2 \int_0^{\pi/2} \left[ 1 - (2x - x^2 \sin^2 t) \right]^{-1/2} \, dt } \right]^{3-2\varepsilon} - \left[ \frac{(2 - x)(1 - x)^{1/2 - \varepsilon}}{1 + (1 - x)^{1/2 - \varepsilon}} \right] \\
= 1 + \left( -\frac{3}{2} + \varepsilon \right) x + \frac{(2\varepsilon - 3)(4\varepsilon - 3)}{16} x^2 + o(x^2) \\
- \left[ 1 + \left( -\frac{3}{2} + \varepsilon \right) x + \frac{(2\varepsilon - 3)^2}{16} x^2 + o(x^2) \right] \\
= -\frac{\varepsilon(3 - 2\varepsilon)}{8} x^2 + o(x^2).
\end{align*}
\] (3.7)

Equations (3.5)–(3.7) imply that for any \( 1 < \varepsilon < 1/2 \) there exist \( \delta_1 = \delta_1(\varepsilon) \in (0, 1) \) and \( \delta_2 = \delta_2(\varepsilon) \in (0, 1) \), such that \( M(1, 1 - x) < S_{1/2 + \varepsilon}(1, 1 - x) \) for \( x \in (0, \delta_1) \) and \( M(1, x) > S_{1-\varepsilon}(1, x) \) for \( x \in (0, \delta_2) \).

**Theorem 3.2.** Inequality \( S_1(a, b) < T(a, b) < S_{3/2}(a, b) \) holds for all \( a, b > 0 \) with \( a \neq b \), and \( S_1(a, b) \) and \( S_{3/2}(a, b) \) are the best possible lower and upper Gini mean bounds for the Toader mean \( T(a, b) \).

**Proof.** From (1.1) and (1.4) we clearly see that both \( S_p(a, b) \) and \( T(a, b) \) are symmetric and homogenous of degree 1. Without loss of generality, we assume that \( a = 1 > b \). Let \( t = b \) and \( r = (1 - t)/(1 + t) \). Then from (1.1), (1.4), and (2.3) we have

\[
\frac{T(a, b)}{S_{3/2}(a, b)} = \frac{2}{\pi} \xi \left( \sqrt{1 - r^2} \right) \cdot \left( 1 + \frac{\sqrt{1 + r}}{1 + t} \right)^2 \\
= \frac{2}{\pi} \xi \left( \frac{2\sqrt{r}}{1 + r} \right) \cdot (1 + r) \cdot \left( \frac{\sqrt{1 + r} + \sqrt{1 - r}}{2} \right)^2 \quad \text{(3.8)}
\]

\[
= \frac{2}{\pi} \left[ 2\xi(r) - r^2 \mathcal{K}(r) \right] \cdot \left( \frac{\sqrt{1 + r} + \sqrt{1 - r}}{2} \right)^2 \\
= \frac{1}{\pi} (1 + r') \left[ 2\xi(r) - r^2 \mathcal{K}(r) \right].
\]

Let

\[
G(r) = \frac{1}{\pi} (1 + r') \left[ 2\xi(r) - r^2 \mathcal{K}(r) \right]. \quad \text{(3.9)}
\]
Then simple computations lead to

\[ G(0) = 1, \quad (3.10) \]

\[
G'(r) = \frac{1}{\pi} \left[ \left( -\frac{r}{r'} \right) \left( 2\mathcal{E}(r) - r'^2\mathcal{K}(r) \right) + (1 + r') \left( \frac{\mathcal{E}(r) - r'^2\mathcal{K}(r)}{r} \right) \right]
\]

\[ = \frac{r'(1 + r')}{\pi r r'} \left[ \mathcal{E}(r) - r'^2\mathcal{K}(r) \right] - r^2 \left[ 2\mathcal{E}(r) - r'^2\mathcal{K}(r) \right] \quad (3.11) \]

\[ = \frac{r}{\pi r'} G_1(r), \]

where

\[ G_1(r) = (1 + r') r' \mathcal{K}(r) \left[ \frac{\mathcal{E}(r) - r'^2\mathcal{K}(r)}{r'^2\mathcal{K}(r)} \right] - \left[ 2\mathcal{E}(r) - r'^2\mathcal{K}(r) \right]. \quad (3.12) \]

It follows from (3.12) and Lemma 2.1(1), (4), and (5) that \( G_1(r) \) is strictly decreasing from \((0, 1)\) onto \((-2, 0)\). Then (3.11) leads to the conclusion that \( G'(r) < 0 \) for \( r \in (0, 1) \). Hence \( G(r) \) is strictly decreasing in \((0, 1)\).

Therefore, \( T(a, b) < S_{3/2}(a, b) \) follows from (3.8)–(3.10) together with the monotonicity of \( G(r) \).

On the other hand, \( T(a, b) > S_1(a, b) = A(a, b) \) follows directly from (1.7).

Next, we prove that \( S_1(a, b) \) and \( S_{3/2}(a, b) \) are the best possible lower and upper Gini mean bounds for the Toader mean \( T(a, b) \).

For any \( 0 < \varepsilon < 1/2 \) and \( 0 < x < 1 \), from (1.1) and (1.4) one has

\[ [T(1, 1-x)]^{1+2\varepsilon} - [S_{3/2-\varepsilon}(1, 1-x)]^{1+2\varepsilon} = \left[ \frac{2}{\pi} \int_0^{\pi/2} \left[ 1 - \left( 2x - x^2 \right) \sin^2 t \right]^{1/2} dt \right]^{1+2\varepsilon} \]
\[ - \left[ \frac{2 - x}{1 + (1-x)^{1/2-\varepsilon}} \right]^2 \]

\[ \lim_{x \to 0} \frac{T(1,x)}{S_{1+\varepsilon}(1,x)} = \lim_{x \to 0} \left[ \frac{2}{\pi} \mathcal{E} \left( \sqrt{1-x^2} \right) \left( \frac{1 + x^2}{1 + x} \right)^{1/(1-\varepsilon)} \right] = \frac{2}{\pi} < 1. \quad (3.14) \]
Letting $x \to 0$ and making use of the Taylor expansion, we get

$$
\frac{2}{\pi} \int_0^{\pi/2} \left[ 1 - \left( 2x - x^2 \right) \sin^2 t \right]^{1/2} dt \left[ 1 + (1 - x)^{1/2} - 1 - \left( \frac{2 - x}{1 + (1 - x)^{1/2}} \right)^2 \right]^{1+2x}
$$

$$
= 1 - \left( \frac{1}{2} + \varepsilon \right) x + \frac{(2\varepsilon + 1)(4\varepsilon + 1)}{16} x^2 + o(x^2)
$$

$$
- \left[ 1 - \left( \frac{1}{2} + \varepsilon \right) x + \frac{(2\varepsilon + 1)^2}{16} x^2 + o(x^2) \right]
$$

$$
= \frac{\varepsilon(2\varepsilon + 1)}{8} x^2 + o(x^2).
$$

Equations (3.13)–(3.15) imply that for any $0 < \varepsilon < 1/2$ there exist $\delta_3 = \delta_3(\varepsilon) \in (0, 1)$ and $\delta_4 = \delta_4(\varepsilon) \in (0, 1)$, such that $T(1, 1 - x) > S_{3/2 - \varepsilon}(1, 1 - x)$ for $x \in (0, \delta_3)$ and $T(1, x) < S_{1 - \varepsilon}(1, x)$ for $x \in (0, \delta_4)$.

**4. Remarks and Corollaries**

**Remark 4.1.** From (3.9) and Lemma 2.1(4) we clearly see that $G(1^-) = 2/\pi$. Then (3.8) and (3.9) together with the monotonicity of $G(r)$ lead to the conclusion that

$$
\frac{2}{\pi} S_{3/2}(a, b) < T(a, b)
$$

(4.1)

for all $a, b > 0$ with $a \neq b$.

**Remark 4.2.** We find that the lower bound $L(a, b)$ in (1.10) and the best possible lower Gini mean bound $S_{1/2}(a, b)$ in Theorem 3.1 are not comparable. In fact, from (1.1) and (1.11) we have

$$
\lim_{x \to +\infty} \frac{S_{1/2}(1, x)}{L(1, x)} = \lim_{x \to +\infty} \left[ \frac{1 + x^{-1}}{1 + x^{-1/2}} \right]^{2/3} \frac{x^{2/3} \log x}{x - 1} = \lim_{x \to +\infty} \frac{\log x}{x^{1/3} - x^{-2/3}} = 0,
$$

$$
S_{1/2}(1, 1 + x) - L(1, 1 + x) = 1 + \frac{1}{2} x - \frac{1}{16} x^2 + o(x^2) - \left[ 1 + \frac{1}{2} x - \frac{1}{12} x^2 + o(x^2) \right]
$$

$$
= \frac{1}{48} x^2 + o(x^2) \quad (x \to 0).
$$
Table 1: Comparison of $K(r)$ with $H(r)$ for some $r \in (0,1)$.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$K(r)$</th>
<th>$H(r)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.5745561517...</td>
<td>1.57455561518...</td>
</tr>
<tr>
<td>0.2</td>
<td>1.586867847...</td>
<td>1.586867848...</td>
</tr>
<tr>
<td>0.3</td>
<td>1.60848620...</td>
<td>1.60848634...</td>
</tr>
<tr>
<td>0.4</td>
<td>1.63999866...</td>
<td>1.640000021...</td>
</tr>
<tr>
<td>0.5</td>
<td>1.685750355...</td>
<td>1.685751528...</td>
</tr>
<tr>
<td>0.6</td>
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</tr>
<tr>
<td>0.7</td>
<td>1.845693998...</td>
<td>1.845732233...</td>
</tr>
<tr>
<td>0.8</td>
<td>1.995302778...</td>
<td>1.995519211...</td>
</tr>
</tbody>
</table>

Remark 4.3. The following two equations show that the best possible upper power mean bound $M_{\log 2/\log(\pi/2)}(a,b)$ in (1.8) and the best possible upper Gini mean bound $S_{3/2}(a,b)$ in Theorem 3.2 are not comparable:

$$\lim_{x \to +\infty} \frac{S_{3/2}(1,x)}{M_{\log 2/\log(\pi/2)}(1,x)} = \frac{2^{\log(\pi/2)/\log 2}}{\pi} = \frac{\pi}{2},$$

$$M_{\log 2/\log(\pi/2)}(1,1+x) - S_{3/2}(1,1+x) = 1 + \frac{1}{2}x + \frac{1}{8} \left[ \frac{\log 2}{\log(\pi/2)} - 1 \right] x^2$$

$$+ o(x^2) - \left[ 1 + \frac{1}{2}x + \frac{1}{16}x^2 + o(x^2) \right]$$

$$= \frac{1}{16} \left[ \frac{2\log 2}{\log(\pi/2)} - 3 \right] x^2 + o(x^2)$$

$$= 0.00436 \cdots \times x^2 + o(x^2) \quad (x \to 0).$$

From Theorem 3.1 we get an upper bound for the complete elliptic integrals of the first kind $K(r)$ as follows.

**Corollary 4.4. Inequality**

$$K(r) < \frac{\pi}{2} \left[ \frac{1 + (1-r^2)^{1/4}}{(1 + \sqrt{1-r^2})(1-r^2)^{1/4}} \right]^{2/3}$$

(4.4)

holds for all $r \in (0,1)$.

Remark 4.5. Computational and numerical experiments show that the upper bound in (4.4) for $K(r)$ is very accurate for some $r \in (0,1)$. In fact, if we let $H(r) = \pi \left[ 1 + (1-r^2)^{1/4} \right]^{2/3} / \left[ 2 \left[(1 + \sqrt{1-r^2})(1-r^2)^{1/4}\right]^{2/3} \right]$, then we have Table 1 via elementary computation.
Table 2: Comparison of $E(r)$ with $J(r)$ for some $r \in (0,1)$.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$E(r)$</th>
<th>$J(r)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
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<td>1.56686192028⋯</td>
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<tr>
<td>0.2</td>
<td>1.554968546⋯</td>
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<tr>
<td>0.6</td>
<td>1.355661136⋯</td>
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<td>0.7</td>
<td>1.276349943⋯</td>
<td>1.276910677⋯</td>
</tr>
<tr>
<td>0.8</td>
<td>1.276349943⋯</td>
<td>1.276910677⋯</td>
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</tbody>
</table>

The following bounds for the complete elliptic integrals of the second kind $E(r)$ follow from Theorem 3.2 and Remark 4.1.

**Corollary 4.6. Inequality**

$$\left[ \frac{1 + \sqrt{1 - r^2}}{1 + (1 - r^2)^{1/4}} \right]^2 < E(r) < \frac{\pi}{2} \left[ \frac{1 + \sqrt{1 - r^2}}{1 + (1 - r^2)^{1/4}} \right]^2$$

(4.5)

holds for all $r \in (0,1)$.

**Remark 4.7.** Computational and numerical experiments show that the upper bound in (4.5) for $E(r)$ is very accurate for some $r \in (0,1)$. In fact, if we let $J(r) = \pi \left[ 1 + \sqrt{1 - r^2} \right]^2 / \left\{ 2(1 + (1 - r^2)^{1/4})^2 \right\}$, then we have Table 2 via elementary computation.

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**References**


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