Research Article

On the Convergence of Absolute Summability for Functions of Bounded Variation in Two Variables

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By adopting some new ideas, we obtain the estimates of an absolute convergence for the functions of the bounded variation in two variables. Our results generalize the related results of Humphreys and Bojanic (1999) and Wang and Yu (2003) from one dimension to two dimensions and can be applied to several summability methods.

1. Introduction

Let $f$ be $2\pi$-periodic functions integrable on $[-\pi, \pi]$. Denote its Fourier series by

$$S(f, x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

(1.1)

where the Fourier coefficients $a_k$ and $b_k$ are defined as follows:

$$a_k := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \, dt, \quad b_k := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt \, dt.$$  (1.2)

Denoted by $S_n(f, x)$ the $n$th partial sums of the Fourier series (1.1), that is,

$$S_n(f, x) := \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx).$$  (1.3)
When $f$ is of bounded variation, Bojanic [1] obtained the following result on the rate of the convergence of approximation by $S_n(f, x)$.

**Theorem A.** If a periodic function $f$ is of bounded variation on the interval $[-\pi, \pi]$, then the following estimate holds for every $x$ and $n = 1, 2, \ldots$:

$$|S_n(f, x) - f(x)| \leq \frac{3}{n} \sum_{k=1}^{n} V_{0}^{x/k} (\varphi_x),$$  \hspace{1cm} (1.4)

where $\varphi_x(t) := f(x+t) + f(x-t) - f(x+0) - f(x-0)$ and $V(\varphi_x, [0, t])$ is the total variation of $\varphi_x$ on $[0, t]$.

There are a lot of interesting generalizations that have been achieved by many authors (see [1–9]). Among them, Jenei [4] and Móricz [5] generalized Theorem A to the double Fourier series of functions of bounded variation; Humphreys and Bojanic [3], Wang and Yu [7] investigated the absolute convergence of Cesàro means of Fourier series.

Let $C_\alpha := (c_{nk}^\alpha)$ be the Cesàro matrix of order $\alpha$, that is,

$$c_{nk}^\alpha := \frac{A_{n-k}^{\alpha-1}}{A_n^\alpha}, \hspace{1cm} k = 0, 1, \ldots, n,$$  \hspace{1cm} (1.5)

where

$$A_n^\alpha := \frac{\Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(n + 1)}, \hspace{1cm} n = 0, 1, \ldots.$$  \hspace{1cm} (1.6)

Then, the so-called Cesàro means of order $\alpha$ of (1.1) is defined by (denote by $\sigma_n^\alpha(f, x)$):

$$\sigma_n^\alpha(f, x) = \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^{\alpha-1} S_k(f, x).$$  \hspace{1cm} (1.7)

Humphreys and Bojanic [3] investigated the rate of the absolute convergence of Cesàro means of the series (1.1). Their result can be read as follows.

**Theorem B.** Let $x \in [0, \pi]$ and $f$ be $2\pi$-periodic functions of bounded variation on $[-\pi, \pi]$. Then for $\alpha > 0$ and $n \geq 2$, one has

$$R_n^\alpha(f, x) \leq \frac{4\alpha}{n\pi} \sum_{k=1}^{\infty} V_{0}^{x/k} (\varphi_x),$$  \hspace{1cm} (1.8)

where

$$R_n^\alpha(f, x) := \sum_{k=n+1}^{\infty} |\sigma_k^\alpha(f, x) - \sigma_{k-1}^\alpha(f, x)|,$$  \hspace{1cm} (1.9)

$$\varphi_x(t) := f(x+t) + f(x-t) - f(x+0) - f(x-0)$$

and $\text{var}_n^\alpha(f)$ be the total variation of $f$ on $[a, b]$. 

However, Wang and Yu [7] showed that Theorem A is not correct when $0 < \alpha < 1$. In fact, they proved the following.

**Theorem C.** Suppose $0 < \alpha < 1$, $x \in [0, \pi]$ and $f$ are $2\pi$-periodic functions of bounded variation on $[-\pi, \pi]$. Then for $n \geq 2$, one has

$$R_n^\alpha(f, x) \leq \frac{100}{\alpha^2 n^a} \sum_{k=1}^{n} k^{\alpha-1} V^{\pi/k}_{0} (\varphi_x), \quad (1.10)$$

and there exists a $2\pi$-periodic function $f^*$ of bounded variation on $[-\pi, \pi]$ and a point $x \in [0, \pi]$ such that

$$R_n^\alpha(f^*, x) > \frac{1}{20000 \alpha n^a} \sum_{k=1}^{n} k^{\alpha-1} V^{\pi/k}_{0} (\varphi_x) \quad (n \geq 8). \quad (1.11)$$

Motivated by Theorems B and C, and the results of Jenei [4] and Móricz [5] on the double Fourier series, we will investigate the absolute convergence of a kind of very general summability of the double Fourier series. Our new results not only generalize Theorems B and C to the double case, but also can be applied to many other classical summability methods. We will present our main results on Section 2. Proofs will be given in Section 3. In Section 4, we will apply our results to some classical summability methods.

**2. The Main Results**

Let $f(x, y)$ be a function periodic in each variable with period $2\pi$ and integrable on the two-dimensional torus $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$ in Lebesgue’s sense, in symbol, $f \in L(\mathbb{T}^2)$. The double Fourier series of a complex-valued function $f \in L(\mathbb{T}^2)$ is defined by

$$f(x, y) \sim \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \hat{f}(k, l) e^{i(kx+ly)}, \quad (2.1)$$

where the $\hat{f}(k, l)$ are the Fourier coefficients of $f$:

$$\hat{f}(k, l) := \frac{1}{4\pi^2} \int_{\mathbb{T}^2} f(u, v) e^{-i(ku+lv)} du dv, \quad (k, l) \in \mathbb{Z}^2. \quad (2.2)$$

Let $R := [a_1, b_1] \times [a_2, b_2]$ be a bounded and closed rectangle on the plane. A function $f(x, y)$ defined on $R$ is said to be of bounded variation over $R$ in the sense of Hardy-Krause, in symbol, $f(x, y) \in BV_{HK}(R)$, if

(i) the total variation $V(f, R)$ of $f(x, y)$ over $R$ is finite, that is,

$$V(f, R) := \sup \sum_{j=1}^{m} \sum_{k=1}^{n} |f(x_j, y_k) - f(x_{j-1}, y_k) - f(x_j, y_{k-1}) + f(x_j, y_k)| < \infty, \quad (2.3)$$
where the supremum is extended for all finite partitions
\begin{equation}
\begin{aligned}
a_1 = x_0 < x_1 < \cdots < x_m = b_1, \quad a_2 = y_0 < y_1 < \cdots < y_n = b_2
\end{aligned}
\end{equation}
of the intervals \([a_1,b_1]\) and \([a_2,b_2]\);
(ii) the marginal functions \(f(\cdot, a_2)\) and \(f(a_1, \cdot)\) are of bounded variation over the intervals \([a_1,b_1]\) and \([a_2,b_2]\), respectively.

For any \(f(x,y) \in BV_H(\mathbb{T})\), define
\begin{equation}
\begin{aligned}
f(x,y) :=& \frac{1}{4} \left( f(x_0, y_0 + 0) + f(x - 0, y + 0) + f(x + 0, y_0 - 0) + f(x_0, y_0 - 0) \right), \\
\varphi_{xy}(u,v) :=& f(x + u, y + v) + f(x - u, y + v) + f(x + u, y - v) + f(x - u, y - v) - 4f(x,y).
\end{aligned}
\end{equation}

For convenience, write
\begin{equation}
\begin{aligned}
V_{00}^{\text{ef}}(\varphi_{xy}) := V(\varphi_{xy}, [0,s] \times [0,t]).
\end{aligned}
\end{equation}

For any double sequence \(\{a_{mn}\}\), define
\begin{equation}
\begin{aligned}
\Delta_{11} a_{mn} :=& a_{mn} - a_{m-1,n} - a_{m,n-1} + a_{m-1,n-1}, \\
\Delta_{10} a_{mn} :=& a_{mn} - a_{m-1,n}, \\
\Delta_{01} a_{mn} :=& a_{mn} - a_{m,n-1}.
\end{aligned}
\end{equation}

For any fourfold sequence \(\{a_{mnjk}\}\), write
\begin{equation}
\begin{aligned}
\Delta_{11} a_{mnjk} :=& a_{mnjk} - a_{m,n,j+1,k} - a_{m,n,j,k+1} + a_{m,n,j,k+1}, \\
\Delta_{01} a_{mnjk} :=& a_{mnjk} - a_{m,n,j,k+1}, \quad \Delta_{10} a_{mnjk} := a_{m,n,j,k} - a_{m,n,j,k+1}.
\end{aligned}
\end{equation}

A doubly infinite matrix \(T := (t_{mnjk})\) is said to be doubly triangular if \(t_{mnjk} = 0\) for \(j > m\) or \(k > n\). The \(mn\)th term of the \(T\)-transform of the double Fourier series (2.1) is defined by
\begin{equation}
\begin{aligned}
T_{mn}(x,y) := \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} f_{mn\mu\nu} S_{\mu\nu}(x,y),
\end{aligned}
\end{equation}
where \(S_{\mu\nu}(x,y)\) is the \(\mu\nu\)th partial sum of (2.1), that is,
\begin{equation}
\begin{aligned}
S_{\mu\nu}(x,y) := \sum_{|k|=0}^{\mu} \sum_{|l|=0}^{\nu} \hat{f}(k,l) e^{ikx+il} := \sum_{|k|=0}^{\mu} \sum_{|l|=0}^{\nu} A_{kl}(x,y).
\end{aligned}
\end{equation}
Let Theorem 2.2.

Then for

\[ f \in BVH(T^2) \]  

Assume that \( T \) is a lower doubly triangular matrix satisfying

\[ \tilde{t}^{(1,1)}_{mnk} = \tilde{t}^{(1,1)}_{mnli} = 0, \quad k, l = 0, 1, \ldots, \]  

\[ \sum_{k=1}^{m} \sum_{l=1}^{n} \tilde{t}^{(1,1)}_{mnkl} = O(1), \]  

and there exist constants \( \alpha, \beta \) \((0 < \alpha, \beta \leq 1)\) such that

\[ \left| \sum_{k=1}^{m} \sum_{l=1}^{n} \tilde{t}^{(1,1)}_{mnkl} \sin kx \sin ly \right| = \begin{cases} 
O\left( t^{1-\beta}xy^{-\beta} \right), & (x, y) \in \left[ 0, \frac{\pi}{m} \right] \times \left[ \frac{\pi}{n}, \pi \right], \\
O\left( k^{-1-\alpha}x^{-\alpha}y \right), & (x, y) \in \left[ \frac{\pi}{m}, \pi \right] \times \left[ 0, \frac{\pi}{n} \right], \\
O\left( k^{-1-\alpha}l^{-1-\beta}x^{-\alpha}y^{-\beta} \right), & (x, y) \in \left[ \frac{\pi}{m}, \pi \right] \times \left[ \frac{\pi}{n}, \pi \right].
\end{cases} \]  

Then for \( n \geq 2, \ (x, y) \in T^2 \), one has

\[ R_{mn}^{(1,1)} (f; x, y) \leq C \frac{1}{m^\alpha n^\beta} \sum_{k=1}^{m} \sum_{l=1}^{n} k^{\alpha-1}l^{\beta-1} V_{00}^0 \frac{\pi/k}{(\pi/l)} (\varphi_{xy}). \]  

**Theorem 2.2.** Let \( f(x, y) \) and \( T \) satisfy all the conditions of Theorem 2.1, except (2.15) is replaced by

\[ \sum_{k=1}^{m} \sum_{l=1}^{n} \left| \Delta_k \tilde{t}^{(1,1)}_{mnkl} \right| = O\left( m^{\tau(k^*)} n^{\tau(l^*)} \right), \]  

\( k^*, l^* \): obtained from (2.15).
where \( k^*, l^* = 0 \) or \( 1 \), \( k^* + l^* \geq 1 \), \( \tau(0) = \sigma(0) = 0 \), \( \tau(1) = \sigma(1) + 1 = 2 \). Then for \( n \geq 2 \), \((x, y) \in \mathbb{T}^2\), one has

\[
R_{mn}^{(1)}(f; x, y) \leq C \frac{1}{mn} \sum_{k=1}^m \sum_{l=1}^n V_{00}^{(\pi/k)(\pi/l)}(\varphi_{xy}).
\]  

(2.18)

3. Proofs of Results

Lemma 3.1 (see [10]). If \( g(x, y) \) is continuous on a rectangle \( R := [a_1, b_1] \times [a_2, b_2] \) and \( f(x, y) \in BV_H(R) \), then

(i) \( g(x, y) \) is integrable with respect to \( f(x, y) \) over \( R \) in the sense of Riemann-Stieltjes integral;

(ii) \( f(x, y) \) is integrable with respect to \( g(x, y) \) over \( R \) in the sense of Riemann-Stieltjes integral and

\[
\int_R f(x, y) \, d_x d_y g(x, y) = \int_R g(x, y) \, d_x d_y f(x, y)
\]

\[
+ \int_{a_1}^{b_1} f(x, b_2) \, d_x g(x, b_1) - \int_{a_1}^{b_1} f(x, a_2) \, d_x g(x, a_1) \\
+ \int_{a_2}^{b_2} f(b_1, y) \, d_y g(b_1, y) - \int_{a_2}^{b_2} f(a_1, y) \, d_y g(a_1, y) \\
- f(b_1, b_2) g(b_1, b_2) + f(a_1, b_2) g(a_1, b_2) \\
+ f(a_2, b_1) g(a_2, b_1) - f(a_1, a_2) g(a_1, a_2).
\]  

(3.1)

Lemma 3.2 (see [10]). If \( f(x, y) \in BV_H(R) \) and \( g(x, y) \) is absolutely continuous on \( R \), that is,

\[
g(x, y) = \int_{a_1}^x \int_{a_2}^y h(u, v) \, du \, dv + C, \quad (x, y) \in R,
\]  

(3.2)

for some function \( h(x, y) \in L(R) \) and constant \( C \), then

\[
\int_R f(x, y) \, d_x d_y g(x, y) = \int_R f(x, y) h(x, y) \, dx \, dy.
\]  

(3.3)

Proof of Theorem 2.1. By the definition of \( T_{mn}(x, y) \) (see (2.9)), we have

\[
T_{jk}(x, y) = \sum_{i_1=0}^j \sum_{i_2=0}^k t_{jk_{i_1 i_2}} \sum_{|\mu|=0}^{i_1} \sum_{|\nu|=0}^{i_2} A_{\mu\nu}(x, y)
\]

\[
= \sum_{|\mu|=0}^j \sum_{|\nu|=0}^k A_{\mu\nu}(x, y) \overline{t}_{jk_{|\mu| |\nu|}}.
\]  

(3.4)
Therefore,

\[
\Delta_{11}T_{jk}(x, y) = \sum_{\mu=0}^{j} \sum_{\nu=0}^{k} A_{\mu\nu}(x, y) \tilde{t}_{jk,\mu\nu}[t] - \sum_{\mu=0}^{j-1} \sum_{\nu=0}^{k-1} A_{\mu\nu}(x, y) \tilde{t}_{j-1,k,\mu\nu}[t] \\
- \sum_{\mu=0}^{j} \sum_{\nu=0}^{k-1} A_{\mu\nu}(x, y) \tilde{t}_{j,k-1,\mu\nu}[t] + \sum_{\mu=0}^{j-1} \sum_{\nu=0}^{k} A_{\mu\nu}(x, y) \tilde{t}_{j-1,k-1,\mu\nu}[t]
\]

\[
= \sum_{\mu=1}^{j} \sum_{\nu=1}^{k} (A_{\mu\nu}(x, y) + A_{-\mu,\nu}(x, y) + A_{\mu,-\nu}(x, y) + A_{-\mu,-\nu}(x, y)) \tilde{t}_{jk,\mu\nu}^{(1,1)} \quad \text{(by (2.11))}
\]

\[
(3.5)
\]

Noting that

\[
A_{\mu\nu}(x, y) + A_{-\mu,\nu}(x, y) + A_{\mu,-\nu}(x, y) + A_{-\mu,-\nu}(x, y)
\]

\[
= \hat{f}(\mu, \nu) e^{i(\mu s + \nu y)} + \hat{f}(-\mu, \nu) e^{i(-\mu s + \nu y)} + \hat{f}(\mu, -\nu) e^{i(\mu s - \nu y)} + \hat{f}(-\mu, -\nu) e^{i(-\mu s - \nu y)}
\]

\[
= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} f(s, t) \cos \mu(s - x) \cos \nu(t - y) ds \, dt,
\]

we have

\[
\Delta_{11}T_{jk}(x, y) = \frac{1}{4\pi^2} \sum_{\mu=1}^{j} \sum_{\nu=1}^{k} \int_{\mathbb{R}^2} f(s, t) \cos \mu(s - x) \cos \nu(t - y) ds \, dt
\]

\[
= \frac{1}{4\pi^2} \sum_{\mu=1}^{j} \sum_{\nu=1}^{k} \int_{0}^{\pi} q_{xy}(s, t) \cos \mu s \cos \nu t ds \, dt \quad \text{(3.7)}
\]

where in the last equality, Lemma 3.2 is applied.

By (3.7) and Lemma 3.1, we have

\[
R_{mn}^{(1,1)}(f; x, y) = \frac{1}{4\pi^2} \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} \left| \sum_{\mu=1}^{j} \sum_{\nu=1}^{k} \tilde{t}_{jk,\mu\nu}^{(1,1)} \int_{0}^{\pi} q_{xy}(s, t) ds \, dt \right| \sin \mu s \sin \nu t
\]

\[
= \frac{1}{4\pi^2} \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} \left| \sum_{\mu=1}^{j} \sum_{\nu=1}^{k} \tilde{t}_{jk,\mu\nu}^{(1,1)} \int_{0}^{\pi} \sin \mu s \sin \nu t ds \, dt \, d_t q_{xy}(s, t) \right|
\]
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$$\leq \frac{1}{4\pi^2} \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} \int_{0}^{\pi/j} \int_{0}^{\pi/k} \left| \sum_{\mu=1}^{i} \sum_{v=1}^{l} \frac{\tilde{f}_{jk\mu v}}{\mu v} \right| \sin \mu s \sin \nu t \left| d_s d_t V_{00}^{st}(\varphi_{xy}) \right|$$

$$= \frac{1}{4\pi^2} \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} \left( \int_{0}^{\pi/j} \int_{0}^{\pi/k} + \int_{\pi/k}^{\pi} \int_{\pi/k}^{\pi} + \int_{\pi/k}^{\pi} \int_{0}^{\pi/j} + \int_{\pi/j}^{\pi} \int_{\pi/j}^{\pi} \right)$$

$$\times \left| \sum_{\mu=1}^{i} \sum_{v=1}^{l} \frac{\tilde{f}_{jk\mu v}}{\mu v} \right| \sin \mu s \sin \nu t \left| d_s d_t V_{00}^{st}(\varphi_{xy}) \right|$$

$$: I_1 + I_2 + I_3 + I_4.$$  \hfill (3.8)

By (2.14), we have (Denote by $\chi((a_1,b_1) \times (a_2,b_2))(x,y)$ and $\chi(a_1,b_1)(x)$ the characteristic functions of $(a_1,b_1) \times (a_2,b_2)$ and $(a_1,b_1)$, respectively.)

$$I_1 = \frac{1}{4\pi^2} \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} \int_{0}^{\pi/j} \int_{0}^{\pi/k} \left| \sum_{\mu=1}^{i} \sum_{v=1}^{l} \frac{\tilde{f}_{jk\mu v}}{\mu v} \right| \sin \mu s \sin \nu t \left| d_s d_t V_{00}^{st}(\varphi_{xy}) \right|$$

$$\leq \frac{1}{4\pi^2} \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} \int_{0}^{\pi/j} \int_{0}^{\pi/k} \left| \sum_{\mu=1}^{i} \sum_{v=1}^{l} \tilde{f}_{jk\mu v} \right| \sin \mu s \sin \nu t \left| d_s d_t V_{00}^{st}(\varphi_{xy}) \right|$$

$$\leq C \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} \int_{0}^{\pi/j} \int_{0}^{\pi/k} \left| \sum_{\mu=1}^{i} \sum_{v=1}^{l} \tilde{f}_{jk\mu v} \right| \sin \mu s \sin \nu t \left| d_s d_t V_{00}^{st}(\varphi_{xy}) \right|$$

$$\leq \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} \int_{0}^{\pi/j} \int_{0}^{\pi/k} \left| \sum_{\mu=1}^{i} \sum_{v=1}^{l} \tilde{f}_{jk\mu v} \right| \sin \mu s \sin \nu t \left| d_s d_t V_{00}^{st}(\varphi_{xy}) \right|$$

$$\leq \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} \int_{0}^{\pi/j} \int_{0}^{\pi/k} \left| \sum_{\mu=1}^{i} \sum_{v=1}^{l} \tilde{f}_{jk\mu v} \right| \sin \mu s \sin \nu t \left| d_s d_t V_{00}^{st}(\varphi_{xy}) \right|$$

$$\leq \frac{C}{m^\alpha n^\beta} \sum_{j=1}^{m} \sum_{k=1}^{n} \sum_{\beta=1}^{\infty} \int_{0}^{\pi/j} \int_{0}^{\pi/k} \left| d_s d_t V_{00}^{st}(\varphi_{xy}) \right|.$$  \hfill (3.9)

For $I_2$, by the first inequality of (2.15),

$$I_2 = \frac{1}{4\pi^2} \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} \int_{0}^{\pi/j} \int_{0}^{\pi/k} \left| \sum_{\mu=1}^{i} \sum_{v=1}^{l} \frac{\tilde{f}_{jk\mu v}}{\mu v} \right| \sin \mu s \sin \nu t \left| d_s d_t V_{00}^{st}(\varphi_{xy}) \right|$$

$$\leq C \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} \int_{0}^{\pi/j} \int_{0}^{\pi/k} \left| \sum_{\mu=1}^{i} \sum_{v=1}^{l} \tilde{f}_{jk\mu v} \right| \sin \mu s \sin \nu t \left| d_s d_t V_{00}^{st}(\varphi_{xy}) \right|$$

$$\leq C \sum_{j=m+1}^{\infty} \sum_{k=n+1}^{\infty} \int_{0}^{\pi/j} \int_{0}^{\pi/k} \left| \sum_{\mu=1}^{i} \sum_{v=1}^{l} \tilde{f}_{jk\mu v} \right| \sin \mu s \sin \nu t \left| d_s d_t V_{00}^{st}(\varphi_{xy}) \right|$$

$$\leq \frac{C}{m^\alpha n^\beta} \sum_{j=1}^{m} \sum_{k=1}^{n} \sum_{\beta=1}^{\infty} \int_{0}^{\pi/j} \int_{0}^{\pi/k} \left| d_s d_t V_{00}^{st}(\varphi_{xy}) \right|.$$


\[ \int_{\pi/(n+1)}^{\pi/m} \sum_{j=m+1}^{\infty} \int_{\pi/j}^{\pi/k} \phi_{xy}(s) \sum_{k=m+1}^{\infty} k^{-1-\beta} t^{-\beta} d_s d_t V_{00}(\phi_{xy}) \]

\[ + C \int_{\pi/(n+1)}^{\pi/m} \sum_{j=m+1}^{\infty} \int_{\pi/j}^{\pi/k} \phi_{xy}(s) \sum_{k=m+1}^{\infty} k^{-1-\beta} t^{-\beta} d_s d_t V_{00}(\phi_{xy}) \]

\[ \leq CV_{00}^{(\pi/m+1)(\pi/(n+1))} (\phi_{xy}) + Cn^{-\beta} \sum_{k=1}^{n} \int_{\pi/(n+1)}^{\pi/k} t^{-\beta} d_s d_t V_{00}(\phi_{xy}) \]

\[ \leq CV_{00}^{(\pi/m+1)(\pi/(n+1))} (\phi_{xy}) + Cn^{-\beta} \sum_{k=1}^{n} k^{-1-\beta} V_{00}^{(\pi/m+1)(\pi/k)} (\phi_{xy}) \]

\[ \leq \frac{C}{m^{\alpha} n^{\beta}} \sum_{j=1}^{m} \sum_{k=1}^{n} j^{-1-\beta} V_{00}^{(\pi/j)(\pi/k)} (\phi_{xy}) . \] (3.10)

Similarly, we also have

\[ I_3 \leq \frac{C}{m^{\alpha} n^{\beta}} \sum_{j=1}^{m} \sum_{k=1}^{n} j^{-1-\beta} V_{00}^{(\pi/j)(\pi/k)} (\phi_{xy}) . \] (3.11)

By (2.15), we deduce that

\[ I_4 \leq C \sum_{j=m+1}^{\infty} \int_{\pi/j}^{\pi/k} j^{-1-\alpha} k^{-1-\beta} s^{-\alpha} t^{-\beta} d_s d_t V_{00}(\phi_{xy}) \]

\[ \leq C \sum_{j=m+1}^{\infty} \sum_{k=m+1}^{\infty} \left( \int_{\pi/j}^{\pi/(n+1)} \int_{\pi/k}^{\pi/(n+1)} \int_{\pi/(n+1)}^{\pi/k} + \int_{\pi/j}^{\pi/(n+1)} \int_{\pi/k}^{\pi/(n+1)} \int_{\pi/(n+1)}^{\pi/k} + \int_{\pi/j}^{\pi/(n+1)} \int_{\pi/k}^{\pi/(n+1)} \int_{\pi/(n+1)}^{\pi/k} \right) \]

\[ \times j^{-1-\alpha} k^{-1-\beta} s^{-\alpha} t^{-\beta} d_s d_t V_{00}(\phi_{xy}) \]

\[ \leq C \int_{\pi/(n+1)}^{\pi/(n+1)} \left( \int_{\pi/j}^{\pi/(n+1)} \int_{\pi/k}^{\pi/(n+1)} \right) \left( \int_{\pi/(n+1)}^{\pi/k} \int_{\pi/(n+1)}^{\pi/k} \int_{\pi/(n+1)}^{\pi/k} \right) d_s d_t V_{00}(\phi_{xy}) \]

\[ + C \int_{\pi/(n+1)}^{\pi} \left( \int_{\pi/j}^{\pi/(n+1)} \int_{\pi/k}^{\pi/(n+1)} \right) \left( \int_{\pi/(n+1)}^{\pi/k} \int_{\pi/(n+1)}^{\pi/k} \int_{\pi/(n+1)}^{\pi/k} \right) d_s d_t V_{00}(\phi_{xy}) \]
Proof of Theorem 2.2. 

By Abel's transformation, we have

\[ + C \int_{\pi/(m+1)}^{\pi} \int_{0}^{\pi/(n+1)} \left( \sum_{j=m+1}^{\infty} j^{-\alpha} S^{-\alpha} \right) \left( \sum_{k=(\pi/t)}^{\infty} k^{-1-\beta} t^{-\beta} \right) d_s d_t V_{00}^{m/n}(\varphi_{xy}) \]

\[ + C \int_{\pi/(m+1)}^{\pi} \int_{\pi/(n+1)}^{\pi} \left( \sum_{j=1}^{\infty} j^{-\alpha} S^{-\alpha} \right) \left( \sum_{k=n+1}^{\infty} k^{-1}\beta t^{-\beta} \right) d_s d_t V_{00}^{m/n}(\varphi_{xy}) \]

\[ \leq CV_{00}^{(\pi/(m+1))(\pi/(n+1))}(\varphi_{xy}) + C m^{-\beta} \sum_{k=1}^{n} k^{-1}_0 V_{00}^{(\pi/(m+1))(\pi/k)}(\varphi_{xy}) \]

\[ + C m^{-\beta} \sum_{j=1}^{m} \sum_{k=1}^{n} j^{-\alpha} V_{00}^{(\pi/j)(\pi/(n+1))}(\varphi_{xy}) \]

\[ + \frac{C}{m^\alpha n^\beta} \sum_{j=1}^{m} \sum_{k=1}^{n} j^{-\alpha} k^{-1}_0 V_{00}^{(\pi/j)(\pi/k)}(\varphi_{xy}) \]

\[ \leq C \frac{C}{m^\alpha n^\beta} \sum_{j=1}^{m} \sum_{k=1}^{n} j^{-\alpha} k^{-1}_0 V_{00}^{(\pi/(m+1))(\pi/(n+1))}(\varphi_{xy}). \]

(3.12)

We get (2.16) by combining (3.8)–(3.12).

Proof of Theorem 2.2. Set

\[ \tilde{D}_i(t) = \sum_{j=1}^{i} \sin jt = \frac{\cos(1/2)t - \cos(i + 1/2)t}{2\sin(1/2)t}, \quad i = 1, 2, \ldots. \]  

(3.13)

By Abel's transformation, we have

\[ \sum_{k=1}^{m} \sum_{l=1}^{n} \tilde{t}_{mnkl}^{(1,1)} \frac{\sin ks \sin lt}{kl} = \sum_{k=1}^{m} \frac{\sin ks}{k} \sum_{l=1}^{n} \frac{\tilde{t}_{mnkl}^{(1,1)}}{l} \tilde{D}_i(t) \]

\[ = \sum_{l=1}^{n} \frac{\sin lt}{l} \sum_{k=1}^{m} \frac{\tilde{t}_{mnkl}^{(1,1)}}{k} \tilde{D}_k(s) \]

(3.14)

\[ = \sum_{k=1}^{m} \sum_{l=1}^{n} \Delta_{kl}^{(1,1)} \tilde{D}_k(s) \tilde{D}_l(t). \]

Therefore, by the following well-known inequalities: \(|\tilde{D}_i(t)| = O(f^{-1}), |\sin t| \leq |t|, \) and condition (2.17), we see that (2.15) holds with \( \alpha = \beta = 1, \) and thus we get Theorem 2.2 from Theorem 2.1.
4. Applications

4.1. Cesàro’s Means

Let $T = (C, \gamma, \delta) = C_\gamma \times C_\delta$ be the double Cesàro matrix of order $(\gamma, \delta)$ with $\gamma, \delta > -1$, that is, $T$ has entries

$$t_{mnjk} = C_{mj}^\gamma \times C_{nk}^\delta = \frac{A_{m-j}^{\gamma-1} A_{n-k}^{\delta-1}}{A_m A_n^\delta}. \tag{4.1}$$

Then for $0 \leq j \leq m-1, 0 \leq k \leq n-1$,

$$\hat{t}^{(1,1)}_{mnjk} = \Delta_{11} t_{m-1,n-1,j,k}$$

$$= \left( \sum_{u=j}^{m-1} \frac{A_{m-1-u}^{\gamma-1}}{A_{m-1}^\gamma} - \sum_{u=j}^{m} A_{m-u}^{\gamma-1} \right) \left( \sum_{v=k}^{n-1} \frac{A_{n-1-v}^{\delta-1}}{A_{n-1}^\delta} - \sum_{v=k}^{n} A_{n-v}^{\delta-1} \right)$$

$$= \left( \frac{A_{m-j}^\gamma}{A_m^\gamma} - \frac{A_{m-j-1}^\gamma}{A_{m-1}^\gamma} \right) \left( \frac{A_{n-k}^\delta}{A_n^\delta} - \frac{A_{n-k-1}^\delta}{A_{n-1}^\delta} \right) \tag{4.2}$$

$$= \frac{j k A_{m-j}^{\gamma-1} A_{n-k}^{\delta-1}}{m n A_m^\gamma A_n^\delta},$$

$$\hat{t}^{(1,1)}_{mnnn} = t_{mnnn} = \frac{1}{A_m^\gamma A_n^\delta}.$$

In particular,

$$\hat{t}^{(1,1)}_{mnn0k} = \hat{t}^{(1,1)}_{mnj0} = 0, \quad j, k = 0, 1, \ldots \tag{4.3}$$

In other words, $T$ satisfies condition (2.13) of Theorem 2.1.

By the well-known inequality (e.g., see [11])

$$A_m^\gamma = \frac{n^\gamma}{\Gamma(\gamma + 1)} \left( 1 + O\left( \frac{1}{n} \right) \right), \tag{4.4}$$

we deduce that

$$\sum_{j=1}^{m} \sum_{k=1}^{n} |\hat{t}^{(1,1)}_{mnjk}| = \left( \sum_{j=1}^{m} \frac{j A_{m-j}^{\gamma-1}}{A_m^\gamma} \right) \left( \sum_{k=1}^{n} \frac{k A_{n-k}^{\delta-1}}{A_n^\delta} \right) + \frac{1}{A_m A_n^\delta}$$

$$= \left( O\left( m^2 \right) \sum_{j=1}^{m/2} j + O\left( m^\gamma \right) \sum_{j=m/2}^{m} A_{m-j}^{\gamma-1} + \frac{1}{A_m} \right)$$

$$= \left( O\left( m^2 \right) \sum_{j=1}^{m/2} j + O\left( m^\gamma \right) \sum_{j=m/2}^{m} A_{m-j}^{\gamma-1} + \frac{1}{A_m} \right)$$
\[\times\left(\mathcal{O}(n^{-2})\sum_{k=1}^{n/2} k + \mathcal{O}(n^{-\delta}) \sum_{k=n/2}^{n} A_{n-k}^{\delta-1} + \frac{1}{A_n^\delta}\right)\]
\[= \mathcal{O}(1), \quad \gamma, \delta > 0, \quad (4.5)\]

which means condition (2.14) of Theorem 2.1.
For \(0 < \gamma < 1\), we have

\[\sum_{k=1}^{n} \frac{A_{n-k}^{\gamma-1}}{nA_n^\gamma} \sin kx = \text{Im}\left(\frac{1}{nA_n^\gamma} \sum_{k=1}^{n} A_{n-k}^{\gamma-1} e^{ikx}\right) = \text{Im}\left(\frac{e^{inx}}{nA_n^\gamma} \sum_{k=1}^{n} A_{n-k}^{\gamma-1} e^{ikx}\right)\]
\[= \text{Im}(J_1 + J_2),\quad (4.6)\]

where

\[J_1 := \frac{e^{inx}}{nA_n^\gamma} (1 - e^{-ix})^{-\gamma}, \quad J_2 := \frac{e^{inx}}{nA_n^\gamma} \sum_{k=1}^{\infty} A_k^{\gamma-1} e^{-ikx}.\quad (4.7)\]

By (4.4), we observe that

\[|J_1| = \mathcal{O}\left(n^{-1-\gamma}x^{-\gamma}\right).\quad (4.8)\]

Since \(A_k^{\gamma-1}\) decreases monotonically to 0, \(J_2\) converges for \(0 < x \leq \pi\), and

\[|J_2| = \mathcal{O}\left(\frac{A_n^{\gamma-1}}{nA_n^\gamma} \left|1 - e^{-ix}\right|^{-1}\right) = \mathcal{O}(n^{-2}x^{-1}) = \mathcal{O}(n^{-1-\gamma}x^{-\gamma})\quad (4.9)\]

for \(t \in [(\pi/n), \pi]\).

By (4.4)–(4.9), we obtain that

\[\sum_{k=1}^{n} \frac{A_{n-k}^{\gamma-1}}{nA_n^\gamma} \sin kx = \mathcal{O}\left(n^{-1-\gamma}x^{-\gamma}\right), \quad x \in \left[\frac{\pi}{n}, \pi\right], \quad 0 < \gamma \leq 1.\quad (4.10)\]

On the other hand, we have

\[\sum_{k=1}^{n} \frac{A_{n-k}^{\gamma-1}}{nA_n^\gamma} \sin kx = \mathcal{O}(x) \sum_{k=1}^{n} \frac{kA_{n-k}^{\gamma-1}}{nA_n^\gamma} = \mathcal{O}(x), \quad x \in \left[0, \frac{\pi}{n}\right], \quad \gamma > 0.\quad (4.11)\]
For $\gamma \geq 1$, we have that

$$\sum_{k=1}^{n-1} \Delta \left( \frac{A_{n-k}^{r-1}}{nA_n^r} \right) = \frac{1}{nA_n^r} \sum_{k=1}^{n-1} (A_{n-k}^{r-1} - A_{n-k-1}^{r-1}) + \frac{1}{nA_n^r} = O\left(n^{-2}\right). \quad (4.12)$$

Now, by (4.10) and (4.11), we deduce that

$$\left| \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{i_{mnjk}}{jk} \sin jx \sin ky \right| \leq \left| \sum_{j=1}^{m} \frac{A_{m-j}^{r-1}}{mA_m^r} \sin jx \right| \left| \sum_{k=1}^{n} \frac{A_{n-k}^{r-1}}{nA_n^r} \sin ky \right|$$

$$= \begin{cases} O(k^{-1-\delta}xy^{-\delta}), & (x, y) \in \left[0, \frac{\pi}{m}\right] \times \left[\frac{\pi}{n}, \pi\right], \\ O(j^{-1-\gamma}xy^{-\delta}), & (x, y) \in \left[0, \frac{\pi}{n}\right] \times \left[\frac{\pi}{m}, \pi\right], \\ O(j^{-1-\gamma}k^{-1-\delta}x^{-1}y^{-\delta}), & (x, y) \in \left[\frac{\pi}{m}, \pi\right] \times \left[\frac{\pi}{n}, \pi\right] \end{cases} \quad (4.13)$$

for $0 < \gamma, \delta \leq 1$.

When $\gamma > 1$, $0 < \delta \leq 1$, by (4.10) and (4.11) again,

$$\left| \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{i_{mnjk}}{jk} \sin jx \sin ky \right| \leq \left| \sum_{j=1}^{m} \frac{A_{m-j}^{r-1}}{mA_m^r} \sin jx \right| \left| \sum_{k=1}^{n} \frac{A_{n-k}^{r-1}}{nA_n^r} \sin ky \right| = O\left(k^{-1-\delta}xy^{-\delta}\right), \quad (4.14)$$

for $(x, y) \in [0, (\pi/m)] \times [(\pi/n), \pi]$. By (4.10)–(4.12), we have

$$\left| \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{i_{mnjk}}{jk} \sin jx \sin ky \right| \leq \left| \sum_{j=1}^{m} \Delta \left( \frac{A_{m-j}^{r-1}}{mA_m^r} \right) D_j(x) \left| \sum_{k=1}^{n} \frac{A_{n-k}^{r-1}}{nA_n^r} \sin ky \right| \right.$$ 

$$= \begin{cases} O(j^{-2}x^{-1}y), & (x, y) \in \left[\frac{\pi}{m}, \pi\right] \times \left[0, \frac{\pi}{n}\right], \\ O(j^{-2}k^{-1-\delta}x^{-1}y^{-\delta}), & (x, y) \in \left[\frac{\pi}{m}, \pi\right] \times \left[\frac{\pi}{n}, \pi\right]. \end{cases} \quad (4.15)$$

Similarly, we have

$$\sum_{j=1}^{m} \sum_{k=1}^{n} \frac{i_{mnjk}}{jk} \sin jx \sin ky = \begin{cases} O(k^{-2}xy^{-1}), & (x, y) \in \left[0, \frac{\pi}{m}\right] \times \left[\frac{\pi}{n}, \pi\right], \\ O(j^{-1-\gamma}x^{-1}y), & (x, y) \in \left[0, \frac{\pi}{n}\right] \times \left[\frac{\pi}{m}, \pi\right], \\ O(j^{-1-\gamma}k^{-2}x^{-1}y^{-1}), & (x, y) \in \left[\frac{\pi}{m}, \pi\right] \times \left[\frac{\pi}{n}, \pi\right]. \end{cases} \quad (4.16)$$
for $\delta > 1, 0 < \gamma \leq 1$, and

$$
\left| \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{f(mnk)}{jk} \sin jx \sin ky \right| = \begin{cases} 
O(k^{-2}xy^{-1}), & (x, y) \in \left[0, \frac{\pi}{m}\right] \times \left[\frac{\pi}{n}, \pi\right], \\
O(j^{-2}x^{-1}y), & (x, y) \in \left[\frac{\pi}{m}, \pi\right] \times \left[0, \frac{\pi}{n}\right], \\
O(j^{-2}\gamma^{-1}x^{-1}y^{-1}), & (x, y) \in \left[\frac{\pi}{m}, \pi\right] \times \left[\frac{\pi}{n}, \pi\right], 
\end{cases} \quad (4.17)
$$

for $\gamma, \delta > 1$.

Setting

$$
\tilde{\gamma} := \begin{cases} 
\gamma, & 0 < \gamma \leq 1, \\
1, & \gamma > 1,
\end{cases}
\quad \tilde{\delta} := \begin{cases} 
\delta, & 0 < \delta \leq 1, \\
1, & \delta > 1,
\end{cases}
$$

then we have

$$
\left| \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{f(mnk)}{jk} \sin jx \sin ky \right| = \begin{cases} 
O(k^{-1-\tilde{\gamma}}xy^{-1}), & (x, y) \in \left[0, \frac{\pi}{m}\right] \times \left[\frac{\pi}{n}, \pi\right], \\
O(j^{-1-\tilde{\gamma}}x^{-1}y), & (x, y) \in \left[\frac{\pi}{m}, \pi\right] \times \left[0, \frac{\pi}{n}\right], \\
O(j^{-1-\tilde{\gamma}}k^{-1-\tilde{\delta}}x^{-1}y^{-1}), & (x, y) \in \left[\frac{\pi}{m}, \pi\right] \times \left[\frac{\pi}{n}, \pi\right], 
\end{cases} \quad (4.19)
$$

by (4.13)–(4.17).

From (4.19), we have the following corollary of Theorem 2.1, which generalizes Theorem B from one dimension to two dimensions.

**Theorem 4.1.** Let $f(x, y)$ be a periodic function and $f(x, y) \in BV_{T}(\mathbb{T}^2)$ and $T = (C, \gamma, \delta)$ be the double Cesàro matrix of order $(\gamma, \delta)$ with $\gamma, \delta > 0$. Then for $n \geq 2$, $(x, y) \in \mathbb{T}^2$, one has

$$
R_{mn}^{(1,1)}(f; x, y) \leq C \frac{1}{m^{\gamma}n^{\delta}} \sum_{k=1}^{m} \sum_{l=1}^{n} k^{\gamma-1}l^{\delta-1}V_{00}^{(\gamma/k)(\delta/l)}(q_{xy}). \quad (4.20)
$$

**4.2. Bernstein-Rogosinski’s Means**

The so-called Bernstein-Rogosinski means of Fourier series (1.1) are defined by

$$
B_n(f, x) := \frac{1}{2} \left( S_n(f, x + \frac{1}{2n+1}\pi) + S_n(f, x - \frac{1}{2n+1}\pi) \right). \quad (4.21)
$$
Now, we introduce the following Bernstein-Rogosinski means of double Fourier series (2.1):

\[
B_{mn}(f, x) := \frac{1}{4} \left( S_{mn} \left( f; x + \frac{\pi}{2m+1}, y + \frac{\pi}{2n+1} \right) + S_{mn} \left( f; x + \frac{\pi}{2m+1}, y - \frac{\pi}{2n+1} \right) + S_{mn} \left( f; x - \frac{\pi}{2m+1}, y + \frac{\pi}{2n+1} \right) + S_{mn} \left( f; x - \frac{\pi}{2m+1}, y - \frac{\pi}{2n+1} \right) \right).
\]

(4.22)

Direct calculations yield that

\[
B_{mn}(f, x) = \sum_{j=0}^{m} \sum_{k=0}^{n} \cos \frac{j\pi}{2m+1} \cos \frac{k\pi}{2n+1} \Delta_{11} S_{jk}(x, y)
\]

\[
= \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \left( \cos \frac{j\pi}{2m+1} - \cos \frac{j+1\pi}{2m+1} \right) \left( \cos \frac{k\pi}{2n+1} - \cos \frac{k+1\pi}{2n+1} \right) S_{jk}(x, y)
\]

\[
+ \sum_{j=0}^{m-1} \left( \cos \frac{j\pi}{2m+1} - \cos \frac{j+1\pi}{2m+1} \right) \cos \frac{n\pi}{2n+1} S_{jn}(x, y)
\]

\[
+ \sum_{k=0}^{n-1} \left( \cos \frac{k\pi}{2n+1} - \cos \frac{k+1\pi}{2n+1} \right) \cos \frac{m\pi}{2m+1} S_{mk}(x, y)
\]

\[
+ \cos \frac{m\pi}{2m+1} \cos \frac{n\pi}{2n+1} S_{mn}(x, y),
\]

(4.23)

Thus, \(B_{mn}(f, x)\) can be regarded as a \(T\)-transformation of Fourier series (1.1) with \(T\) of entries defined as

\[
t_{mnjk} = \left( \cos \frac{j\pi}{2m+1} - \cos \frac{j+1\pi}{2m+1} \right) \left( \cos \frac{k\pi}{2n+1} - \cos \frac{k+1\pi}{2n+1} \right), \quad 0 \leq j < m, \ 0 \leq k < n,
\]

\[
t_{mnjn} = \left( \cos \frac{j\pi}{2m+1} - \cos \frac{j+1\pi}{2m+1} \right) \cos \frac{n\pi}{2n+1}, \quad 0 \leq j < m,
\]

\[
t_{mmnk} = \left( \cos \frac{k\pi}{2n+1} - \cos \frac{k+1\pi}{2n+1} \right) \cos \frac{m\pi}{2m+1}, \quad 0 \leq k < n,
\]

\[
t_{mmnn} = \cos \frac{m\pi}{2m+1} \cos \frac{n\pi}{2n+1}.
\]

(4.24)

By direct calculations, we have

\[
\tilde{t}_{mnjk}^{(1,1)} = \left( \cos \frac{j\pi}{2m-1} - \cos \frac{j+1\pi}{2m+1} \right) \left( \cos \frac{k\pi}{2n-1} - \cos \frac{k+1\pi}{2n+1} \right), \quad 0 \leq j < m, \ 0 \leq k < n,
\]

(4.25)
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Theorem 2.2. By direct calculations, we deduce that

\[
\tilde{t}_{mnjk}^{(1,1)} = \left( \cos \frac{j}{2m-1} \pi - \cos \frac{j}{2m+1} \pi \right) \cos \frac{n\pi}{2n+1}, \quad 0 \leq j < m, \quad (4.26)
\]

\[
\tilde{t}_{mnmk}^{(1,1)} = \left( \cos \frac{k}{2n-1} \pi - \cos \frac{k}{2n+1} \pi \right) \cos \frac{m\pi}{2m+1}, \quad 0 \leq k < n, \quad (4.27)
\]

\[
\tilde{t}_{mnmn}^{(1,1)} = \cos \frac{m\pi}{2m+1} \cos \frac{n\pi}{2n+1}. \quad (4.28)
\]

Therefore, (4.26) and (4.27) imply (2.13), while from (4.25)–(4.28), we get

\[
\sum_{j=1}^{m} \sum_{k=1}^{n} |\tilde{t}_{mnjk}^{(1,1)}| \leq \sum_{j=1}^{m-1} \sum_{k=1}^{n-1} \left| \cos \frac{j}{2m-1} \pi - \cos \frac{j}{2m+1} \pi \right| \left| \cos \frac{k}{2n-1} \pi - \cos \frac{k}{2n+1} \pi \right| \\
+ \sum_{j=1}^{m-1} \left| \cos \frac{j}{2m-1} \pi - \cos \frac{j}{2m+1} \pi \right| \cos \frac{n\pi}{2n+1} \\
+ \sum_{k=1}^{n-1} \left| \cos \frac{k}{2n-1} \pi - \cos \frac{k}{2n+1} \pi \right| \cos \frac{m\pi}{2m+1} + \cos \frac{m\pi}{2m+1} \cos \frac{n\pi}{2n+1} \\
\leq 4 \sum_{j=1}^{m-1} \sum_{k=1}^{n-1} \sin \frac{2mj}{4m^2-1} \pi \sin \frac{j}{4m^2-1} \pi \sin \frac{2nk}{4n^2-1} \pi \sin \frac{k}{4n^2-1} \pi \\
+ 2 \sum_{j=1}^{m-1} \sin \frac{2mj}{4m^2-1} \pi \sin \frac{j}{4m^2-1} \pi + 2 \sum_{k=1}^{n-1} \sin \frac{2nk}{4n^2-1} \pi \sin \frac{k}{4n^2-1} \pi + 1 \\
\leq 4 \pi^2 \sum_{j=1}^{m-1} \sum_{k=1}^{n-1} \frac{2mj^2}{(4m^2-1)^2} \frac{2nk^2}{(4n^2-1)^2} + 2 \pi^2 \sum_{j=1}^{m-1} \frac{2mj^2}{(4m^2-1)^2} + 2 \pi^2 \sum_{k=1}^{n-1} \frac{2nk^2}{(4n^2-1)^2} + 1 \\
= O(1), \quad (4.29)
\]

which implies (2.14).

Finally, we verify that \( T \) satisfies condition (2.17); thus \( T \) satisfies all the conditions of Theorem 2.2. By direct calculations, we deduce that

\[
\sum_{j=1}^{m} \sum_{k=1}^{n} \left| \Delta_{01} \frac{\tilde{t}_{mnjk}^{(1,1)}}{k} \right| = \left( \sum_{j=1}^{m} \left| \cos \frac{j}{2m-1} \pi - \cos \frac{j}{2m+1} \pi \right| + \cos \frac{m\pi}{2m+1} \right) \\
\times \left( \sum_{k=1}^{n-1} \left| \cos \frac{(k/(2n-1))\pi}{k} - \cos \frac{(k/(2n+1))\pi}{k} \right| \right) \\
- \cos \frac{(k+1)/(2n-1)\pi}{k+1} - \cos \frac{(k+1)/(2n+1)\pi}{k+1} + \frac{1}{n} \cos \frac{n\pi}{2n+1}
\]
\[ \frac{n-1}{\sum_{k=1}^{n} \left| \cos \left( \frac{k}{2n-1} \pi - \cos \left( \frac{k}{2n+1} \pi \right) \right) \right| + n^2} = O \left( \sum_{k=1}^{n-1} \left| \cos \left( \frac{k}{2n-1} \pi - \cos \left( \frac{k}{2n+1} \pi \right) \right) \right| + n^2 \right) \]

\[ = O(1) \sum_{k=1}^{n-1} \left( \left| \cos \frac{k+1}{2n-1} \pi - \cos \frac{k+1}{2n+1} \pi \right| \right) \]

\[ = O(1) \sum_{k=1}^{n-1} \left( \left| \frac{1}{k} - \frac{1}{k+1} \right| \cos \frac{k+1}{2n-1} \pi - \cos \frac{k+1}{2n+1} \pi \right) \]

\[ = \sum_{k=1}^{n-1} \left| \left( \sin \frac{2nk}{4n^2-1} - \sin \frac{2n(k+1)}{4n^2-1} \right) \left( \sin \frac{k}{4n^2-1} \pi \right) + \left( \sin \frac{k}{4n^2} - \sin \frac{k}{4n^2} \pi \right) \left( \sin \frac{2n(k+1)}{4n^2-1} \pi \right) \right| \]

\[ + \sum_{k=1}^{n-1} \frac{k}{k(k+1)} \sin \frac{2n(k+1)}{4n^2} - \sin \frac{(k+1)}{4n^2} \pi \]

\[ = O(1) \sum_{k=1}^{n-1} \sin \frac{k}{4n^2} \pi - \sin \frac{(k+1)}{4n^2} \pi \]

\[ = O(1) \sum_{k=1}^{n-1} n^{-3} = O \left( n^{-2} \right). \]

(4.30)

Analogously,

\[ \sum_{j=1}^{m} \sum_{k=1}^{n} \left| \Delta_{10} \frac{\tilde{t}_{mnjk}^{(1,1)}}{j} \right| = O \left( m^{-2} \right). \]

(4.31)

\[ \sum_{j=1}^{m} \sum_{k=1}^{n} \left| \Delta_{11} \frac{\tilde{t}_{mnjk}^{(1,1)}}{j} \right| = O \left( m^{-2} n^{-2} \right). \]

Now, (4.30) and (4.31) imply (2.17).

### 4.3. Riesz’s Means

Let \( \{p_n\} \) and \( \{q_n\} \) be positive sequences such that \( P_n = p_0 + p_1 + \cdots + p_n \to \infty \), \( Q_n = q_0 + q_1 + \cdots + q_n \to \infty \), and let \( T := (t_{mnjk}) \) be a lower triangular matrix with entries \( t_{mnjk} = p_j q_k / P_m Q_n \), \( j = 0, 1, \ldots m \), \( k = 0, 1, \ldots n \); \( m, n = 1, 2, \ldots \). Then the \( T \)-transformation of Fourier series (2.1) is known as Riesz’s mean.
Proposition 4.2. Let \( \{p_n\} \) and \( \{q_n\} \) be positive nondecreasing sequences such that (one says \( A_n \sim B_n \), if there are two positive constants \( C_1 \) and \( C_2 \) such that \( C_1 \leq (A_n / B_n) \leq C_2 \).

\[
np_n \sim P_n, \quad nq_n \sim Q_n,
\]

\[
\sum_{k=1}^n \frac{p_k}{k} = O(p_n), \quad \sum_{k=1}^n \frac{q_k}{k} = O(q_n).
\]

Then \( T \) satisfies all the conditions of Theorem 2.2.

Proof. Direct calculations yield that (set \( P_{-1} = Q_{-1} = 0)\)

\[
\tilde{t}_{mnj}^{(1,1)} = \frac{\sum_{i=j}^{m} p_i}{P_m} - \frac{\sum_{i=j}^{m-1} p_i}{P_{m-1}} \left( \sum_{i=k}^{n} q_i - \sum_{i=k}^{n-1} q_i \right) = \frac{p_m q_n P_{j-1} Q_{k-1}}{P_m P_{m-1} Q_n Q_{n-1}}, \quad 0 \leq j < m, \ 0 \leq k < n,
\]

(4.34)

\[
\tilde{t}_{mmnk}^{(1,1)} = \frac{\sum_{i=k}^{m} q_i}{Q_m} - \frac{\sum_{i=k}^{m-1} q_i}{Q_{m-1}} = \frac{p_m q_n Q_{k-1}}{P_m Q_n Q_{n-1}}, \quad 0 \leq k < n,
\]

(4.35)

\[
\tilde{t}_{mnjn}^{(1,1)} = \left( \sum_{i=j}^{m} p_i - \sum_{i=j}^{m-1} p_i \right) \frac{q_n}{Q_n} = \frac{p_m q_n P_{j-1}}{P_m P_{m-1} Q_n}, \quad 0 \leq j < m,
\]

(4.36)

It follows from (4.35) that \( T \) satisfies condition (2.13).

Since \( \{p_n\} \) and \( \{q_n\} \) are nondecreasing, by (4.32), (4.34)–(4.36), we have

\[
\sum_{j=1}^{m} \sum_{k=1}^{n} \tilde{t}_{mnj}^{(1,1)} = \frac{p_m q_n}{P_m P_{m-1} Q_n Q_{n-1}} \sum_{j=1}^{m} \sum_{k=1}^{n} P_{j-1} Q_{k-1} = O\left( \frac{mnp_m q_n}{P_m Q_n} \right) = O(1).
\]

(4.37)

Hence, \( T \) satisfies condition (2.14).

Since \( \{p_n\} \) and \( \{q_n\} \) are nondecreasing, by (4.32)–(4.36), we have

\[
\sum_{j=1}^{m} \sum_{k=1}^{n} \Delta_0 \left( \frac{\tilde{t}_{mnj}^{(1,1)}}{k} \right) = \left( \frac{p_m}{P_m} \sum_{j=1}^{m-1} P_{j-1} + \frac{p_m}{P_m} \right) \times \left( \frac{q_n}{Q_n Q_{n-1}} \sum_{k=1}^{n} \frac{Q_{k-1}}{k} - \frac{Q_k}{k+1} + \frac{q_n}{nP_n} \right)
\]
\begin{align*}
&= O(1) \left( \frac{q_n}{Q_n} \left( \frac{n-1}{k} \sum_{k=1}^{n-1} \frac{Q_{k-1}}{(k+1)} + \sum_{k=2}^{n-1} \frac{q_k}{k+1} \right) + O(n^{-2}) \right) \\
&= O(1) \left( \frac{1}{nQ_{n-1}} \sum_{k=1}^{n-1} \frac{q_k}{k} + n^{-2} \right) \\
&= O\left( \frac{q_{n-1}}{nQ_{n-1}} \right) \\
&= O\left( n^{-2} \right). 
\end{align*}

(4.38)

Similarly,

\begin{align*}
\sum_{j=1}^{m} \sum_{k=1}^{n} \left| \Delta_{10} \hat{t}_{j}^{(1,1)}_{mnjk} \right| &= O\left( m^{-2} \right), \\
\sum_{j=1}^{m} \sum_{k=1}^{n} \left| \Delta_{11} \hat{t}_{j}^{(1,1)}_{mnjk} \right| &= O\left( m^{-2} n^{-2} \right). 
\end{align*}

(4.39)

Now, by (4.38) and (4.39), we show that \( T \) also satisfies condition (2.17).

\[ \square \]

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**References**


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