Research Article

Differential Subordination Results for Certain Integrodifferential Operator and Its Applications

M. A. Kutbi$^1$ and A. A. Attiya$^{2,3}$

$^1$ Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
$^2$ Department of Mathematics, Faculty of Science, University of Mansoura, Mansoura 35516, Egypt
$^3$ Department of Mathematics, College of Science, University of Hail, Hail, Saudi Arabia

Correspondence should be addressed to A. A. Attiya, aattiy@mans.edu.eg

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We introduce an integrodifferential operator $J_{s,b}(f)$ which plays an important role in the Geometric Function Theory. Some theorems in differential subordination for $J_{s,b}(f)$ are used. Applications in Analytic Number Theory are also obtained which give new results for Hurwitz-Lerch Zeta function and Polylogarithmic function.

1. Introduction

Let $A$ denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disc $U = \{ z \in \mathbb{C} : |z| < 1 \}$.

Also, let $\mu$ denote the class of analytic functions in the form

$$r(z) = 1 + \sum_{k=1}^{\infty} a_k z^k.$$
We begin by recalling that a general Hurwitz-Lerch Zeta function $\Phi(z, s, b)$ defined by (cf., e.g., [1, P. 121 et seq.])

$$
\Phi(z, s, b) = \sum_{k=0}^{\infty} \frac{z^k}{(k + b)^s},
$$

(1.3)

($b \in \mathbb{C} \setminus \mathbb{Z}_{0}^{\circ}, \mathbb{Z}_{0}^{\circ} = \mathbb{Z}^{-} \cup \{0\} = \{0, -1, -2, \ldots\}, s \in \mathbb{C}$ when $z \in U$, $\text{Re}(s) > 1$ when $|z| = 1$)

which contains important functions of Analytic Number Theory, as the Polylogarithmic function:

$$
Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} = z\Phi(z, s, 1),
$$

(1.4)

($s \in \mathbb{C}$ when $z \in U$, $\text{Re}(s) > 1$ when $|z| = 1$).

Several properties of $\Phi(z, s, b)$ can be found in the recent papers, for example Choi et al. [2], Ferreira and López [3], Gupta et al. [4], and Luo and Srivastava [5]. See, also [6–16].

Recently, Srivastava and Attiya [8] introduced the operator $J_{s,b}(f)$ which makes a connection between Geometric Function Theory and Analytic Number Theory, defined by

$$
J_{s,b}(f)(z) = G_{s,b}(z) \ast f(z),
$$

(1.5)

($z \in U; f \in A; b \in \mathbb{C} \setminus \mathbb{Z}_{0}^{\circ}; s \in \mathbb{C}$),

where

$$
G_{s,b}(z) = (1 + b)^s[\Phi(z, s, b) - b^{-s}] 
$$

(1.6)

and $\ast$ denotes the Hadamard product (or convolution).

Furthermore, Srivastava and Attiya [8] showed that

$$
J_{s,b}(f)(z) = z + \sum_{k=2}^{\infty} \left(\frac{1 + b}{k + b}\right)^s a_k z^k.
$$

(1.7)
As special cases of $J_{s,b}(f)$, Srivastava and Attiya [8] introduced the following identities:

$$
J_{0,b}(f)(z) = f(z),
$$

$$
J_{1,0}(f)(z) = \int_0^z \frac{f(t)}{t} dt = A(f)(z),
$$

$$
J_{1,1}(f)(z) = \frac{2}{z} \int_0^z f(t) dt = \mathcal{L}(f)(z),
$$

$$
J_{1,\gamma}(f)(z) = \frac{1+\gamma}{z^\gamma} \int_0^z f(t) t^{1-\gamma} dt = \mathcal{L}_\gamma(f)(z) \quad (\gamma \text{ real; } \gamma > -1),
$$

$$
J_{0,1}(f)(z) = \frac{2\sigma}{z^\Gamma(\sigma)} \int_0^z \left(\log\left(\frac{z}{t}\right)\right)^{\sigma-1} f(t) dt = I^\sigma(f)(z) \quad (\sigma \text{ real; } \sigma > 0),
$$

(1.8)

where, the operators $A(f)$ and $\mathcal{L}(f)$ are the integral operators introduced earlier by Alexander [17] and Libera [18], respectively, $\mathcal{L}_\gamma(f)$ is the generalized Bernardi operator, $\mathcal{L}_\gamma(f)(\gamma \in \mathbb{N} = \{1, 2, \ldots\})$ introduced by Bernardi [19], and $I^\sigma(f)$ is the Jung-Kim-Srivastava integral operator introduced by Jung et al. [20].

Moreover, in [8], Srivastava and Attiya defined the operator $J_{s,b}(f)$ for $b \in \mathbb{C} \setminus \mathbb{Z}^-$, by using the following relationship:

$$
J_{s,0}(f)(z) = \lim_{b \to 0} J_{s,b}(f)(z).
$$

(1.9)

Some applications of the operator $J_{s,b}(f)$ to certain classes in Geometric Function Theory can be found in [21, 22].

In our investigations we need the following definitions and lemma.

**Definition 1.1.** Let $f(z)$ and $F(z)$ be analytic functions. The function $f(z)$ is said to be subordinate to $F(z)$, written $f(z) \prec F(z)$, if there exists a function $w(z)$ analytic in $U$, with $w(0) = 0$ and $|w(z)| \leq 1$, and such that $f(z) = F(w(z))$. If $F(z)$ is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

**Definition 1.2.** Let $\Psi : \mathbb{C}^2 \times U \to \mathbb{C}$ be analytic in domain $\mathbb{D}$, and let $h(z)$ be univalent in $U$. If $p(z)$ is analytic in $U$ with $(p(z), z p'(z)) \in \mathbb{D}$ when $z \in U$, then we say that $p(z)$ satisfies a first order differential subordination if

$$
\Psi(p(z), z p'(z); z) < h(z) \quad (z \in U).
$$

(1.10)

The univalent function $q(z)$ is called dominant of the differential subordination (1.10), if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.10), if $\tilde{q}(z) \prec q(z)$ for all dominant of (1.10), then we say that $\tilde{q}(z)$ is the best dominant of (1.10).

**Lemma 1.3** (see [8]). If $z \in U, f \in A, b \in \mathbb{C} \setminus \mathbb{Z}^-$ and $s \in \mathbb{C}$, then

$$
zJ_{s+1,b}^f(f)(z) = (1 + b)J_{s,b}(f)(z) - bJ_{s+1,b}(f)(z).
$$

(1.11)
The purpose of the present paper is to extend the use of $J_{s,b}(f)$ as integrodifferential operator, and some theorems in differential subordination for $J_{s,b}(f)$ are used. Applications in *Analytic Number Theory* are also obtained which give new results for Hurwitz-Lerch Zeta function and Polylogarithmic function.

### 2. Making Use of $J_{s,b}(f)$ as a Differential Operator

From the definition of $J_{s,b}(f)$ in (1.5) and using (1.7), we obtain the following identities.

For $z \in U, f \in A, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $b \in \mathbb{C} \setminus \mathbb{Z}^-$, we have

\[
J^{-1,0}(f)(z) = zf'(z),
J^{-1,1}(f)(z) = \frac{1}{2} \{f(z) + zf'(z)\},
J^{-1,1/(1-\lambda)}(f)(z) = \lambda f(z) + (1 - \lambda)zf''(z) \quad (\lambda \neq 1),
J^{-n,0}(f)(z) = D^n(f)(z),
J^{-n,(1/\lambda)-1}(f)(z) = D^n_{\lambda}(f)(z) \quad (\lambda \neq 0),
J^{-n,1}(f)(z) = I^n_{\lambda}(f)(z) \quad (\lambda > -1),
J^{-n,1}(f)(z) = I_n(f)(z),
\]

where $D^n(f)$ is the Sălăgean differential operator which introduced by Sălăgean [23], $D^n_{\lambda}(f)$ is the generalized of operator, $D^n_{1}(f) (\lambda > 0; \text{ real})$ introduced by Al-Oboudi [24], $I^n_{\lambda}(f)$ was studied by Cho and Srivastava [25] and by Cho and Kim [26], and the operator $I_n(f)$ was studied by Uralegaddi and Somanatha [27].

Also, we note that

\[
J^{-n,0}(f)(z) = Li_{-n}(z) * f(z) \quad (n \in \mathbb{N}_0; f \in A),
J^{-n,1}(f)(z) = \frac{Li_{-n}(z)}{z} * f(z) \quad (n \in \mathbb{N}_0; f \in A),
\]

where $Li_n(z)$ is the Polylogarithmic function defined by (1.4).

Now, we prove the following lemma.

**Lemma 2.1.** If $z \in U, f \in A, n \in \mathbb{N}_0$ and $b \in \mathbb{C} \setminus \mathbb{Z}^-$, then

\[
J^{-n,b}(f)(z) = \frac{1}{(1 + b)^n} (zd + b)^n f(z) \left( D := \frac{d}{dz} \right),
\]
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where \((zD + b)^n = (zD + b) \circ (zD + b) \circ \cdots \circ (zD + b)\) to \(n\)-times, and \(\circ\) denotes the composition \((I \circ f)(z) = I(J(f(z)))\).

**Proof.** Putting \(s = -n\) \((n \in \mathbb{N}_0)\) in (1.11), we have

\[
(1 + b)(J_{-n,b})(f)(z) = \left[ z \frac{d}{dz} J_{-n+1,b}(f)(z) + b J_{-n+1,b}(f)(z) \right] = (zD + b) J_{-n+1,b}(f)(z) \quad \left( D := \frac{d}{dz} \right),
\]

therefore,

\[
J_{-n,b}(f)(z) = \frac{1}{(1 + b)} (zD + b) J_{-n+1,b}(f)(z). \tag{2.5}
\]

Noting that the relation (2.5) is a recurrence relation, by using mathematical induction, we get (2.3), which completes the proof of the lemma.

Putting \(f(z) = f_0(z) = z/(1 - z)\) in Lemma 2.1, we obtain the following properties for both Hurwitz-Lerch Zeta function \(\Phi(z,s,b)\) and Polylogarithmic function \(\text{Li}_s(z)\).

**Corollary 2.2.** Let \(\Phi(z,s,b)\) and \(\text{Li}_s(z)\) be the Hurwitz-Lerch Zeta function and Polylogarithmic function defined by (1.3) and (1.4), respectively, then we have

\[
\Phi(z,-n,b) = b^n + \left( z \frac{d}{dz} + b \right)^n \left( \frac{z}{1 - z} \right) \quad (|z| < 1),
\]

\[
\text{Li}_{-n}(z) = z \left\{ 1 + \left( z \frac{d}{dz} + 1 \right)^n \left( \frac{z}{1 - z} \right) \right\} \quad (|z| < 1),
\]

where \(b \in \mathbb{C} \setminus \mathbb{Z}_0^-\) and \(n \in \mathbb{N}_0\).

**Example 2.3.** Using Corollary 2.2, we have the following well known results for \(z(z \in \mathbb{C}; |z| < 1)\).

(i) \(\Phi(z,0,b) = 1/(1 - z)\).

(ii) \(\Phi(z,-1,b) = b + ((1 + b)z - bz^2)/(1 - z)^2\).

(iii) \(\Phi(z,-2,b) = b^2 + ((1 + b)^2z + (1 - 2b - 2bz^2)z^2 + b^2z^3)/(1 - z)^3\).

(iv) \(\text{Li}_0(z) = z/(1 - z)\).

(v) \(\text{Li}_{-1}(z) = z/(1 - z)^2\).

(vi) \(\text{Li}_{-2}(z) = z(1 + z)/(1 - z)^3\).

3. **Applications of Differential Subordination for** \(J_{s,b}(f)\)

To prove our results, we need the following lemmas due to Hallenbeck and Ruscheweyh [28] and Miller and Mocanu [29], respectively, see also Miller and Mocanu [30].
Lemma 3.1. Let \( h(z) \) be convex univalent in \( U \), with \( h(0) = 1, \gamma \neq 0 \) and \( \text{Re}(\gamma) \geq 0 \). If \( q(z) \in \mu \) and
\[
q(z) - \frac{zq'(z)}{\gamma} < h(z),
\]
then
\[
q(z) < S(z) < h(z),
\]
where
\[
S(z) = \frac{\gamma}{2\gamma} \int_{0}^{z} h(t) t^{\gamma - 1} dt.
\]
The function \( S(z) \) is convex univalent and is the best dominant.

Lemma 3.2. Let \( \lambda > 0 \), and let \( \beta = \beta_{0}(\lambda) \) be the root of the equation as follows:
\[
\beta \pi = \frac{3\pi}{2} - \tan^{-1}(\lambda \beta).
\]
In addition, let \( \alpha = \alpha(\beta, \lambda) = \beta + (2/\pi)\tan^{-1}(\lambda \pi) \), for \( 0 < \beta \leq \beta_{0} \).
If \( p(z) \in \mu \) and
\[
p(z) + \lambda z p'(z) < \left[ \frac{1 + z}{1 - z} \right]^{\alpha},
\]
then
\[
p(z) < \left[ \frac{1 + z}{1 - z} \right]^{\beta}.
\]

Now, we define the function \( L(f)(z) := L_{(s,b,\lambda)}(f)(z) \) as the following:
\[
L(f)(z) = (1 - \lambda - \lambda b) J_{s,b}(f)(z) + \lambda (1 + b) J_{s-1,b}(f)(z) \quad (z \in U),
\]
(3.7)
(\( z \in U; f \in A; b \in \mathbb{C} \setminus \mathbb{Z}^{\times}; \{ s, \lambda \in \mathbb{C}; \lambda \neq 0; \text{Re}(\lambda) \geq 0 \} \)).

Theorem 3.3. Let the function \( L(f)(z) \) defined by (3.7) and for some \( \alpha(0 \leq \alpha < 1) \). If
\[
\text{Re}\left\{ \frac{L(f)(z)}{z} \right\} > \alpha,
\]
(3.8)
then
\[
\text{Re} \left\{ \frac{J_{s,b}(f)(z)}{z} \right\} > (2\alpha - 1) + 2(1 - \alpha) \, _2F_1 \left( 1, \frac{1}{\lambda}; \frac{1}{\lambda} + 1, -1 \right). \quad (3.9)
\]

The constant \((2\alpha - 1) + 2(1 - \alpha) \, _2F_1(1, 1/\lambda; (1/\lambda) + 1, -1)\) is the best estimate.

**Proof.** Defining the function \(q(z) = J_{s,b}(f)(z)/z\), then we have \(q(z) \in \mu\).

If we take \(\gamma = 1/\lambda\), and the convex univalent function \(h(z)\) defined by
\[
h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}, \quad 0 \leq \alpha < 1,
\]
then, we have
\[
q(z) + \frac{zq'(z)}{\gamma} = (1 - \lambda) \frac{J_{s,b}(f)(z)}{z} + \lambda J_{s,b}^\prime(f)(z). \quad (3.11)
\]

Using Lemma 1.3 and (3.7), therefore (3.11) can be written as
\[
q(z) + \frac{zq'(z)}{\gamma} = \frac{L(f)(z)}{z},
\]
then,
\[
q(z) + \frac{zq'(z)}{\gamma} < h(z),
\]
where \(h(z)\) is defined by (3.10) satisfying \(h(0) = 1\).

Applying Lemma 3.1, we obtain that \(J_{s,b}(f)(z)/z < S(z)\), where the convex univalent function \(S(z)\) defined by
\[
S(z) = \frac{1}{\lambda z^{1/\lambda}} \int_0^z \frac{1 + (2\alpha - 1)t}{1 + t} t^{(1/\lambda) - 1} dt. \quad (3.14)
\]

Since \(\text{Re}\{h(z)\} > 0\) and \(S(z) < h(z)\), we have \(\text{Re}\{S(z)\} > 0\).

This implies that
\[
\inf_{z \in U} \text{Re}\{S(z)\} = S(1) = (2\alpha - 1) + \frac{2}{\lambda}(1 - \alpha) \int_0^1 \frac{u^{(1/\lambda) - 1}}{1 + u} du
\]
\[
= (2\alpha - 1) + 2(1 - \alpha) \int_0^1 \frac{dt}{1 + t^\lambda}
\]
\[
= (2\alpha - 1) + 2(1 - \alpha) \, _2F_1 \left( 1, \frac{1}{\lambda}; \frac{1}{\lambda} + 1, -1 \right). \quad (3.15)
\]
Hence, the constant \((2\alpha - 1) + 2(1 - \alpha)\) cannot be replace by any larger one.

This completes the proof of Theorem 3.3.

**Theorem 3.4.** Let the function \(L(f)(z)\) with \(\lambda > 0\) real, defined by (3.7), and let \(\beta_0\) satisfy the following equation:

\[
\beta_0 \pi + \tan^{-1}\left(\frac{\beta_0}{2}\right) = \frac{3\pi}{2},
\]

(3.16)

If

\[
\frac{L(f)(z)}{z} < \left[\frac{1 + z}{1 - z}\right]^{\beta + (2/\pi)\tan^{-1}(\lambda\beta)},
\]

(3.17)

then

\[
\frac{J_{s,b}(f)(z)}{z} < \left[\frac{1 + z}{1 - z}\right]^\beta \quad (0 < \beta \leq \beta_0).
\]

(3.18)

**Proof.** Defining the function \(p(z) = J_{s,b}(f)(z)/z \in \mu\), then we have

\[
p(z) + \lambda z p'(z) = (1 - \lambda) \frac{J_{s,b}(f)(z)}{z} + \lambda J'_{s,b}(f)(z).
\]

(3.19)

Using Lemma 1.3 and (3.7), therefore (3.11) can be written as

\[
p(z) + \lambda z p'(z) = \frac{L(f)(z)}{z}.
\]

(3.20)

This completes the proof of Theorem 3.4 after applying Lemma 3.2.

### 4. Applications in Analytic Number Theory

Putting \(f(z) = f_0(z) = z/(1 - z)\) in Theorem 3.3, then we have the following property of Hurwitz-Lerch Zeta function.

**Corollary 4.1.** Let the function \(G_{s,b}(z)\) defined by (1.6). If

\[
\text{Re}\left\{\frac{(1 - \lambda - \lambda b)G_{s,b}(z) + \lambda (1 + b)G_{s-1,b}(z)}{z}\right\} > \alpha,
\]

(4.1)

then

\[
\text{Re}\left\{\frac{G_{s,b}(z)}{z}\right\} > (2\alpha - 1) + 2(1 - \alpha)\ _2\!F_1\left(1, \frac{1}{\lambda}; \frac{1}{\lambda} + 1, -1\right),
\]

(4.2)

where \(z \in \mathbb{U}, 0 < \alpha < 1, b \in \mathbb{C} \setminus \mathbb{Z}^- \) and \(s, \lambda \in \mathbb{C}; \lambda \neq 0; \text{Re} \lambda \geq 0\).
The constant $(2\alpha - 1) + 2(1 - \alpha) \ {}^2F_1(1, 1/\lambda; (1/\lambda) + 1, -1)$ is the best estimate.

Putting $f(z) = f_0(z) = z/(1 - \bar{z})$ in Theorem 3.4, then we have another property of Hurwitz-Lerch Zeta function.

**Corollary 4.2.** Let the function $G_{s,b}(z)$ defined by (1.6), and let $\beta_0$ satisfy the following equation:

$$\beta_0 \pi + \tan^{-1}(\beta_0) = \frac{3\pi}{2}. \quad (4.3)$$

If

$$\frac{(1 - \lambda - \lambda b)G_{s,b}(z) + \lambda(1 + b)G_{s-1,b}(z)}{z} < \frac{1 + z}{1 - z}^{\beta + (2/\pi)\tan^{-1}(\beta)}, \quad (4.4)$$

then

$$\frac{G_{s,b}(z)}{z} < \left[\frac{1 + z}{1 - z}\right]^\beta \quad (0 < \beta < \beta_0), \quad (4.5)$$

where $z \in \mathbb{U}$, $b \in \mathbb{C} \setminus \mathbb{Z}^-$, $s \in \mathbb{C}$ and $\lambda > 0$; real.

Putting $f(z) = f_0(z) = z/(1 - z)$ and $b = 1$ in Theorem 3.3, then we have the following property of Polylogarithmic function.

**Corollary 4.3.** Let the function $H_s(z)$ defined by

$$H_s(z) = 2^s \left[\frac{Li_s(z)}{z} - 1\right]. \quad (4.6)$$

If

$$\text{Re}\left\{\frac{(1 - 2\lambda)H_s(z) + 2\lambda H_{s-1}(z)}{z}\right\} > \alpha, \quad (4.7)$$

then

$$\text{Re}\left\{\frac{H_s(z)}{z}\right\} > (2\alpha - 1) + 2(1 - \alpha) \ {}^2F_1\left(1, \frac{1}{\lambda}; \frac{1}{\lambda} + 1, -1\right), \quad (4.8)$$

where $z \in \mathbb{U}$, $0 \leq \alpha < 1$ and \{s, $\lambda \in \mathbb{C}$; $\lambda \neq 0$; $\text{Re} \lambda \geq 0$\}.

The constant $(2\alpha - 1) + 2(1 - \alpha) \ {}^2F_1(1, 1/\lambda; (1/\lambda) + 1, -1)$ is the best estimate.

Putting $f(z) = f_0(z) = z/(1 - z)$ and $b = 1$ in Theorem 3.4, then we have the following property of Polylogarithmic function.
Corollary 4.4. Let the functions $G_{s,b}(z)$ and $H_s(z)$ defined by (1.6) and (4.6), respectively, and let $\beta_0$ satisfy the following:

$$\beta_0 \pi + \tan^{-1}(\lambda \beta_0) = \frac{3\pi}{2}. \quad (4.9)$$

If

$$\frac{(1 - 2\lambda)H_s(z) + 2\lambda H_{s-1}(z)}{z} < \left[ \frac{1 + z}{1 - z} \right]^{\frac{\beta + (2/\pi) \tan^{-1}(\lambda \beta)}{2}}, \quad (4.10)$$

then

$$\frac{G_{s,b}(z)}{z} < \left[ \frac{1 + z}{1 - z} \right]^\beta \quad (0 < \beta \leq \beta_0), \quad (4.11)$$

where $z \in \mathbb{U}$, $s \in \mathbb{C}$ and $\lambda > 0$; real.

Setting $f(z) = f_0(z) = z/(1 - z)$, $b = 1$ and $\lambda = 1/2$ in Theorem 3.3, then we have the following property of Polylogarithmic function.

Corollary 4.5. Let the function $H_s(z)$ defined by (4.6).

If

$$\operatorname{Re}\left\{ \frac{H_{s-1}(z)}{z} \right\} > \alpha, \quad (4.12)$$

then

$$\operatorname{Re}\left\{ \frac{H_s(z)}{z} \right\} > 2(2\ln 2 - 1)\alpha + (3 - 4\ln 2), \quad (4.13)$$

where $z \in \mathbb{U}$, $0 \leq \alpha < 1$ and $s \in \mathbb{C}$.

The constant $2(2\ln 2 - 1)\alpha + (3 - 4\ln 2)$ is the best estimate.

Taking $f(z) = f_0(z) = z/(1 - z)$, $b = 1$ and $\lambda = 1/2$ in Theorem 3.4, then we have the following property of polylogarithmic function.

Corollary 4.6. Let the function $H_s(z)$ defined by (4.6).

If

$$\frac{H_{s-1}(z)}{z} < \left[ \frac{1 + z}{1 - z} \right]^{\beta + (2/\pi) \tan^{-1}(\beta)}, \quad (4.14)$$

then

$$\frac{H_s(z)}{z} < \left[ \frac{1 + z}{1 - z} \right]^\beta \quad (0 < \beta \leq 1.3148754023\ldots), \quad (4.15)$$

where $z \in \mathbb{U}$ and $s \in \mathbb{C}$. 
Corollary 4.7. Let the function $H_s(z)$ defined by (4.6) as follows:

If

$$\frac{H_{s-1}(z)}{z} < \left[ \frac{1 + z}{1 - z} \right]^{3/2},$$

then

$$\text{Re}\left\{ \frac{H_s(z)}{z} \right\} > 1 - (4 \ln 2 - 2)^n \quad (n \in \mathbb{N}_0),$$

where $z \in U$ and $s \in \mathbb{C}$.

Proof. Let $H_{s-1}(z)$ satisfy the condition (4.16). Also, putting $f(z) = f_0(z) = z/(1 - z)$, $b = 1$, $\lambda = 1/2$ and $\beta = 1$ in Theorem 3.4.

Using (4.16), then we have

$$\frac{H_s(z)}{z} < \left[ \frac{1 + z}{1 - z} \right],$$

therefore

$$\text{Re}\left\{ \frac{H_s(z)}{z} \right\} > 0.$$

Corollary 4.5, gives

$$\text{Re}\left\{ \frac{H_{s+1}(z)}{z} \right\} > 3 - 4 \ln 2.$$  

Applied (4.11) again and to $n$-times, which gives (4.17). This completes the proof of Corollary 4.7. \qed

Finally, we can put Corollary 4.7 in the following form.

Corollary 4.8. Let the function $H_s(z)$ defined by (4.6).

If

$$\left| \text{Arg} \left( \frac{H_{s-1}(z)}{z} \right) \right| < \frac{3\pi}{4},$$

then

$$\text{Re}\left\{ \frac{H_{s+n}(z)}{z} \right\} > 1 - (4 \ln 2 - 2)^n \quad (n \in \mathbb{N}_0),$$

where $z \in U$ and $s \in \mathbb{C}$. 

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