Research Article

The Backward Euler Fully Discrete Finite Volume Method for the Problem of Purely Longitudinal Motion of a Homogeneous Bar

Ziwen Jiang and Deren Xie

School of Mathematical Sciences, Shandong Normal University, Jinan, Shandong 250014, China

Correspondence should be addressed to Ziwen Jiang, ziwenjiang@163.com

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We present a linear backward Euler fully discrete finite volume method for the initial-boundary-value problem of purely longitudinal motion of a homogeneous bar and give optimal order error estimates in $L^2$ and $H^1$ norms. Furthermore, we obtain the superconvergence error estimate of the generalized projection of the solution $u$ in $H^1$ norm. Numerical experiment illustrates the convergence and stability of this scheme.

1. Introduction

We consider the following mixed boundary-initial value problem:

\begin{align*}
(a) \quad u_t &= u_{xxt} + f(u_x)_x, \quad (x, t) \in (0, 1) \times [0, T], \\
(b) \quad u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in (0, 1), \\
(c) \quad u(0, t) = u(1, t) = 0, \quad t \in [0, T].
\end{align*}

This problem (1.1) arises when one considers the purely longitudinal motion of a homogeneous bar [1]. The displacement of the cross-section of the bar at time $t$ is denoted by $u(x, t)$. When both ends of the bar are fixed, $u(0, t) = u(1, t) = 0, t \in [0, T]$, that is the boundary condition. $u(x, 0) = u_0(x)$ and $u_t(x, 0) = u_1(x)$ are the initial data.

In theoretical analysis, the problem (1.1) was first treated by Greenberg et al. [2], by assuming that the function $f$ was monotonic, that is,

\begin{equation}
 f'(s) > 0, \quad \forall s \in (-\infty, +\infty),
\end{equation}

1.1
and that the initial data was smooth, specifically
\[ u_0 \in C^4([0, 1]), \quad u_1 \in C^2([0, 1]). \] (1.3)

Under these assumptions they showed the existence of a unique smooth solution which decays the zero solution as \( t \to \infty \). (See also Greenberg [3] and Greenberg and MacCamy [4].) Andrews [1] made the hypotheses that \( u_0 \in W_0^{1, \infty}(0, 1), \ u_1 \in H_0^1(0, 1), \) the function \( f \) is locally Lipschitz continuous, and there exists a constant \( a_0 > 0 \) such that \((f(s_1) - f(s_2))(s_1 - s_2) > 0\) whenever \(|s_1 - s_2| \geq a_0\); then he proved the existence of a unique global weak solution. Under the hypotheses that \( u_0, u_1 \in H^2(0, 1) \cap H_0^1(0, 1), \) and there exists a constant \( a_1 > 0 \) such that \( f'(s) \geq a_1, \ s \in (-\infty, +\infty), \) Y. Liu and D. Liu [5] proved the existence of a unique global strong solution of the problem (1.1).

Numerical simulation methods for the problem (1.1) are recently studied by several authors ([6–11]). In [6], Gao et al. studied a finite difference method of the problem (1.1) in the domain \([0, 1] \times [0, T] \) \( (T > 0) \) and proved the convergence of the method by using discrete functional analysis and prior estimate. In [7, 8], Jiang et al. studied two finite element methods of (1.1) and obtained the optimal error estimates of this finite element scheme in \( L^2 \) and \( H^1 \) norms. In [9], Z. Jiang and Y. Jiang, and in [10], Jiang and Li, studied a mixed finite element method and a expanded mixed finite element method, respectively, and obtained the optimal error estimates of these schemes. However, few work on finite volume methods of (1.1) was found (see [11]). As we know, finite volume methods (also called generalized difference methods) were proposed in eighties last century and developed very quickly. Now this kind of method becomes one of the main numerical methods for solving differential equations, for example, convection diffusion equations [12–14] and Navier-Stokes equations [15].

In this paper we want to make further study of finite volume methods for the problem (1.1). First in Section 2, we derive a finite volume weak form of (1.1) in the case that \( f(s) \) is nonlinear, then propose a linear backward Euler fully discrete finite volume scheme of (1.1). Existence and uniqueness of the solution of this scheme are proved. Next, in Section 3, we give optimal error estimates in \( L^2 \) and \( H^1 \) norms and superconvergence in \( H^1 \) norm by using new defined projections. Numerical experiments and computational results are presented in Section 4, which confirm our theoretical analysis.

2. The Linear Backward Euler Fully Discrete Finite Volume Scheme

In this section, we construct the finite volume method of the problem (1.1) and prove the existence and uniqueness of the solution of this finite volume scheme.

Firstly, let \( T_h \) be a partition for the interval \( I = [0, 1] \), with its nodes \( 0 = x_0 < x_1 < \cdots < x_r = 1 \). The length of the element \( I_i = [x_{i-1}, x_i] \) is denoted by \( h_i = x_i - x_{i-1}, i = 1, 2, \ldots, r \), \( h = \max_{0 \leq i \leq r} h_i \) is maximum of \( h_i \). We suppose \( T_h \) is regular, that is, there exists a positive constant \( \mu > 0 \) such that \( h_i \geq \mu h, i = 1, 2, \ldots, r \). For the definition of the finite volume scheme, the dual partition \( T_h^* \) of \( T_h \) is needed, which is \( 0 = x_0 < x_{1/2} < x_{3/2} < \cdots < x_{r-1/2} < x_r = 1 \). The dual elements are denoted by \( I_0^* = [x_0, x_{1/2}], I_j^* = [x_{j-1/2}, x_{j+1/2}], j = 1, 2, \ldots, r - 1, I_r^* = [x_{r-1/2}, x_r], \) where \( x_{j-1/2} = (1/2)(x_j + x_{j-1}), \ j = 1, 2, \ldots, r \).

Secondly, we define the piecewise linear trial function space \( U_h \) over the partition \( T_h \), satisfying \( U_h \subset H_0^1(I) \), where \( H_0^1(I) \) is the Sobolev space on \( I \). Then \( u_h(x) = \sum_{i=1}^{r} u_h(x_i) \varphi_i(x) \) where \( \varphi_i(x) \) are the linear basis functions defined on \( I_i \). After we get the numerical solution \( u_h \), we get the error \( u_h - u \).
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for all \( u_h(x) \in \mathcal{U}_h \), where \( \varphi_i(x) \) is the basis function associated with the nodes \( x_i \) (\( i = 1, 2, \ldots, r-1 \)),

\[
\varphi_i(x) = \begin{cases} 
1 - \frac{x - x_i}{h_i}, & x \in I_i, \\
1 - \frac{x_i - x}{h_{i+1}}, & x \in I_{i+1}, \\
0, & x \notin I_i \cup I_{i+1}.
\end{cases} \tag{2.1}
\]

It is easy to know that the derivative of \( u_h \) with respect to \( x \) is

\[
u_{hx}(x) = \frac{u_h(x) - u_h(x_{i-1})}{h_i}, \quad x_{i-1} \leq x \leq x_i, \quad i = 1, 2, \ldots, r. \tag{2.2}\]

The test function space \( V_h \subset L^2(I) \) associated with the dual partition \( T^*_h \) is defined as the set of all piecewise constants with \( v_h(0) = v_h(1) = 0 \) for all \( v_h(x) \in V_h \). We may choose the basis function \( \varphi_j(x) \) of \( V_h \) in such a way that \( \varphi_j(x) \) is the characteristic function of \( I^*_j \), that is,

\[
\varphi_j(x) = \begin{cases} 
1, & x \in I^*_j, \\
0, & x \notin I^*_j,
\end{cases} \quad j = 1, 2, \ldots, r-1. \tag{2.3}
\]

Then for any \( v_h(x) \in V_h \) can be expressed as \( v_h(x) = \sum_{j=1}^{r-1} v_h(x_j) \varphi_j(x) \).

Obviously,

\[
\mathcal{U}_h \subset H^1_0(I), \quad V_h \subset L^2(I),
\]

\[
dim \mathcal{U}_h = \dim V_h = r - 1. \tag{2.4}
\]

Meanwhile, \( \mathcal{U}_h \subset W^{1,\infty}(I) \).

Thirdly, for the time interval \([0, T]\), we give an isometric partition and denote the nodes \( t_i = i\tau, i = 0, 1, \ldots, N, \tau = T/N \).

We introduce some notations for functions \( u(x, t) \) and \( f(u_x(x, t)) \):

\[
u^n = u(x, t_n), \quad u_j = u(x_j, t), \quad u^n_j = u(x_j, t_n), \quad \partial_t u^n = \frac{u^{n+1} - u^n}{\tau},
\]

\[
\partial^n_{t} u^n = \frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2}, \quad u^{n+1/2} = \frac{u^{n+1} + u^n}{2},
\]

\[
u^{n+1/4} = \frac{u^{n+1} + 2u^n + u^{n-1}}{4}, \quad f^{1/2}(u^n_x) = \frac{f(u^{n+1}_x) + f(u^{n-1}_x)}{2},
\]

\[
f^{1/4}(u^n_x) = \frac{f(u^{n+1}_x) + 2f(u^n_x) + f(u^{n-1}_x)}{4}, \quad f^{(1/4)}(u^n_x) = \frac{3f(u^n_x) + f(u^{n-1}_x)}{4}. \tag{2.5}
\]
Then we can get
\[
\frac{\partial^2 u^n}{\partial t^2} = \frac{\partial u^n - \partial t u^{n-1}}{\tau}, \quad u^{n1/4} = \frac{u^{n+1/2} + u^{n-1/2}}{2},
\]
\[
\frac{1}{2} \left( \partial_t u^n + \partial_t u^{n-1} \right) = \frac{1}{2} (u^{n+1/2} - u^{n-1/2}),
\] (2.6)

Let \( u \) be the solution of (1.1). Integrating (1.1) a over the dual element \( I_j^* \in T_h^* \), we obtain
\[
\int_{x_{j-1/2}}^{x_{j+1/2}} u_t dx + u_x (x_{j-1/2}) - u_x (x_{j+1/2}) = f(x_u (x_{j-1/2})) - f(x_u (x_{j+1/2})) = 0,
\] (2.7)
where \( j = 0,1,\ldots,r \), \( x_{-1/2} = x_0 \), \( x_{r+1/2} = x_r \), and \( 0 < t \leq T \). The problem (2.7) can be rewritten in a variational form. For any arbitrary \( \psi_h \in V_h \), we multiply the integral relation in (2.7) by \( \psi_h (x_j) \) and sum over all \( j = 0,1,\ldots,r \) to obtain
\[
(u_t, \psi_h) + a^*(u_t, \psi_h) + b^*(f(u), \psi_h) = 0, \quad \forall \psi_h \in V_h, \quad t \in (0,T],
\]
\[
0 = u_0, \quad u_t(0) = u_1,
\] (2.8)
where for any arbitrary \( \omega = \sum_{j=1}^{r-1} w_j \varphi_j \in V_h \), the bilinear forms \( a^*(\varphi, \omega) \) and \( b^*(f(\varphi), \omega) \) are defined by
\[
a^*(\varphi, \omega) = \sum_{j=1}^{r-1} w_j a^*(\varphi, \varphi_j), \quad a^*(\varphi, \varphi_j) = v_x (x_{j-1/2}) - v_x (x_{j+1/2}),
\]
\[
b^*(f(\varphi), \omega) = \sum_{j=1}^{r-1} w_j b^*(f(\varphi), \varphi_j), \quad b^*(f(\varphi), \varphi_j) = f(v_x (x_{j-1/2})) - f(v_x (x_{j+1/2})).
\] (2.9)

Since \( f \) is a nonlinear function, we will consider the following linear finite volume scheme: find \( u^n_h \in U_h, \ n = 0,1,2,\ldots,N \) such that
\[
\left( \frac{\partial^2 u^n_h}{\partial t^2}, \psi_h \right) + a^*(\partial_t u^n_h, \psi_h) + b^*(f^{(1/4)}(u^n_h), \psi_h) = 0, \quad \forall \psi_h \in V_h, \ n = 1,\ldots,N-1,
\]
\[
u^n_0 = \Pi_h u_0, \quad u^n_1 = \Pi_h u_0 + \tau \Pi_h u_1,
\] (2.10)
where \( \Pi_h u_0 \) and \( \Pi_h u_1 \) are the interpolation projection of \( u_0 \) and \( u_1 \) onto the trial function space \( U_h \), respectively, and the interpolation operator \( \Pi_h \) is defined as \( \Pi_h : H^1_0(I) \rightarrow U_h \)
\[
\Pi_h \omega = \sum_{i=1}^{r-1} w_i \varphi_i, \quad \forall \omega \in H^1_0(I).
\] (2.11)
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In this paper we adopt the standard notation $W^{m,p}$ with the norm $L^q$ by the index $m$. Hence the existence and uniqueness of the solution $u$ are equivalent to the existence and uniqueness of $u_{h_j}$, choosing $\tau h_j$. After all these denotations, we give the existence and uniqueness of the solution of the finite volume scheme 

$$\Pi_h^m w = \sum_{i=1}^{r-1} w_i \phi_i, \quad \forall w \in H^1_0(I).$$  \hspace*{1cm} (2.12)

By Sobolev’s interpolation theory, we know that

$$|w - \Pi_h w|_{m,p} \leq c h^{2-m} |w|_{2,p}, \quad m = 0, 1, 1 \leq p \leq \infty.$$ \hspace*{1cm} (2.13)

In this paper we adopt the standard notation $W^{m,p}(I)$ for Sobolev space on $I$ with norm $\|\cdot\|_{m,p}$ and seminorm $|\cdot|_{m,p}$. In order to simplify the notations, we denote $W^{m,2}(I)$ by $H^m(I)$ and skip the index $p = 2$ when possible, that is, $|\cdot|_{m,p} = |\cdot|_m$, $\|\cdot\|_{m,p} = \|\cdot\|_m$, $\|\cdot\|_0 = \|\cdot\|$. We denote by $L^q(0,T;W^{m,p}(I))$ the Banach space of all $L^q$ integrable functions from $(0,T)$ into $W^{m,p}(I)$ with the norm $\|\cdot\|_{L^q(W^{m,p})} = (\int_0^T \|\cdot\|_{m,p}^q dt)^{1/q}$ for $q \in [1, \infty)$ and the standard modification for $q = \infty$. After all these denotations, we give the existence and uniqueness of the solution of the finite volume scheme (2.10).

**Theorem 2.1.** The solution of the finite volume scheme (2.10) is existent and unique.

**Proof.** Let $u^n_h = \sum_{i=1}^{r-1} u^n_{hi} \phi_i(x) \in U_h$ ($n = 0, 1, 2, \ldots, N$) be the solution of (2.10). According to $u^0_h = \Pi_h u_0$ and $u^1_h = \Pi_h u_0 + \tau \Pi_h u_1$, $u^0_{h_j} = u_0(x_j)$, $u^n_{h_j} = u_0(x_j) + \tau u_1(x_j)$, $j = 1, 2, \ldots, r - 1$ are known. Hence the existence and uniqueness of the solution $u^n_{hi}$ ($n = 0, 1, 2, \ldots, N$) of scheme (2.10) are equivalent to the existence and uniqueness of $u^n_{hi}$ ($n = 0, 1, 2, \ldots, N$) of scheme (2.10).

For $n = 2, \ldots, N$, choosing $v_h = \phi_j$, $j = 1, 2, \ldots, r - 1$. By (2.2) we have

$$\left( \partial^2_n u^{n-1}_h, v_h \right) = \int_0^{x_{j+1/2}} \partial^2_n u^{n-1}_h \phi_j(x) dx = \int_{x_{j-1/2}}^{x_{j+1/2}} \partial^2_n u^{n-1}_h dx$$

$$= \int_{x_{j-1/2}}^{x_{j+1/2}} \left( u^n_h - 2u^{n-1}_h + u^{n-2}_h \right) dx$$

$$= \int_{x_{j-1/2}}^{x_{j+1/2}} \sum_{i=1}^{r-1} \left( u^n_{hi} - 2u^{n-1}_{hi} + u^{n-2}_{hi} \right) \phi_i(x) dx$$

$$= \frac{1}{\tau^2} \left\{ \left( u^n_{hj-1} - 2u^{n-1}_{hj-1} + u^{n-2}_{hj-1} \right) \int_{x_{j-1/2}}^{x_{j+1/2}} \phi_{j-1}(x) dx \right.$$  

$$+ \left( u^n_{hj} - 2u^{n-1}_{hj} + u^{n-2}_{hj} \right) \int_{x_{j-1/2}}^{x_{j+1/2}} \phi_j(x) dx \right.$$  

$$+ \left( u^n_{hj+1} - 2u^{n-1}_{hj+1} + u^{n-2}_{hj+1} \right) \int_{x_{j-1/2}}^{x_{j+1/2}} \phi_{j+1}(x) dx \right\}$$
Using (2.10) we obtain

\[
\frac{1}{\tau^2} \left\{ \frac{1}{8} h_j \left( u_{hj-1}^n - 2u_{hj-1}^{n-1} + u_{hj-1}^{n-2} \right) \right. \\
+ \frac{3}{8} \left( h_j + h_{j+1} \right) \left( u_{hj}^n - 2u_{hj}^{n-1} + u_{hj}^{n-2} \right) \\
+ \frac{1}{8} h_{j+1} \left( u_{hj+1}^n - 2u_{hj+1}^{n-1} + u_{hj+1}^{n-2} \right) \right\},
\]

\[
a^*(\partial_t u_{hj-1}^{n-1}, v_h) = a^*(\partial_t u_{hj-1}^{n-1}, q_j) = \partial_t u_{hj}^{n-1} (x_{j-1/2}) - \partial_t u_{hj+1}^{n-1} (x_{j+1/2})
\]

\[
= \frac{u_{hj-1}^{n-1} - u_{hj-1}^{n-2}}{h_j} - \frac{u_{hj}^{n-1} - u_{hj}^{n-2}}{h_{j+1}}
\]

\[
= \frac{1}{\tau h_j} \left( u_{hj-1}^n - u_{hj-1}^{n-1} - u_{hj}^n + u_{hj}^{n-1} \right)
\]

\[
- \frac{1}{\tau h_{j+1}} \left( u_{hj+1}^n - u_{hj+1}^{n-1} - u_{hj}^n + u_{hj}^{n-1} \right),
\]

\[
b^* \left( f^{(1/4)*} (u_{hj-1}^{n-1}), v_h \right) = b^* \left( f^{(1/4)*} (u_{hj}^{n-1}), q_j \right)
\]

\[
= f^{(1/4)*} \left( u_{hj}^{n-1} (x_{j-1/2}) \right) - f^{(1/4)*} \left( u_{hj+1}^{n-1} (x_{j+1/2}) \right)
\]

\[
= f^{(1/4)*} \left( \frac{u_{hj}^{n-1} - u_{hj-1}^{n-1}}{h_j} \right) - f^{(1/4)*} \left( \frac{u_{hj+1}^{n-1} - u_{hj}^{n-1}}{h_{j+1}} \right)
\]

\[
= \frac{3}{4} f \left( \frac{u_{hj-1}^{n-1} - u_{hj-1}^{n-2}}{h_j} \right) + \frac{1}{4} f \left( \frac{u_{hj}^{n-2} - u_{hj-1}^{n-2}}{h_j} \right)
\]

\[
- \frac{3}{4} f \left( \frac{u_{hj+1}^{n-1} - u_{hj-1}}{h_{j+1}} \right) - \frac{1}{4} f \left( \frac{u_{hj+1}^{n-2} - u_{hj-1}}{h_{j+1}} \right) + \frac{1}{2} f \left( \frac{u_{hj+1}^{n-2} - u_{hj-1}}{h_{j+1}} \right).
\]

\[\text{(2.14)}\]
In this section, we will prove the optimal error estimates in the

\[ \text{3. Error Estimates} \]

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that is,

\[ \begin{align*}
&+ \frac{3}{4} f \left( \frac{u_{hj}^{n-1} - u_{hj-1}^{n-1}}{h_j} \right) + \frac{1}{4} f \left( \frac{u_{hj}^{n-2} - u_{hj-1}^{n-2}}{h_j} \right) \\
&- \frac{3}{4} f \left( \frac{u_{hj+1}^{n-1} - u_{hj}^{n-1}}{h_{j+1}} \right) - \frac{1}{4} f \left( \frac{u_{hj+1}^{n-2} - u_{hj}^{n-2}}{h_{j+1}} \right) = 0,
\end{align*} \]

(2.15)

that is,

\[ \left( \frac{h_j}{8} - \frac{\tau}{h_j} \right) u_{hj-1}^n + \left[ \frac{3}{8} (h_j + h_{j+1}) + \tau \left( \frac{1}{h_j} + \frac{1}{h_{j+1}} \right) \right] u_{hj}^n + \left( \frac{h_{j+1} - \tau}{h_{j+1}} \right) u_{hj+1}^n \\
= \left( \frac{h_j}{4} - \frac{\tau}{h_j} \right) u_{hj-1}^{n-1} + \left[ \frac{3}{4} (h_j + h_{j+1}) + \tau \left( \frac{1}{h_j} + \frac{1}{h_{j+1}} \right) \right] u_{hj}^{n-1} \\
+ \left( \frac{h_{j+1}}{4} - \frac{\tau}{h_{j+1}} \right) u_{hj+1}^{n-1} + \frac{3 \tau^2}{4} \left[ f \left( \frac{u_{hj+1}^{n-1} - u_{hj-1}^{n-1}}{h_{j+1}} \right) - f \left( \frac{u_{hj}^{n-1} - u_{hj-2}^{n-1}}{h_j} \right) \right] \\
- \frac{1}{8} h_j u_{hj-1}^{n-2} - \frac{3}{8} (h_j + h_{j+1}) u_{hj}^{n-2} - \frac{1}{8} h_{j+1} u_{hj+1}^{n-2} \\
+ \frac{\tau^2}{4} \left[ f \left( \frac{u_{hj+1}^{n-2} - u_{hj-2}^{n-2}}{h_{j+1}} \right) - f \left( \frac{u_{hj}^{n-2} - u_{hj-2}^{n-2}}{h_j} \right) \right], \quad j = 1, 2, \ldots, r - 1.
\]

(2.16)

Noticing \( u_{h0}^n = u_{hn}^n = 0 \), the coefficient matrix of the system (2.16) is strictly diagonally dominant matrix. So, when \( \{u_{hj}^{n-1} \}_{j=1}^{r-1} \) and \( \{u_{hj}^{n-2} \}_{j=1}^{r-1} \) are known, the solution \( \{u_{hj}^n \}_{j=1}^{r-1} \) is existent and unique. Combining the above conditions, the solution \( \{u_{hj}^n \}_{j=1}^{r-1} \) (\( n = 2, \ldots, N \)) is existent and unique when \( \{u_{hj}^0 \}_{j=1}^{r-1} \) and \( \{u_{hj}^{n-1} \}_{j=1}^{r-1} \) are known. This completes the proof.

3. Error Estimates

In this section, we will prove the optimal error estimates in the \( L^2 \) and \( H^1 \) norms as well as the superconvergence error estimates in the \( H^1 \) norm. This needs some assumptions about the data.

\( (H_1) \) The nonlinear function \( f(s) \) satisfies

\[ 0 < l \leq f'(s) \leq L, \quad \max \{ |f''(s)|, |f'''(s)| \} \leq M, \quad s \in (-\infty, +\infty). \]

(3.1)

\( (H_2) \) The initial functions \( u_0 \) and \( u_1 \) satisfy

\[ u_0 \in H^2_0(I) \cap H^2(I), \quad u_1 \in H_0^1(I) \cap W^{3,1}(I). \]

(3.2)
(H₃) The solution $u$ of (1.1) satisfies
\[ u \in L^\infty \left( 0, T; W^{3,1} (I) \right), \quad u_t \in L^\infty \left( 0, T; W^{2, \infty} (I) \cap W^{3,1} (I) \right), \]
\[ u_{tt} \in L^\infty \left( 0, T; W^{3,1} (I) \right), \quad u_{tttt} \in L^2 \left( 0, T; L^2 (I) \right). \]

(3.3)

For any $u_h, v_h \in \mathcal{U}_h$, $u_h = \sum_{i=1}^{r-1} u_{hi} \varphi_i (x)$, $v_h = \sum_{i=1}^{r-1} v_{hi} \varphi_i (x)$, where $u_{hi} = u_h (x_i)$, $v_{hi} = v_h (x_i)$, $i = 1, 2, \ldots, r - 1$. Using the definition of the interpolation operator $\Pi_h$, we know that $\Pi_h u_h = \sum_{i=1}^{r-1} u_{hi} \varphi_i (x)$, $\Pi_h v_h = \sum_{i=1}^{r-1} v_{hi} \varphi_i (x) \in V_h$. Noting $u_{h0} = u_{hr} = v_{h0} = v_{hr} = 0$ and [11, Lemma 3.1] we have
\[ (u_h, \Pi_h^* v_h) = (v_h, \Pi_h u_h) = \frac{1}{8} \sum_{i=1}^{r-1} [h_i u_{hi-1} v_{hi} + 3 (h_i + h_{i+1}) u_{hi} v_{hi} + h_{i+1} u_{hi+1} v_{hi}]. \]

Hence
\[ (u_h, \Pi_h u_h) = \frac{1}{8} \sum_{i=1}^{r-1} [h_i (u_{hi} + u_{hi+1})^2 + 2 (h_i + h_{i+1}) u_{hi}^2] + h_1 u_{h1}^2. \]

(3.5)

So, when $u_h \neq 0$, we can get $(u_h, \Pi_h u_h) > 0$. Let $\| u_h \|_0 = (u_h, \Pi_h u_h)^{1/2}$. By [11, Lemma 3.2], we have
\[ \sqrt{\frac{2}{3}} \| u_h \|_0 \leq \| u_h \| \leq \sqrt{\frac{4}{3}} \| u_h \|_0, \quad \forall u_h \in \mathcal{U}_h. \]

(3.6)

On the other hand, using [11, Lemma 3.3], we also have
\[ \alpha^* (u_h, \Pi_h u_h) = \sum_{i=1}^{r} \frac{1}{h_i} (u_{hi} - u_{hi-1})^2 = | u_h^1 |^2. \]

(3.7)

For error estimate, we will use the generalized adjoint finite volume element projection $\tilde{u}^{[11]}$ of the solution $u$ of (1.1), that is, $\tilde{u} : [0, T] \to \mathcal{U}_h$ satisfies
\[ \alpha^* (u_t - \tilde{u}_t, v_h) + b^* (f (u) - f (\tilde{u}), v_h) = 0, \quad \forall v_h \in V_h, \quad 0 < t \leq T, \]
\[ \tilde{u}(0) = \Pi_h u_0. \]

(3.8)

Let $u$ and $\tilde{u}$ denote the solutions of (1.1) and (3.8), respectively. Under the assumptions (H₁), (H₂), and (H₃) together with [11, Theorems 3.4-3.5] and the one-dimensional imbedding theorem in Sobolev space, then we can obtain the following.
Lemma 3.1. Let \( u \) and \( \bar{u} \) be the solutions of (1.1) and (3.8), respectively. If the assumptions \((H_1), (H_2), \) and \((H_3)\) hold, then one has

\[
\| \Pi_h u - \bar{u} \|_1 + |u - \bar{u}|_1 + |u_t - \bar{u}_t|_1 + |u_{tt} - \bar{u}_{tt}|_1 \leq ch, \quad t \in [0, T],
\]

\[
\| u - \bar{u} \| + \| u_t - \bar{u}_t \| + \| u_{tt} - \bar{u}_{tt} \| \leq ch^2, \quad t \in [0, T].
\]

Next, we give the corresponding error estimates. Choosing \( t = t_{n-1}, t_n, t_{n+1} \) in the first equation of (2.8), then multiplying the three equations by \( 1/4, 1/2, 1/4, \) respectively, and adding them, we have

\[
\left( u_t^{n,1/4}, v_h \right) + a^\ast \left( u_t^{n,1/4}, v_h \right) + b^\ast \left( f^{1/4}(u^n), v_h \right) = 0, \quad \forall v_h \in V_h, \ n = 1, 2, \ldots, N - 1.
\]

(3.11)

Similarly, taking \( t = t_{n-1}, t_n, t_{n+1} \) in the first equation of (3.8), then multiplying the three equations by \( 1/4, 1/2, 1/4, \) respectively, and adding them, we obtain

\[
a^\ast \left( u_t^{n,1/4} - \bar{u}_t^{n,1/4}, v_h \right) + b^\ast \left( f^{1/4}(u^n) - f^{1/4}(\bar{u}^n), v_h \right) = 0, \quad \forall v_h \in V_h, \ n = 1, 2, \ldots, N - 1.
\]

(3.12)

Subtracting (3.11) from (2.10) and using (3.12), we get the error equation

\[
\left( \partial_t^2 u_h^n - u_t^{n,1/4}, v_h \right) + a^\ast \left( \partial_t u_h^n - \bar{u}_t^{n,1/4}, v_h \right) + b^\ast \left( f^{1/4}(u_h^n) - f^{1/4}(\bar{u}^n), v_h \right) = 0, \quad \forall v_h \in V_h,
\]

(3.13)

where \( n = 1, 2, \ldots, N - 1. \) Let

\[
u^n_t - u^n = \theta^n + \rho^n, \quad \theta^n = u^n_h - \bar{u}^n, \quad \rho^n = \bar{u}^n - u^n, \quad n = 1, 2, \ldots, N - 1.
\]

(3.14)

Then the error equation (3.13) can be rewritten as

\[
\left( \partial_t^2 \theta^n, v_h \right) + a^\ast \left( \partial_t \theta^n, v_h \right) + b^\ast \left( f^{1/4}(u_h^n) - f^{1/4}(\bar{u}^n), v_h \right)
\]

\[
- \left( \partial_t^2 \rho^n, v_h \right) + \left( u_t^{n,1/4} - \bar{u}_t^{n,1/4}, v_h \right) + a^\ast \left( \rho_t^{n,1/4} - \partial_t \rho^n, v_h \right)
\]

\[
+ a^\ast \left( u_t^{n,1/4} - \partial_t u^n, v_h \right), \quad \forall v_h \in V_h, \ n = 1, 2, \ldots, N - 1.
\]

Theorem 3.2. Let \( \{u_h^n\}_{n=0}^N \) and \( \bar{u} \) be the solutions of (2.10) and (3.8), respectively. If the assumptions \((H_1), (H_2), \) and \((H_3)\) hold, then there exists a positive constant \( h_0, \) such that when \( h < h_0 \)

\[
|u_h^n - \bar{u}^n|_1 = |\theta^n|_1 \leq c \left( h^2 + \tau^2 \right),
\]

(3.16)

\[
\| \partial_t (u_h^0 - \bar{u}^0) \| = |\partial_t \theta^0| \leq c \left( h^2 + \tau \right).
\]

(3.17)
Proof. Firstly, we estimate $|\theta^1|_1$. Applying the Taylor formula,

$$
\tilde{u}^1 = \tilde{u}(\tau) = \tilde{u}(0) + \tau \tilde{u}_t(0) + \frac{1}{2} \tau^2 \tilde{u}_{tt}(\xi) = \tilde{u}^0 + \tau \tilde{u}_t^0 + \frac{1}{2} \tau^2 \tilde{u}_{tt}^0(\xi), \quad \xi \in (0, \tau),
$$

(3.18)

and noting the definitions of $u_h^1$ and $\tilde{u}^0$ in (2.10) and (3.8), respectively, we know

$$
u_h^1 - \tilde{u}^1 = \theta^1 = \tau \left( \Pi_h u_1 - \tilde{u}_t^0 \right) - \frac{1}{2} \tau^2 \tilde{u}_{tt}^0(\xi), \quad \xi \in (0, \tau),
$$

(3.19)

so

$$
|\theta^1|_1^2 = \left| \tau \left( \Pi_h u_1 - \tilde{u}_t^0 \right) - \frac{1}{2} \tau^2 \tilde{u}_{tt}^0(\xi) \right|^2 \leq 4\tau^2 \left( |\Pi_h u_1 - u_t|^2 + |u_t - \tilde{u}_t^0|^2_1 \right) + \frac{1}{2} \tau^4 |\tilde{u}_{tt}^0(\xi)|^2_1.
$$

(3.20)

Notice that $u_t = u_t(0) = u_t^0$, (2.13) and (3.9), there exists a positive constant $h_0$ such that, when $h < h_0$ we have

$$
|\theta^1|^2 \leq c\tau^2 h^2 + \frac{1}{2} \tau^4 (|\tilde{u}_{tt}^0(\xi)|^2_1 + c h)^2 \leq c \left( \tau^2 h^2 + \tau^4 \right) \leq c \left( h^4 + \tau^4 \right).
$$

(3.21)

This implies (3.16) holds. Secondly, noting $\theta^0 = 0$ and (3.19), we obtain

$$
\partial_t \theta^0 = \frac{\theta^1 - \theta^0}{\tau} = \Pi_h u_1 - \tilde{u}_t^0 - \frac{1}{2} \tau \tilde{u}_{tt}^0(\xi), \quad \xi \in (0, \tau),
$$

(3.22)

hence, by (2.16), (3.10), and $h < h_0$ we have

$$
\left\| \partial_t \theta^0 \right\|^2 = \left\| (\Pi_h u_1 - u_t) + \left( u_t - \tilde{u}_t^0 \right) - \frac{1}{2} \tau \tilde{u}_{tt}^0(\xi) \right\|^2 \\
\quad \leq 4 \left\| (\Pi_h u_1 - u_t) \right\|^2 + 4 \left\| u_t - \tilde{u}_t^0 \right\|^2 + \frac{1}{2} \tau^2 \left\| \tilde{u}_{tt}^0(\xi) \right\|^2 \\
\quad \leq c h^4 + \frac{1}{2} \tau^2 \left( \left\| u_t(\xi) \right\|^2 + c h^2 \right) \leq c \left( h^4 + \tau^4 \right).
$$

(3.23)

This completes the proof of (3.17).
Theorem 3.3. Let $u_n$, $u^n_n$, and $\bar{u}$ be the solutions of (1.1), (2.10), and (3.8), respectively. Assume that (H1), (H2), and (H3) hold; then for any $n = 1, 2, \ldots, N - 1$, when $\tau$ is sufficiently small and $h < h_0$, one has

\begin{equation}
\begin{aligned}
b^* \left(f^{(1/4)}(u^n_h) - f^{1/4}(\bar{u}^n), \Pi^* \partial_t \theta^n\right) \\
\geq \frac{1}{4\tau} \sum_{j=1}^{n-1} a_{j-1/2} \left[a_{j-1/2}^{n+1} (\theta_j^{n+1} - \theta_j^{n+1})^2 - a_{j-1/2}^{n} (\theta_j^{n} - \theta_j^{n-1})^2\right] - \frac{3}{4} |\partial_t \theta^n|_1^2 \\
- c \left[1 + |\theta_n^{1/2}|^2 + |\theta_n|_1^2 + |\theta_n^{3/2}|_1^2\right] - c \tau^2,
\end{aligned}
\end{equation}

(3.24)

where

\begin{equation}
a^k_{j-1/2} = \int_0^1 f'(u^k_{x(j-1/2)}) - s \theta^k_x(x_{j-1/2}) dx, \quad k = n, n+1, j = 1, 2, \ldots, r - 1.
\end{equation}

(3.25)

Proof. Expanding the term $b^* \left(f^{(1/4)}(u^n_h) - f^{1/4}(\bar{u}^n), \Pi^* \partial_t \theta^n\right)$, we have

\begin{equation}
\begin{aligned}
b^* \left(f^{(1/4)}(u^n_h) - f^{1/4}(\bar{u}^n), \Pi^* \partial_t \theta^n\right) \\
= b^* \left(f^{1/4}(u^n_h) - f^{1/4}(\bar{u}^n), \Pi^* \partial_t \theta^n\right) + b^* \left(\frac{1}{4} \left[f(u^n_h) - f \left(u^{n+1}_h\right)\right], \Pi^* \partial_t \theta^n\right) \\
= \frac{1}{2\tau} \left[b^* \left(f^{1/2}(u^n_h) - f^{1/2}(\bar{u}^n), \Pi^* \theta^{n+1}\right) - b^* \left(f^{1/2}(u^{n+1}_h) - f^{1/2}(\bar{u}^{n+1}), \Pi^* \theta^{n+1}\right)\right] \\
- \frac{1}{2\tau} \left[b^* \left(f^{1/2}(u^n_h) - f^{1/2}(\bar{u}^n), \Pi^* \theta^n\right) - b^* \left(f^{1/2}(u^{n+1}_h) - f^{1/2}(\bar{u}^{n+1}), \Pi^* \theta^n\right)\right] \\
+ \frac{1}{4\tau} \left[b^* \left(f \left(u^n_h\right) - f \left(u^{n+1}_h\right), \Pi^* \theta^{n+1}\right) - b^* \left(f \left(u^n_h\right) - f \left(u^{n+1}_h\right), \Pi^* \theta^n\right)\right].
\end{aligned}
\end{equation}

(3.26)

For any $v_h \in V_h$, since $v_h = \sum_{i=1}^{r-1} v_{hi} \delta(x)$ and $v_{h_0} = v_{hr} = 0$, then

\begin{equation}
b^* \left(f(u), v_h\right) = \sum_{i=1}^{r-1} v_{hi} \left[f(u(x_{i-1/2})) - f(u(x_{i+1/2}))\right] = \sum_{i=1}^{r} f(u(x_{i-1/2})) (v_{hi} - v_{hi-1}).
\end{equation}

(3.27)

Let $\partial^i_j(x) = \theta^i_j$, thus $\Pi^* \partial^i_j = \sum_{i=1}^{r-1} \delta^i_j \delta^i_j(x) \in V_h$. Applying transformation of variable, we know

\begin{equation}
\begin{aligned}
b^* \left(f^{1/2}(u^n_h) - f^{1/2}(\bar{u}^n), \Pi^* \theta^n\right) \\
= \sum_{j=1}^{r} \left(\partial^i_j - \theta^i_j\right) \left[f^{1/2}(u^{n}_h(x_{j-1/2})) - f^{1/2}(\bar{u}^n(x_{j-1/2}))\right]
\end{aligned}
\end{equation}

(3.28)
\begin{align}
\frac{1}{2} \sum_{j=1}^{r} (\theta_j^i - \theta_{j-1}^i) \left[ f\left( u_{h^m}^n(x_{j-1/2}) \right) - f\left( \tilde{u}_x^{m+1}(x_{j-1/2}) \right) \right] \\
\quad + f\left( u_{h^m}^n(x_{j-1/2}) \right) - f\left( \tilde{u}_x^m(x_{j-1/2}) \right) \right] \\
= \frac{1}{2} \sum_{j=1}^{r} \left[ a_{j-1/2}^{m+1} \partial_x^{m+1}(x_{j-1/2}) + a_{j-1/2}^m \partial_x^m(x_{j-1/2}) \right] \\
= \frac{1}{2} \sum_{j=1}^{r} \frac{1}{h_j} \left[ a_{j-1/2}^{m+1} \left( \theta_j^i - \theta_{j-1}^i \right) \left( \theta_{j-1}^{m+1} - \theta_{j-1}^m \right) \right. \\
\quad + a_{j-1/2}^m \left( \theta_j^i - \theta_{j-1}^i \right) \left( \theta_{j-1}^m - \theta_{j-1}^m \right] \right.
\end{align}

(3.28)

where

\begin{align}
a_{j-1/2}^k = \int_0^1 f'\left( \tilde{u}_x^k(x_{j-1/2}) + s \theta_x^k(x_{j-1/2}) \right) ds, \\
\theta_x^k(x_{j-1/2}) = \frac{1}{h_j} \left( \theta_j^k - \theta_{j-1}^k \right), \quad k = m, m + 1.
\end{align}

By (3.28), we have

\begin{align}
b^*\left( f^{1/2}(u_h^n) - f^{1/2}(\tilde{u}^n), \Pi_h^n \theta^{n+1} \right) = \frac{1}{2} \sum_{j=1}^{r} \frac{1}{h_j} \left[ a_{j-1/2}^{n+1} \left( \theta_j^n - \theta_{j-1}^{n+1} \right) \right. \\
\quad + a_{j-1/2}^n \left( \theta_j^n - \theta_{j-1}^n \right) \left( \theta_{j-1}^{n+1} - \theta_{j-1}^n \right) \right. \\
\end{align}

(3.30)
On the other hand

\[ b^* \left( f(u^n_h) - f(u^{n+1}_h), \Pi^*_h \theta^{n+1} \right) - b^* \left( f(u^n_h) - f(u^{n+1}_h), \Pi^*_h \theta^n \right) \]

\[ = \sum_{j=1}^r \left[ f(u^n_{hx}(x_{j-1/2})) - f(u^{n+1}_{hx}(x_{j-1/2})) \right] \left[ (\theta^n_j - \theta^{n+1}_j) - (\theta^n_{j-1} - \theta^n_{j-1}) \right] \]

\[ = \sum_{j=1}^r a^n_{j-1/2}(x_{j-1/2}) \left( u^n_{hx}(x_{j-1/2}) - u^{n+1}_{hx}(x_{j-1/2}) \right) \left( \theta^n_j - \theta^n_{j-1} - (\theta^n_{j-1} - \theta^n_{j-1}) \right) \]

\[ = -r^2 \sum_{j=1}^r a^n_{j=1/2} \partial_t u^n_{hx}(x_{j-1/2}) \left( \partial_t \theta^n - \partial_t \theta^n_{j-1} \right), \]

where

\[ a^n_{j=1/2} = \int_0^1 f'(u^n_{hx}(x_{j-1/2})) + s \left( u^n_{hx}(x_{j-1/2}) - u^{n+1}_{hx}(x_{j-1/2}) \right) ds. \]

Combining the above five equalities with (3.26), we can get

\[ b^* \left( f^{(1/4)}(u^n_h) - f^{(1/4)}(u^n), \Pi^*_h \theta^n \right) \]

\[ = \frac{1}{4r} \sum_{j=1}^r \frac{1}{h_j} \left[ a^n_{j-1/2} \left( \theta^n_j - \theta^{n+1}_j \right)^2 - a^n_{j-(1/2)} \left( \theta^n_j - \theta^n_{j-1} \right)^2 \right] \]

\[ + \frac{1}{4r} \sum_{j=1}^r \frac{1}{h_j} a^n_{j-1/2} \left( \theta^n_j - \theta^n_{j-1} \right) \left[ \theta^n_{j+1} - \theta^n_{j-1} - (\theta^n_{j} - \theta^n_{j-1}) \right] \]

\[ + \frac{1}{4r} \sum_{j=1}^r \frac{1}{h_j} \left[ \theta^n_{j+1} - \theta^n_{j-1} - (\theta^n_{j} - \theta^n_{j-1}) \right] \]

\[ - \frac{1}{4r} \sum_{j=1}^r \frac{1}{h_j} \left( a^n_{j+1/2} - a^n_{j-1/2} \right) \left( \theta^n_{j+1} - \theta^n_{j-1} \right) \left( \theta^n_{j} - \theta^n_{j-1} \right) \]

\[ - \frac{r}{4} \sum_{j=1}^r a^n_{j=1/2} \partial_t \theta^n_j - \partial_t \theta^n_{j-1} \partial_t u^n_{hx}(x_{j-1/2}) \]

\[ = B_1 + B_2 + B_3 + B_4 + B_5. \]

Using the \( \varepsilon \)-inequality, we find

\[ -B_2 = \frac{1}{4r} \sum_{j=1}^r \frac{1}{h_j} a^n_{j-1/2} \left( \theta^n_j - \theta^n_{j-1} \right) \left[ \theta^n_{j+1} - \theta^n_{j-1} - (\theta^n_{j} - \theta^n_{j-1}) \right] \]

\[ = -\frac{r}{4} \sum_{j=1}^r \frac{1}{h_j} a^n_{j=1/2} \left( \theta^n_j - \theta^n_{j-1} \right) \left( \partial_t \theta^n_j - \partial_t \theta^n_{j-1} \right) \]
\[
\leq c \sum_{j=1}^{r} \left( \frac{\theta^n_j - \theta^{n-1}_j}{h_j} \right)^2 + \frac{1}{16} \sum_{j=1}^{r} \left( \frac{\partial_t \theta^n_j - \partial_t \theta^{n-1}_j}{h_j} \right)^2 = c |\theta^{n-1}_1|^2 + \frac{1}{16} |\partial_t \theta^n_1|^2,
\]

\[
-B_3 = \frac{1}{4\tau} \sum_{j=1}^{r} \frac{1}{h_j} a_{j-1/2}^n \left( \theta^{n-1}_j - \theta^{n-1}_j \right) \left[ \theta^n_j - \theta^{n-1}_j - \left( \theta^{n+1}_j - \theta^{n+1}_j \right) \right]
\]

\[
= -\frac{1}{4} \sum_{j=1}^{r} \frac{1}{h_j} a_{j-1/2}^n \left( \theta^{n-1}_j - \theta^{n-1}_j \right) \left( \partial_t \theta^n_j - \partial_t \theta^{n-1}_j \right) \leq c |\theta^{n-1}_1|^2 + \frac{1}{16} |\partial_t \theta^n_1|^2,
\]

\[
-B_5 = \frac{\tau}{4} \sum_{j=1}^{r} a_{j-1/2}^n \left( \partial_t \theta^n_j - \partial_t \theta^{n-1}_j \right) \partial_t u^n_{hx}(x_{j-1/2})
\]

\[
\leq \frac{1}{16} |\partial_t \theta^n_1|^2 + c \tau^2 \sum_{j=1}^{r} h_j(\partial_t u^n_{hx}(x_{j-1/2}))^2 = \frac{1}{16} |\partial_t \theta^n_1|^2 + I_1.
\]

(3.34)

Now we estimate \(I_1\). Noting (2.2), we have

\[
u^n_{hx}(x_{j-1/2}) = \frac{u^n_{hx} - u^n_{hx-1}}{h_j} = \frac{1}{h_j} \int_{x_{j-1/2}}^{x_j} u^n_{hx}(x) dx.
\]

(3.35)

Furthermore,

\[
\left( \partial_t u^n_{hx}(x_{j-1/2}) \right)^2 = \left( \frac{1}{h_j} \int_{x_{j-1/2}}^{x_j} \partial_t u^n_{hx}(x) dx \right)^2 \leq \frac{1}{h_j} \int_{x_{j-1}}^{x_j} \left( \partial_t u^n_{hx}(x) \right)^2 dx.
\]

(3.36)

So

\[
I_1 \leq c \tau^2 \sum_{j=1}^{r} \int_{x_{j-1}}^{x_j} \left( \partial_t u^n_{hx}(x) \right)^2 dx = c \tau^2 |\partial_t u^n|_1^2
\]

(3.37)

\[
\leq c \tau^2 |\partial_t \theta^n|_1^2 + c \tau^2 |\partial_t \bar{u}^n|_1^2 \leq c_1 \tau^2 |\partial_t \theta^n|_1^2 + c \tau^2 |\partial_t \bar{u}^n|_1^2.
\]

When \(\tau\) is sufficiently small, we can take \(\tau\) suitable such that \(c_1 \tau^2 \leq 1/16\), from the above equalities we have

\[
I_1 \leq \frac{1}{16} |\partial_t \theta^n|_1^2 + c \tau^2 |\partial_t \bar{u}^n|_1^2.
\]

(3.38)

Moreover, in view of \(\partial_t \bar{u}^n = 1/(\tau) \int_{t_n}^{t_{n+1}} \bar{u}_t dt\) and (3.9), we obtain

\[
|\partial_t \bar{u}^n|_1^2 = \int_0^1 \left( \partial_t \bar{u}^n(x) \right)^2 dx = \frac{1}{\tau^2} \int_0^1 \left( \int_{t_n}^{t_{n+1}} \bar{u}^n dt \right)^2 dx \leq \frac{1}{\tau} \int_{t_n}^{t_{n+1}} \left( \bar{u}^n dt \right)^2 dx
\]

(3.39)

\[
= \frac{1}{\tau} \int_{t_n}^{t_{n+1}} |\bar{u}|_1^2 dt \leq \frac{1}{\tau} \int_{t_n}^{t_{n+1}} (ch + |u|_1)^2 dt \leq c.
\]
Substituting the estimate of $\xi_n$ from the assumption

$$I_1 \leq \frac{1}{16} |\partial_t \theta^n|_1^2 + c r^2. \quad (3.40)$$

Substituting the estimate of $I_1$ into (3.34), we have

$$-B_5 \leq \frac{1}{8} |\partial_t \theta^n|_1^2 + c r^2,$$

$$-B_4 = \frac{1}{4r} \sum_{j=1}^{r} \int_{0}^{1} \left( a_{j-1/2}^{n+1} - a_{j-1/2}^{n} \right) \left( \theta_{j-1}^{n+1} - \theta_{j-1}^{n} \right) \left( \theta_{j}^{n} - \theta_{j}^{n-1} \right). \quad (3.41)$$

Since

$$a_{j-1/2}^{n+1} - a_{j-1/2}^{n} = \int_{0}^{1} \left[ f' \left( \tilde{u}^{n+1}_x(x_{j-1/2}) + s \theta_x^{n+1}(x_{j-1/2}) \right) - f' \left( \tilde{u}^n_x(x_{j-1/2}) + s \theta_x^n(x_{j-1/2}) \right) \right] ds$$

$$= \int_{0}^{1} f'' (\xi^n_j) \left[ \tilde{u}^{n+1}_x(x_{j-1/2}) - \tilde{u}^n_x(x_{j-1/2}) + s \left( \theta_x^{n+1}(x_{j-1/2}) - \theta_x^n(x_{j-1/2}) \right) \right] ds$$

$$= \tau \int_{0}^{1} f'' (\xi^n_j) ds \partial_t \tilde{u}^n_x(x_{j-1/2}) + \tau \int_{0}^{1} f'' (\xi^n_j) ds \partial_t \theta_x^n(x_{j-1/2}),$$

where $\xi^n_j$ lies between $\tilde{u}^{n+1}_x(x_{j-1/2}) + s \theta_x^{n+1}(x_{j-1/2})$ and $\tilde{u}^n_x(x_{j-1/2}) + s \theta_x^n(x_{j-1/2})$, by (2.13), (3.9), and the inverse estimate, we obtain

$$|\tilde{u}_t|_{1,1} \leq |u_t|_{1,1} + |u_t - \tilde{u}_t|_{1,1} \leq |u_t|_{1,1} + |u_t - \Pi h u_t|_{1,1} + |\Pi h u_t - \tilde{u}_t|_{1,1}$$

$$\leq |u_t|_{1,1} + ch |u_t|_{2,\infty} + ch^{-1/2} |\Pi h u_t - \tilde{u}_t|_1 \leq |u_t|_{1,1} + ch |u_t|_{2,\infty} + ch^{1/2}. \quad (3.43)$$

Thus, when $h < h_0$ we have $||\tilde{u}_t||_{L^\infty(L^\infty)} \leq c$, then

$$||\partial_t \tilde{u}_x^n(x_{j-1/2})|| = ||\tilde{u}_t(x_{j-1/2}, t_n + cr)| \leq ||\tilde{u}_t||_{L^\infty(L^\infty)} \leq c. \quad (3.44)$$

From the assumption (H1), we find that $\int_{0}^{1} f'' (\xi^n_j) ds$ is bounded: this leads to the boundedness of $\int_{0}^{1} f'' (\xi^n_j) ds \partial_t \tilde{u}^n_x(x_{j-1/2})$ and $\int_{0}^{1} f'' (\xi^n_j) ds$. So

$$-B_4 = \frac{1}{4} \sum_{j=1}^{r} \int_{0}^{1} f'' (\xi^n_j) ds \partial_t \tilde{u}^n_x(x_{j-1/2}) + \int_{0}^{1} f'' (\xi^n_j) ds \partial_t \theta_x^n(x_{j-1/2})$$

$$\times \left( \theta_{j}^{n+1} - \theta_{j-1}^{n+1} \right) \left( \theta_{j}^{n} - \theta_{j-1}^{n} \right)$$
Theorem 3.4. Let $u, \{u^n_{j}\}_{n=0}^{N}$ and $\tilde{u}$ be the solutions of (1.1), (2.10), and (3.8), respectively. Assume that (H$_1$), (H$_2$), and (H$_3$) hold; then for any $n = 1, 2, \ldots, N - 1$, when $h$ and $\tau$ are sufficiently small, one has

$$\max_{1 \leq n \leq N-1} ||\partial_t \theta^n|| + |\theta^n|_1 \leq c \left( h^2 + \tau \right).$$

(3.46)

Proof. Taking $v_h = \Pi_h^* \partial_t \theta^n$ in the error equation (3.15) and making a simple calculation yield

$$\begin{align*}
&\left( \partial_t^2 \theta^n, \Pi_h^* \partial_t \theta^n \right) + a^* \left( \partial_t \theta^n, \Pi_h^* \partial_t \theta^n \right) + b^* \left( f^{(1/4)*}(u^n_{t1}), \Pi_h^* \partial_t \theta^n \right) \\
&\quad= - \left( \partial_t^2 \rho^n, \Pi_h^* \partial_t \theta^n \right) + \left( u^{n,1/4}_{t1} - \partial_t^2 u^n, \Pi_h^* \partial_t \theta^n \right) \\
&\quad+ a^* \left( \rho^{n,1/4}_{t1} - \partial_t \rho^n, \Pi_h^* \partial_t \theta^n \right) + a^* \left( u^{n,1/4}_t - \partial_t u^n, \Pi_h^* \partial_t \theta^n \right) .
\end{align*}$$

(3.47)

Form (3.4), we derive that

$$\begin{align*}
&\left( \partial_t \theta^n, \Pi_h^* \partial_t \theta^n \right) - 2 \left( \partial_t \theta^{n-1}, \Pi_h^* \partial_t \theta^n \right) + \left( \partial_t \theta^{n-1}, \Pi_h^* \partial_t \theta^{n-1} \right) \\
&\quad= n \left( \partial_t \theta^n - \partial_t \theta^{n-1}, \Pi_h^* \left( \partial_t \theta^n - \partial_t \theta^{n-1} \right) \right) \geq 0 .
\end{align*}$$

(3.48)
Then
\[
\left( \partial_t^n \theta^n, \Pi_h \partial_t \theta^n \right) = \frac{1}{\tau} \left( \partial_t \theta^n - \partial_t \theta^{n-1}, \Pi_h^* \partial_t \theta^n \right) \\
= \frac{1}{2\tau} \left\{ \left( \partial_t \theta^n, \Pi_h^* \partial_t \theta^n \right) - \left( \partial_t \theta^{n-1}, \Pi_h^* \partial_t \theta^{n-1} \right) \right. \\
+ \left[ \left( \partial_t \theta^n, \Pi_h^* \partial_t \theta^n \right) - 2 \left( \partial_t \theta^{n-1}, \Pi_h^* \partial_t \theta^n \right) + \left( \partial_t \theta^{n-1}, \Pi_h^* \partial_t \theta^{n-1} \right) \right] \right. \\
\geq \frac{1}{2\tau} \left\{ \| \partial_t \theta^n \|_0^2 - \| \partial_t \theta^{n-1} \|_0^2 \right\}.
\]
(3.49)

By (3.7), we get
\[
a^* \left( \partial_t \theta^n, \Pi_h^* \partial_t \theta^n \right) = \| \partial_t \theta^n \|_1^2.
\]
(3.50)

Applying Taylor’s formula with integral-type remainder,
\[
f(t) = f(t_n) + f'(t_n)(t - t_n) + \cdots + \frac{f^{(k)}(t_n)}{k!}(t - t_n)^k + \frac{1}{k!} \int_{t_n}^{t} f^{(k+1)}(s)(t - s)^k ds,
\]
(3.51)
we have
\[
\partial_t^n \rho^n = \frac{1}{\tau^2} \left[ \int_{t_n}^{t_{n+1}} (t_{n+1} - s) \rho_H(s) ds + \int_{t_n}^{t_{n-1}} (t_{n-1} - s) \rho_H(s) ds \right]
\]
\[
= \frac{1}{\tau^2} \int_{-\tau}^{\tau} (\tau - |s|) \rho_H(t_n + s) ds = \frac{1}{\tau^2} \int_{t_{n-1}}^{t_{n+1}} (\tau - |t_n - s|) \rho_H(s) ds,
\]
\[
u_t^{n,1/4} - \partial_t^2 u^n = \int_{-\tau}^{\tau} (\tau - |s|) \left[ \frac{1}{4} - \frac{(\tau - |s|)^2}{6\tau^2} \right] u_{tttt}(t_n + s) ds
\]
\[
= \int_{t_{n-1}}^{t_{n+1}} (\tau - |t_n - s|) \left[ \frac{1}{4} - \frac{(\tau - |t_n - s|)^2}{6\tau^2} \right] u_{tttt}(s) ds,
\]
(3.52)
\[
u_t^{n,1/4} - \partial_t u^n = \frac{1}{4} \left[ \int_{t_n}^{t_{n+1}} u_H(s) ds + \int_{t_n}^{t_{n-1}} u_H(s) ds \right] - \int_{t_n}^{t_{n+1}} u_H(s) \frac{t_{n+1} - s}{\tau} ds,
\]
\[
\rho_t^{n,1/4} - \partial_t \rho^n = \frac{1}{4} \left[ \int_{t_n}^{t_{n+1}} \rho_H(s) ds + \int_{t_n}^{t_{n-1}} \rho_H(s) ds \right] - \int_{t_n}^{t_{n+1}} \rho_H(s) \frac{t_{n+1} - s}{\tau} ds.
\]

Note that \( \| \Pi_h^* \partial_t \theta^n \| \leq c \| \partial_t \theta^n \| \), we get
\[
\left( \partial_t^2 \rho^n, \Pi_h^* \partial_t \theta^n \right) \leq \| \partial_t^2 \rho^n \| \| \Pi_h^* \partial_t \theta^n \| \leq c \| \partial_t^2 \rho^n \|^2 + c \| \partial_t \theta^n \|^2
\]
\[
\leq c \tau^{-1} \int_{t_{n-1}}^{t_{n+1}} \| \rho_H \|^2 dt + c \| \partial_t \theta^n \|^2,
\]
\begin{equation}
\left( u_{n}^{1/4} - \partial_t^2 u^n, \Pi_h^* \partial_t \theta^n \right) \leq \left\| u_{n}^{1/4} - \partial_t^2 u^n \right\| \Pi_h^* \partial_t \theta^n \right\| \leq c \left\| u_{n}^{1/4} - \partial_t^2 u^n \right\|^2 + \left\| \partial_t \theta^n \right\|^2 \\
\leq c T^3 \int_{t_{n-1}}^{t_{n}} \left\| u_{n} \right\|^2 dt + c \left\| \partial_t \theta^n \right\|^2.
\end{equation}

(3.53)

For \( \Pi_h^* v = \sum_{j=1}^{r-1} v_j \varphi_j(x) \in V_h, v_0 = v_r = 0 \), we have

\begin{equation}
\begin{aligned}
a^*(u, \Pi_h^* v) &= \sum_{j=1}^{r-1} v_j \left( u(x_{j-1/2}) - u(x_{j+1/2}) \right) = \sum_{j=1}^{r-1} u_x(x_{j-1/2}) (v_j - v_{j-1}) \\
\leq \left( \sum_{j=1}^{r} \left( \frac{v_j - v_{j-1}}{h_j} \right)^2 \right)^{1/2} \left( \sum_{j=1}^{r} h_j u_x^2(x_{j-1/2}) \right)^{1/2}.
\end{aligned}
\end{equation}

(3.54)

Noting that

\begin{align*}
u_x(x_{j-1/2}) &= u_x(x) - \int_{x_{j-1/2}}^{x} u_{xx}(s) ds, \quad x \in [x_{j-1}, x_j], \\
u_x^2(x_{j-1/2}) &= \left( u_x(x) - \int_{x_{j-1/2}}^{x} u_{xx}(s) ds \right)^2 \leq 2u_x^2(x) + 2 \left( \int_{x_{j-1/2}}^{x} u_{xx}(s) ds \right)^2 \\
&\leq 2u_x^2(x) + 2 \left( \int_{x_{j-1}}^{x_j} |u_{xx}(x)| dx \right)^2 \leq 2u_x^2(x) + 2h_j \left( \int_{x_{j-1}}^{x_j} u_{xx}^2(x) dx \right),
\end{align*}

we see that

\begin{equation}
\sum_{j=1}^{r} h_j u_x^2(x_{j-1/2}) = \sum_{j=1}^{r} \int_{x_{j-1}}^{x_j} u_x^2(x_{j-1/2}) dx \leq 2 \sum_{j=1}^{r} \int_{x_{j-1}}^{x_j} u_x^2(x) dx \\
+ 2 \sum_{j=1}^{r} h_j^2 \int_{x_{j-1}}^{x_j} u_{xx}^2(x) dx \leq 2|u|^2 + 2h^2|u|^2,
\end{equation}

(3.56)

so

\begin{align*}
a^*(u_{n}^{1/4} - \partial_t u^n, \Pi_h^* \partial_t \theta^n) &\leq 2 \left( \left| u_{n}^{1/4} - \partial_t u^n \right|_1^2 + h^2 \left| u_{n}^{1/4} - \partial_t u^n \right|_2^2 \right)^{1/2} \left| \partial_t \theta^n \right|_1 \\
&\leq c \left( \left| u_{n}^{1/4} - \partial_t u^n \right|_1^2 + h^2 \left| u_{n}^{1/4} - \partial_t u^n \right|_2^2 \right)^{1/2} + \frac{1}{8} \left| \partial_t \theta^n \right|_1^2 \\
&\leq c T \int_{t_{n-1}}^{t_{n}} \left( |u_{tt}|_1^2 + h^2 |u_{tt}|_2^2 \right) dt + \frac{1}{8} \left| \partial_t \theta^n \right|_1^2,
\end{align*}

\begin{equation}
\end{equation}
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\[ a^*(\rho_i^{n+1/4} - \partial_t \rho_i^n, \Pi_i \partial_t \theta^n) \leq c\tau \int_{t_n-1}^{t_{n+1}} \left( |\rho_{i1}'|^2 + h^2 |\rho_{i2}'|^2 \right) dt + \frac{1}{8} |\partial_t \theta^n|^2_{l_1} \]

= \frac{c\tau}{2} \int_{t_n-1}^{t_{n+1}} \left( |\rho_{i1}'|^2 + h^2 |u_{i2}'|^2 \right) dt + \frac{1}{8} |\partial_t \theta^n|^2_{l_1}.

(3.57)

Substituting (3.24), (3.49)–(3.57) into (3.47) and using the Lemma 3.1, we obtain

\[
\frac{1}{2\tau} \left\{ \|\partial_t \theta^n\|_0^2 - \|\partial_t \theta^{n-1}\|_0^2 \right\} + \frac{1}{4\tau} \sum_{j=1}^{r} \frac{1}{h_j} a_{n+1}^{j-1/2} \left( \theta_j^{n+1} - \theta_j^{n+1} \right)^2 \\
\leq c\|\partial_t \theta^n\|^2 + c \left( 1 + |\theta^n|_{1,\infty}^2 \right) \left| \theta^{n+1} \right|_{1}^2 + |\theta^n|_{1}^2 + \left| \theta^{n-1} \right|_{1}^2 \\
+ c \left( \tau^2 + \tau^4 + \tau^2 h^2 + h^4 \right).
\]

(3.58)

Multiplying the above inequation by \( \tau \) and then summing it with respect to \( n \) from 1 to \( n \), we have

\[
\frac{1}{2} \left\| \partial_t \theta^n \right\|_0^2 + \frac{1}{4} \sum_{j=1}^{r} \frac{1}{h_j} a_{n+1}^{j-1/2} \left( \theta_j^{n+1} - \theta_j^{n+1} \right)^2 \\
\leq \frac{1}{2} \left\| \partial_t \theta^n \right\|_0^2 + \frac{1}{4} \sum_{j=1}^{r} \frac{1}{h_j} a_{j-1/2}^1 \left( \theta_j^{1} - \theta_j^{1} \right)^2 \\
+ c\tau \sum_{j=1}^{n} \left\| \partial_t \theta^n \right\|_0^2 \left( 1 + |\theta^n|_{1,\infty}^2 \right) \left| \theta^{n+1} \right|_{1}^2 + \left| \theta^n \right|_{1}^2 + \left| \theta^{n-1} \right|_{1}^2 + c \left( \tau^2 + h^4 \right).
\]

(3.59)

Now we make the induction hypothesis:

\[
\max_{1 \leq k \leq n} \left| \theta^k_{x,\infty} \right|_{1,\infty} \leq 1.
\]

(3.60)

Substituting (3.60) into (3.59) and noting (3.6) and the assumption (H_{1}), we conclude

\[
c^2 \left( \left\| \partial_t \theta^n \right\|_0^2 + \left| \theta^{n+1} \right|_{1}^2 \right) \leq \frac{1}{2} \left\| \partial_t \theta^n \right\|_0^2 + \frac{1}{4} \left| \theta^{n+1} \right|_{1}^2 \\
\leq \frac{1}{2} \left\| \partial_t \theta^n \right\|_0^2 + \frac{1}{4} \sum_{j=1}^{r} \frac{1}{h_j} a_{j-1/2}^1 \left( \theta_j^{1} - \theta_j^{1} \right)^2
\]

\[
\leq \frac{1}{2} \sqrt{\frac{3}{2} \| \partial_t \theta^0 \|^2 + \frac{1}{4} \| \theta^1 \|^2}
+ c \tau \sum_{i=1}^{n} \| \partial_t \theta^i \|^2 + c \tau \sum_{i=0}^{n+1} \| \theta^i \|^2 + c \left( \tau^2 + h^4 \right)
\]
\[
\leq c_3 \left( \| \partial_t \theta^0 \|^2 + \| \theta^1 \|^2 + \tau^2 + h^4 + \tau \sum_{i=1}^{n} \| \partial_t \theta^i \|^2 + \tau \sum_{i=0}^{n+1} \| \theta^i \|^2 \right).
\]
(3.61)

Taking \( \tau \) is sufficiently small, such that \( c_3 \tau < c_2 \) and \( c_1 \tau^2 \leq 1/16 \) (these ensure that Theorem 3.3. holds), combining the above equalities and Theorem 3.2. we have
\[
\| \partial_t \theta^n \|^2 + \| \theta^{n+1} \|^2 \leq c \left( \tau^2 + h^4 \right) + \frac{c_3 \tau}{c_2} \left( \sum_{i=1}^{n} \| \partial_t \theta^i \|^2 + \sum_{i=0}^{n+1} \| \theta^i \|^2 \right).
\]
(3.62)

By the discrete Gronwall’s lemma, it follows that
\[
\| \partial_t \theta^n \|^2 + \| \theta^{n+1} \|^2 \leq c \left( \tau^2 + h^4 \right), \quad n = 1, 2, \ldots, N - 1.
\]
(3.63)

To complete our proof we must verify that (3.60) holds. In fact, by considering the inverse hypothesis \( |\theta^n|_{1, \infty} \leq c_4 h^{-1/2} |\theta^n|_1, n = 1, 2, \ldots, N - 1 \) and (3.16), we have
\[
|\theta^1|_{0, \infty} = |\theta^1|_{1, \infty} \leq c_4 h^{-1/2} c \left( \tau^2 + h^2 \right) \leq c_5 h^{-1/2} \left( \tau^2 + h^2 \right).
\]
(3.64)

Thus, we can choose \( h, \tau \) such that \( h, \tau \) satisfy \( c_5 h^{-1/2} (\tau^2 + h^2) \leq 1 \) for \( h, \tau \) are sufficiently small, that is, (3.60) holds for \( n = 1 \). From the proof of (3.63), as \( h, \tau \) are sufficiently small and satisfy
\[
c_1 \tau^2 \leq \frac{1}{16}, \quad c_3 \tau < c_2, \quad c_5 h^{-1/2} \left( \tau^2 + h^2 \right) \leq 1,
\]
(3.65)
then (3.63) holds for \( n = 1 \). Thus
\[
|\theta^2|_1 \leq c_6 \left( h^2 + \tau \right).
\]
(3.66)

Hence
\[
|\theta^2|_{0, \infty} = |\theta^2|_{1, \infty} \leq c_4 h^{-1/2} |\theta^2|_1 \leq c_4 c_6 h^{-1/2} \left( h^2 + \tau \right).
\]
(3.67)
Theorem 3.5. Under the assumption of Theorem 3.4, for inequality, we obtain the following.

\[ h \]

Similarly, letting \( h, \tau \) be sufficiently small such that \( h, \tau \) satisfy \( c_4 \tau h^{-1/2}(h^2 + \tau) \leq 1 \), then we get \( |\theta_2^n_{\tau}|_{0,\infty} \leq 1 \). Furthermore, setting \( h, \tau \) to satisfy

\[ c_1 \tau^2 \leq \frac{1}{16}, \quad c_3 \tau < c_2, \quad c_5 h^{-1/2}(\tau^2 + h^2) \leq 1, \quad c_4 c_6 h^{-1/2}(h^2 + \tau) \leq 1, \quad (3.68) \]

we conclude that (3.60) holds for \( n = 2 \). Using the above facts, we find that (3.63) holds for \( n = 2 \) as \( h, \tau \) are sufficiently small and satisfy (3.68). By the induction argument for \( n = 1, 2, \ldots, N - 1 \), we deduce that (3.60) holds as \( h, \tau \) satisfy (3.68). Then (3.63) holds for \( n = 1, 2, \ldots, N - 1 \), which implies (3.46) is valid. This completes the proof of the theorem. \( \square \)

Note that \( \theta_0^n = 0 \) and (2.13), by Theorems 3.2 and 3.4, Lemma 3.1, and the triangle inequality, we obtain the following.

**Theorem 3.5.** Under the assumption of Theorem 3.4, for \( h, \tau \) are sufficiently small, one has

\[
\max_{0 \leq n \leq N} \left\| u_h^n - u^n \right\| \leq c \left( h^2 + \tau \right),
\]
\[
\max_{0 \leq n \leq N} |u_h^n - u^n|_{1} \leq c(h + \tau),
\]
\[
\max_{0 \leq n \leq N} |u_h^n - \bar{u}\|^1 \leq c \left( h^2 + \tau \right). \tag{3.69}
\]

**Remark 3.6.** The optimal order error estimates in \( L^2, H^1 \) norms and superconvergence error estimates in \( H^1 \) norm of the solution of the finite volume scheme (2.10) are achieved in Theorem 3.5.
4. Numerical Experiments

To test the finite volume scheme (2.10) and the theoretical analysis, we do some experiments by considering the following problem:

\begin{align}
(a) & \quad u_{it} = u_{xxt} + \frac{\pi^2 - 1}{\pi^2} u_{xx}, \quad (x, t) \in (0, 1) \times (0, 1), \\
(b) & \quad u(x, 0) = \sin(\pi x), \quad u_t(x, 0) = -\sin(\pi x), \quad x \in (0, 1), \\
(c) & \quad u(0, t) = u(1, t) = 0, \quad t \in [0, 1].
\end{align}

The exact solution of the problem (4.1) is \( u = e^{-t} \sin(\pi x), (x, t) \in [0, 1] \times [0, 1]. \)

For simplicity, we choose a constant spatial step \( h \) and a constant time step \( \tau \). The corresponding nodes are denoted by \( x_i = ih, i = 0, 1, \ldots, 2r \) and the time levels by \( t_j = j\tau, j = 0, 1, \ldots, 2N \).

The discrete forms \( \|u\|_{0,h} \) and \( |u|_{1,h} \) of forms \( \|u\| \) and \( |u|_1 \) are calculated by the compound Simpson formula. Here we choose \( r = N \); this yields \( h = \tau \). The program is written in MATLAB and run in Windows XP. By the calculated approximate solution \( u_{ht} \), we get the discrete norms \( \|u - u_{ht}\|_{0,h} \) and \( |u - u_{ht}|_{1,h} \), which are given in Tables 1 and 2.

From Tables 1 and 2, we can see that the scheme (2.10) is indeed efficient. For \( h = \tau \), we get the first-order optimal convergence under the discrete norm \( \|u\|_{0,h} \) and the discrete norm \( |u|_{1,h} \), respectively (see Table 3). Meanwhile, the scheme (2.10) is stable.

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References


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