Research Article

Strong Convergence of an Implicit S-Iterative Process for Lipschitzian Hemicontractive Mappings

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We establish the strong convergence for the implicit S-iterative process associated with Lipschitzian hemicontractive mappings in Hilbert spaces.

1. Introduction

Let $H$ be a Hilbert space and let $T : H \to H$ be a mapping. The mapping $T$ is called Lipschitzian if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in H. \quad (1.1)$$

If $L = 1$, then $T$ is called nonexpansive and if $0 \leq L < 1$, then $T$ is called contractive. The mapping $T$ is said to be pseudocontractive ([1, 2]) if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H, \quad (1.2)$$

and the mapping $T$ is said to be strongly pseudocontractive if there exists $k \in (0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H. \quad (1.3)$$
Let $F(T) := \{ x \in H : Tx = x \}$ and the mapping $T$ is called hemiconttractive if $F(T) \neq \emptyset$ and
\[
\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + \|x - Tx\|^2, \quad \forall x \in H, x^* \in F(T).
\] (1.4)

It is easy to see the class of pseudocontractive mappings with fixed points is a subclass of the class of hemiconttractive mappings. For the importance of fixed points of pseudocontractions the reader may consult [1].

In 1974, Ishikawa [3] proved the following result.

**Theorem 1.1.** Let $K$ be a compact convex subset of a Hilbert space $H$ and let $T : K \to K$ be a Lipschitzian pseudocontractive mapping.

For arbitrary $x_1 \in K$, let $\{x_n\}$ be a sequence defined iteratively by
\[
x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTy_n,
\]
\[
y_n = (1 - \beta_n)x_n + \beta_nTx_n, \quad n \geq 1,
\] (1.5)

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying the conditions:

(i) $0 \leq \alpha_n \leq \beta_n \leq 1$,
(ii) $\lim_{n \to \infty} \beta_n = 0$,
(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point of $T$.

Another iteration scheme which has been studied extensively in connection with fixed points of pseudocontractive mappings.


Let $K$ be a nonempty convex subset of a normed space $X$ and let $T : K \to K$ be a mapping. Then, for arbitrary $x_1 \in K$, the $S$-iterative process is defined by
\[
x_{n+1} = Ty_n,
\]
\[
y_n = (1 - \beta_n)x_n + \beta_nTx_n, \quad n \geq 1,
\] (1.6)

where $\{\beta_n\}$ is a real sequence in $[0,1]$.

In this paper, we establish the strong convergence for the implicit $S$-iterative process associated with Lipschitzian hemiconttractive mappings in Hilbert spaces.

**2. Main Results**

We need the following lemma.

**Lemma 2.1** (see [6]). For all $x, y \in H$ and $\lambda \in [0,1]$, the following well-known identity holds
\[
\|(1 - \lambda)x + \lambda y\|^2 = (1 - \lambda)\|x\|^2 + \lambda\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.
\] (2.1)
Now we prove our main results.

**Theorem 2.2.** Let $K$ be a compact convex subset of a real Hilbert space $H$ and let $T : K \to K$ be a Lipschitzian hemicontractive mapping satisfying

$$
\|x - Ty\| \leq \|Tx - Ty\|, \quad \forall x, y \in K.
$$

(C)

Let $\{\beta_n\}$ be a sequence in $[0, 1]$ satisfying

(iv) $\sum_{n=1}^{\infty} \beta_n = \infty,$

(v) $\sum_{n=1}^{\infty} \beta_n^2 < \infty.$

For arbitrary $x_0 \in K$, let $\{x_n\}$ be a sequence defined iteratively by

$$
x_n = Ty_n,
$$

$$
y_n = (1 - \beta_n)x_{n-1} + \beta_n Tx_n, \quad n \geq 1.
$$

(2.2)

Then the sequence $\{x_n\}$ converges strongly to the fixed point $x^*$ of $T$.

**Proof.** From Schauder’s fixed point theorem, $F(T)$ is nonempty since $K$ is a convex compact set and $T$ is continuous, let $x^* \in F(T)$. Using the fact that $T$ is hemicontractive we obtain

$$
\|Tx_n - x^*\|^2 \leq \|x_n - x^*\|^2 + \|x_n - Tx_n\|^2,
$$

(2.3)

$$
\|Ty_n - x^*\|^2 \leq \|y_n - x^*\|^2 + \|y_n - Ty_n\|^2.
$$

(2.4)

Now by (v), there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$
\beta_n \leq \min \left\{ \frac{1}{3}, \frac{1}{L^2} \right\},
$$

(2.5)

which implies that

$$
\frac{2\beta_n}{1 - \beta_n} \leq 1.
$$

(2.6)
With the help of (2.2), (2.3), and Lemma 2.1, we obtain the following estimates:

\[
\|y_n - x^*\|^2 = \|(1 - \beta_n)x_{n-1} + \beta_nTx_n - x^*\|^2 \\
= \|(1 - \beta_n)(x_{n-1} - x^*) + \beta_n(Tx_n - x^*)\|^2 \\
= (1 - \beta_n)\|x_{n-1} - x^*\|^2 + \beta_n\|Tx_n - x^*\|^2 \\
- \beta_n(1 - \beta_n)\|x_{n-1} - Tx_n\|^2 \\
\leq (1 - \beta_n)\|x_{n-1} - x^*\|^2 + \beta_n\left(\|x_n - x^*\|^2 + \|x_n - Tx_n\|^2\right) \\
- \beta_n(1 - \beta_n)\|x_{n-1} - Tx_n\|^2, \\
\tag{2.7}
\]

\[
\|y_n - Ty_n\|^2 = \|(1 - \beta_n)x_{n-1} + \beta_nTx_n - Ty_n\|^2 \\
= \|(1 - \beta_n)(x_{n-1} - Ty_n) + \beta_n(Tx_n - Ty_n)\|^2 \\
= (1 - \beta_n)\|x_{n-1} - Ty_n\|^2 + \beta_n\|Tx_n - Ty_n\|^2 \\
- \beta_n(1 - \beta_n)\|x_{n-1} - Tx_n\|^2.
\]

Substituting (2.7) in (2.4) we obtain

\[
\|Ty_n - x^*\|^2 \leq (1 - \beta_n)\|x_{n-1} - x^*\|^2 + \beta_n\left(\|x_n - x^*\|^2 + \|x_n - Tx_n\|^2\right) \\
+ (1 - \beta_n)\|x_{n-1} - Ty_n\|^2 + \beta_n\|Tx_n - Ty_n\|^2 \\
- 2\beta_n(1 - \beta_n)\|x_{n-1} - Tx_n\|^2. \\
\tag{2.8}
\]

Also with the help of condition (C) and (2.8), we have

\[
\|x_{n+1} - x^*\|^2 = \|Ty_n - x^*\|^2 \\
\leq (1 - \beta_n)\|x_{n-1} - x^*\|^2 + \beta_n\left(\|x_n - x^*\|^2 + \|x_n - Tx_n\|^2\right) \\
+ (1 - \beta_n)\|x_{n-1} - Ty_n\|^2 + \beta_n\|Tx_n - Ty_n\|^2 \\
- 2\beta_n(1 - \beta_n)\|x_{n-1} - Tx_n\|^2 \\
\leq (1 - \beta_n)\|x_{n-1} - x^*\|^2 + \beta_n\|x_n - x^*\|^2 + (1 - \beta_n)\|x_{n-1} - Ty_n\|^2 \\
+ 2\beta_n\|Tx_n - Ty_n\|^2 - 2\beta_n(1 - \beta_n)\|x_{n-1} - Tx_n\|^2.
\tag{2.9}
\]
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which implies that

\[ \|x_{n+1} - x^*\|^2 \leq \|x_{n-1} - x^*\|^2 + \|x_{n-1} - Ty_n\|^2 \]

\[ + \frac{2\beta_n}{1 - \beta_n} \|Tx_n - Ty_n\|^2 - 2\beta_n \|x_{n-1} - Tx_n\|^2 \]

\[ \leq \|x_{n-1} - x^*\|^2 + \|x_{n-1} - Ty_n\|^2 + \|Tx_n - Ty_n\|^2 \]

\[ - 2\beta_n \|x_{n-1} - Tx_n\|^2 , \]

where

\[ \|x_{n-1} - Ty_n\|^2 \leq \|Tx_{n-1} - Ty_n\|^2 \]

\[ \leq L^2 \|x_{n-1} - y_n\|^2 \]

\[ = L^2 \beta_n^2 \|x_{n-1} - Tx_n\|^2 , \]

\[ \|Tx_n - Ty_n\|^2 \leq L^2 \|x_n - y_n\|^2 \]

\[ \leq L^2 (\|x_n - x_{n-1}\| + \|x_{n-1} - y_n\|)^2 \]

\[ \leq L^2 (\|x_n - x_{n-1}\| + \beta_n \|x_{n-1} - Tx_n\|)^2 \]

\[ \leq L^2 (\|x_n - x_{n-1}\| + \beta_n M)^2 , \]

\[ \|x_n - x_{n-1}\| = \|x_{n-1} - Ty_n\| \]

\[ \leq \|Tx_{n-1} - Ty_n\| \]

\[ \leq L \|x_{n-1} - y_n\| \]

\[ = L \beta_n \|x_{n-1} - Tx_n\| \]

\[ \leq L \beta_n M \]

and consequently from (2.12), we obtain

\[ \|Tx_n - Ty_n\|^2 \leq L^2 (1 + L)^2 M^2 \beta_n^2 , \]

(2.13)
Hence by (2.5), (2.10), (2.11), and (2.13), we have
\[
\|x_n - x^*\|^2 \leq \|x_{n-1} - x^*\|^2 + L^2 \beta_n^2 \|x_{n-1} - Tx_n\|^2 \\
+ L^2 (1 + L)^2 M^2 \beta_n^2 - 2 \beta_n \|x_{n-1} - Tx_n\|^2 \\
= \|x_{n-1} - x^*\|^2 + L^2 (1 + L)^2 M^2 \beta_n^2 \\
- \beta_n \left(2 - L^2 \beta_n\right) \|x_{n-1} - Tx_n\|^2 \\
\leq \|x_{n-1} - x^*\|^2 + L^2 (1 + L)^2 M^2 \beta_n^2 - \beta_n \|x_{n-1} - Tx_n\|^2,
\] (2.14)
which implies that
\[
\beta_n \|x_{n-1} - Tx_n\|^2 \leq \|x_{n-1} - x^*\|^2 - \|x_{n-1} - x^*\|^2 + L^2 (1 + L)^2 M^2 \beta_n^2,
\] (2.15)
so that
\[
\frac{1}{2} \sum_{j=N}^{n} \beta_j \|x_j - Tx_j\|^2 \leq \|x_N - x^*\|^2 - \|x_n - x^*\|^2 + L^2 (1 + L)^2 M^2 \sum_{j=N}^{n} \beta_j^2.
\] (2.16)
Hence by conditions (iv) and (v), we get
\[
\sum_{j=0}^{\infty} \|x_{j-1} - Tx_j\|^2 < \infty.
\] (2.17)
It implies that
\[
\lim_{n \to \infty} \|x_{n-1} - Tx_n\| = 0.
\] (2.18)
Consider
\[
\|x_n - Tx_n\| \leq \|x_n - x_{n-1}\| + \|x_{n-1} - Tx_n\|,
\] (2.19)
which implies that
\[
\lim_{n \to \infty} \|x_n - Tx_n\| = 0.
\] (2.20)

The rest of the argument follows exactly as in the proof of Theorem of [3]. This completes the proof. \(\square\)

**Theorem 2.3.** Let \(K\) be a compact convex subset of a real Hilbert space \(H\) and let \(T : K \to K\) be a Lipschitzian hemicontractive mapping satisfying the condition (C). Let \(\{\beta_n\}\) be a sequence in \([0, 1]\) satisfying the conditions (iv) and (v).
Assume that $P_K : H \to K$ be the projection operator of $H$ onto $K$. Let $\{x_n\}$ be a sequence defined iteratively by

$$
x_n = P_K(Ty_n),
$$

$$
y_n = P_K((1 - \beta_n)x_{n-1} + \beta_nTx_n), \quad n \geq 1.
$$

Then the sequence $\{x_n\}$ converges strongly to a fixed point of $T$.

**Proof.** The operator $P_K$ is nonexpansive (see, e.g., [2]). $K$ is a Chebyshev subset of $H$ so that, $P_K$ is a single-valued mapping. Hence, we have the following estimate:

$$
\|x_n - x^*\|^2 = \|P_K(Ty_n) - P_Kx^*\|^2
\leq \|Ty_n - x^*\|^2
\leq \|x_{n-1} - x^*\|^2 + L^2(1 + L)^2M^2\beta_n^2 - \beta_n\|x_{n-1} - Tx_n\|^2.
$$

The set $K = K \cup T(K)$ is compact and so the sequence $\{\|x_n - Tx_n\|\}$ is bounded. The rest of the argument follows exactly as in the proof of Theorem 2.2. This completes the proof. 

**Remark 2.4.** In main results, the condition (C) is not new and it is due to Liu et al. [7].

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**References**


