Research Article

A New Approach to the Approximation of Common Fixed Points of an Infinite Family of Relatively Quasinonexpansive Mappings with Applications

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By using a specific way of choosing the indexes, we propose an iteration algorithm generated by the monotone CQ method for approximating common fixed points of an infinite family of relatively quasinonexpansive mappings. A strong convergence theorem without the stronger assumptions of the AKTT condition and the \(^{\ast}\)AKTT condition imposed on the involved mappings is established in the framework of Banach space. As application, an iterative solution to a system of equilibrium problems is studied. The result is more applicable than those of other authors with related interest.

1. Introduction

Let \( C \) be a nonempty and closed convex subset of a real Banach space \( E \). A mapping \( T : C \to E \) is said to be nonexpansive if

\[
\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.1}
\]

A mapping \( T \) is said to be quasi-nonexpansive if \( F(T) := \{x \in C : x = Tx\} \neq \emptyset \) and

\[
\|Tx - p\| \leq \|x - p\|, \quad \forall x \in C, \ p \in F(T). \tag{1.2}
\]
It is easy to see that if $T$ is nonexpansive with $F(T) \neq \emptyset$, then it is quasi-nonexpansive. There are many methods for approximating fixed points of quasi-nonexpansive mappings. In 1953, Mann [1] introduced the iteration as follows: a sequence $\{x_n\}$ is defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n,$$  \hspace{1cm} (1.3)

where the initial element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a sequence of real numbers in $[0,1]$. Approximation of fixed points of nonexpansive mappings via Mann’s algorithm has extensively been investigated. One of the fundamental convergence results was proved by Reich [2]. In infinite-dimensional Hilbert spaces, Mann iteration can yield only weak convergence (see [3, 4]).

Attempts to modify the Mann iteration method (1.3) for strong convergence have recently been made. Nakajo and Takahashi [5] proposed the following modification of Mann iteration method (1.3) for a nonexpansive mapping $T$ from $C$ into itself in a Hilbert space: from an arbitrary $x_0 \in C$,

$$y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n,$$

$$C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\},$$

$$Q_n = \{z \in C : (x_n - z, x_0 - x_n) \geq 0\},$$

$$x_{n+1} = P_{C_n \cap Q_n} x_0, \hspace{1cm} \forall n \geq 0,$$  \hspace{1cm} (1.4)

where $P_K$ denotes the metric projection from a Hilbert space $H$ onto a closed convex subset $K$ of $H$. They proved that the sequence $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Recently, Su and Qin [6] introduced a monotone CQ method for nonexpansive mapping, defined as follows: from an arbitrary $x_0 \in C$,

$$y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n,$$

$$C_0 = \{z \in C : \|y_0 - z\| \leq \|x_0 - z\|\}, \hspace{1cm} Q_0 = C,$$

$$C_n = \{z \in C_{n-1} \cap Q_{n-1} : \|y_n - z\| \leq \|x_n - z\|\},$$

$$Q_n = \{z \in C_{n-1} \cap Q_{n-1} : (x_n - z, x_0 - x_n) \geq 0\},$$

$$x_{n+1} = P_{C_n \cap Q_n} x_0, \hspace{1cm} \forall n \geq 0,$$  \hspace{1cm} (1.5)

and it proved that the sequence $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

We now recall some definitions concerning relatively quasi-nonexpansive mappings. Let $E$ be a real smooth Banach space with norm $\|\cdot\|$ and let $E^*$ be the dual of $E$. The normalized duality mapping $J$ from $E$ to $E^*$ is defined by

$$Jx = \{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \}, \hspace{1cm} \forall x \in E,$$  \hspace{1cm} (1.6)
where \( \langle \cdot, \cdot \rangle \) denotes the pairing between \( E \) and \( E^* \). Readers are directed to [7] (and its review [8]), where the properties of the duality mapping and several related topics are presented. The function \( \phi : E \times E \rightarrow \mathbb{R} \) is defined by

\[
\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E.
\]

Let \( T \) be a mapping from \( C \) into \( E \). A point \( p \) in \( C \) is said to be an asymptotic fixed point [9] of \( T \) if \( C \) contains a sequence \( \{x_n\} \) which converges weakly to \( p \) and \( \lim_{n \to \infty} (x_n - Tx_n) = 0 \). The set of asymptotic fixed points of \( T \) is denoted by \( \hat{F}(T) \).

We say that the mapping \( T \) is relatively nonexpansive (see [10]) if the following conditions are satisfied:

\[
\begin{align*}
(R1) \quad & F(T) \neq \emptyset; \\
(R2) \quad & \phi(p, Tx) \leq \phi(p, x), \quad \forall p \in F(T); \\
(R3) \quad & F(T) = \hat{F}(T).
\end{align*}
\]

If \( T \) satisfies (R1) and (R2), then \( T \) is called relatively quasi-nonexpansive.

Several articles have provided methods for approximating fixed points of relatively quasi-nonexpansive mappings [11–16]. Employing the ideas of Su and Qin [6], and of Aoyama et al. [17], in 2008, Nilsrakoo and Saejung [18] used the following iterations to obtain strong convergence theorems for common fixed points of a countable family of relatively quasi-nonexpansive mappings in a Banach space

\[
\begin{align*}
x_0 & \in C, \quad C_{-1} = Q_{-1} = C; \\
y_n & = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JTx_n), \\
C_n & = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \\
Q_n & = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\
x_{n+1} & = \Pi_n x_0, \quad \forall n \geq 0.
\end{align*}
\]

However, the results were obtained under two stronger assumption conditions, namely, the AKTT-condition and the \(^\star\)AKTT-condition imposed on the involved mappings.

Inspired and motivated by those studies mentioned above, in this paper, we use a modified type of the iteration scheme (1.8) for approximating common fixed points of an infinite family of relatively quasi-nonexpansive mappings; without stronger assumptions imposed on the involved mappings, a strong convergence theorem in Banach spaces is obtained for solving a system of equilibrium problems. The results improve those of other authors with related interest.
2. Preliminaries

Throughout the paper, let $E$ be a real Banach space. We say that $E$ is strictly convex if the following implication holds for $x, y \in E$:

$$\|x\| = \|y\| = 1, \quad x \neq y \implies \left\| \frac{x + y}{2} \right\| < 1. \quad (2.1)$$

It is also said to be uniformly convex if for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \quad \|x - y\| \geq \epsilon \implies \left\| \frac{x + y}{2} \right\| \leq 1 - \delta. \quad (2.2)$$

It is known that if $E$ is uniformly convex Banach space, then $E$ is reflexive and strictly convex. A Banach space $E$ is said to be smooth if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S(E) := \{x \in E : \|x\| = 1\}$. In this case, the norm of $E$ is said to be Gâteaux differentiable. The space $E$ is said to have uniformly Gâteaux differentiable norm if for each $y \in S(E)$; the limit (2.3) is attained uniformly for $x \in S(E)$. The norm of $E$ is said to be Fréchet differentiable if for each $x \in S(E)$; the limit (2.3) is attained uniformly for $y \in S(E)$. The norm of $E$ is said to be uniformly Fréchet differentiable (and $E$ is said to be uniformly smooth) if the limit (2.3) is attained uniformly for $x, y \in S(E)$.

We also know the following properties (see, e.g., [19] for details).

1. $E$ ($E^*$, resp.) is uniformly convex $\iff E^*$ ($E$, resp.) is uniformly smooth.
2. $Jx \neq \emptyset$ for each $x \in E$.
3. If $E$ is reflexive, then $J$ is a mapping from $E$ onto $E^*$.
4. If $E$ is strictly convex, then $Jx \cap Jy = \emptyset$ as $x \neq y$.
5. If $E$ is smooth, then $J$ is single-valued.
6. If $E$ has a Fréchet differentiable norm, then $J$ is norm-to-norm continuous.
7. If $E$ is uniformly smooth, then $J$ is uniformly norm-to-norm continuous on each bounded subset of $E$.
8. If $E$ is a Hilbert space, then $J$ is the identity operator.

Let $E$ be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}^+$ is defined by

$$\phi(x, y) := \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2. \quad (2.4)$$

It is obvious from the definition of the function $\phi$ that

$$\left(\|x\| - \|y\|\right)^2 \leq \phi(x, y) \leq \left(\|x\| + \|y\|\right)^2. \quad (2.5)$$

Moreover, we know the following results.
Lemma 2.1 (see [13]). Let $E$ be a strictly convex and smooth Banach space, then $\phi(x, y) = 0$ if and only if $x = y$.

Lemma 2.2 (see [11]). Let $E$ be a uniformly convex and smooth Banach space and let $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and

$$g(||x|| - ||y||) \leq \phi(x, y)$$

for all $x, y \in B_r := \{z \in E : Pz \leq r\}$.

Let $C$ be a nonempty and closed convex subset of $E$. Suppose that $E$ is reflexive, strictly convex, and smooth. It is known in [20] that for any $x \in E$, there exists a unique point $x^* \in C$ such that

$$\phi(x^*, x) = \min_{y \in C} \phi(y, x).$$

Following Alber [21], we denote such an $x^*$ by $\Pi_C x$. The mapping $\Pi_C$ is called the \textit{generalized projection} from $E$ onto $C$. It is easy to see that in a Hilbert space, the mapping $\Pi_C$ coincides with the metric projection $P_C$. What follows are the well-known facts concerning the generalized projection.

Lemma 2.3 (see [20]). Let $C$ be a nonempty closed convex subset of a smooth Banach space $E$ and let $x \in E$. Then

$$x^* = \Pi_C x \iff \langle x^* - y, Jx - Jx^* \rangle \geq 0, \quad \forall y \in C.$$  \hfill (2.8)

Lemma 2.4 (see [20]). Let $E$ be a reflexive, strictly convex, and smooth Banach space, let $C$ be a nonempty closed convex subset of $E$, and let $x \in E$. Then

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$  \hfill (2.9)

Dealing with the generalized projection from $E$ onto the fixed point set of a relatively quasi-nonexpansive mapping, we have the following result.

Lemma 2.5 (see [18]). Let $E$ be a strictly convex and smooth Banach space, let $C$ be a nonempty and closed convex subset of $E$, and let $T$ be a relatively quasi-nonexpansive mapping from $C$ into $E$. Then $F(T)$ is closed and convex.

Let $C$ be a subset of a Banach space $E$ and let $\{T_n\}$ be a family of mappings from $C$ into $E$. For a subset $B$ of $C$, we say that

(i) $\{\{T_n\}, B\}$ satisfies AKTT-condition if

$$\sum_{n=1}^{\infty} \sup \{||T_{n+1}z - T_nz|| : z \in B\} < \infty;$$  \hfill (2.10)
(ii) \( (\{T_n\}, B) \) satisfies *AKTT-condition if
\[
\sum_{n=1}^{\infty} \sup \{ \| JT_{n+1}z - JT_n z \| : z \in B \} < \infty.
\] (2.11)

3. Main Results

Recall that an operator \( T \) in a Banach space is closed if \( x_n \to x \) and \( Tx_n \to y \) as \( n \to \infty \), then \( Tx = y \).

**Theorem 3.1.** Let \( E \) be a uniformly convex and uniformly smooth Banach space, \( C \) a nonempty and closed convex subset of \( E \). Let \( \{T_i\}_{i=1}^{\infty} : C \to E \) be a sequence of closed and relatively quasi-nonexpansive mappings with \( F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset \). Starting from an arbitrary \( x_1 \in C \), the sequence \( \{x_n\} \) is define by

\[
\begin{align*}
x_1 & \in C, \quad C_0 = Q_0 = C; \\
y_n & = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT_n x_n), \\
C_n & = \{ z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n) \}, \\
Q_n & = \{ z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_1 - Jx_n \rangle \geq 0 \}, \\
x_{n+1} & = \Pi_n x_1, \quad \forall n \geq 1,
\end{align*}
\]

where \( \Pi_n := \Pi_{C_n \cap Q_n} \) and \( \{\alpha_n\} \) is a sequence in \( [0, 1) \) with \( \lim \sup_{n \to \infty} \alpha_n < 1 \); \( i_n \) is the solution to the positive integer equation: \( n = i + (m - 1)m/2 \quad (m \geq i, n = 1, 2, \ldots) \), that is, for each \( n \geq 1 \), there exists a unique \( i_n \) such that

\[
i_1 = 1, \quad i_2 = 1, \quad i_3 = 2, \quad i_4 = 1, \quad i_5 = 2, \quad i_6 = 3, \quad i_7 = 1, \quad i_8 = 2,
\]

\[
(3.2)
\]

Then \( \{x_n\} \) converges strongly to \( \Pi_F x_1 \).

**Proof.** We first claim that both \( C_n \) and \( Q_n \) are closed and convex. This follows from the fact that \( \phi(z, y_n) \leq \phi(z, x_n) \) is equivalent to the following:

\[
2\langle z, Jx_n - Jy_n \rangle \leq \|x_n\|^2 - \|y_n\|^2. \tag{3.3}
\]

It is clear that \( F \subset C = C_0 \cap Q_0 \). Next, we show that

\[
F \subset C_n \cap Q_n, \quad \forall n \geq 1. \tag{3.4}
\]
Suppose that $F \subset C_{k-1} \cap Q_{k-1}$ for some $k \geq 2$. Letting $p \in F$, we then have

$$\phi(p, y_k) = \phi(p, J^{-1}(\alpha_k Jx_k + (1 - \alpha_k)JT_k - x_k))$$

$$= \|p\|^2 - 2\langle p, \alpha_k Jx_k + (1 - \alpha_k)JT_kx_k \rangle + \|\alpha_k Jx_k + (1 - \alpha_k)JT_kx_k\|^2$$

$$\leq \|p\|^2 - 2\alpha_k \langle p, Jx_k \rangle - 2(1 - \alpha_k) \langle p, JT_kx_k \rangle + \alpha_k \|x_k\|^2 + (1 - \alpha_k)\|T_kx_k\|^2$$

$$= \alpha_k \left( \|p\|^2 - 2\langle p, Jx_k \rangle + \|x_k\|^2 \right) + (1 - \alpha_k) \left( \|p\|^2 - 2\langle p, JT_kx_k \rangle + \|T_kx_k\|^2 \right)$$

(3.5)

$$= \alpha_k \phi(p, x_k) + (1 - \alpha_k)\phi(p, T_kx_k)$$

$$\leq \alpha_k \phi(p, x_k) + (1 - \alpha_k)\phi(p, x_k)$$

$$= \phi(p, x_k).$$

This implies that $F \subset C_k$. It follows from $x_k = \Pi_{k-1}x_1$ and Lemma 2.3 that

$$\langle x_k - z, Jx_1 - Jx_k \rangle \geq 0, \quad \forall z \in C_{k-1} \cap Q_{k-1}. \quad (3.6)$$

Particularly,

$$\langle x_k - z, Jx_1 - Jx_k \rangle \geq 0, \quad \forall p \in F \quad (3.7)$$

and hence $F \subset Q_k$, which yields that

$$F \subset C_k \cap Q_k. \quad (3.8)$$

By induction, (3.4) holds. This implies that $\{x_n\}$ is well defined. It follows from the definition of $Q_n$ and Lemma 2.3 that $x_n = \Pi_{Q_n}x_1$. Since $x_{n+1} = \Pi_{Q_n}x_1 \in Q_n$, we have

$$\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1), \quad \forall n \geq 1. \quad (3.9)$$

Therefore, $\{\phi(x_n, x_1)\}$ is nondecreasing. Using $x_n = \Pi_{Q_n}x_1$ and Lemma 2.4, we have

$$\phi(x_n, x_1) = \phi(\Pi_{Q_n}x_1, x_1) \leq \phi(p, x_1) - \phi(p, x_n) \leq \phi(p, x_1)$$

(3.10)

for all $p \in F$ and for all $n \geq 1$, that is, $\{\phi(x_n, x_1)\}$ is bounded. Then

$$\lim_{n \to \infty} \phi(x_n, x_1) \text{ exists.} \quad (3.11)$$
In particular, by (2.5), the sequence \( \{ (\|x_n\| - \|x_1\|)^2 \} \) is bounded. This implies that \( \{x_n\} \) is bounded. Note again that \( x_n = \Pi_{Q_n} x_1 \) and for any positive integer \( k, x_{n+k} \in Q_{n+k-1} \subset Q_n \). By Lemma 2.4,

\[
\phi(x_{n+k}, x_n) = \phi(x_{n+k}, \Pi_{Q_n} x_1) \\
\leq \phi(x_{n+k}, x_1) - \phi(\Pi_{Q_n} x_1, x_1) \\
= \phi(x_{n+k}, x_1) - \phi(x_n, x_1).
\]

(3.12)

By Lemma 2.2, we have, for any positive integers \( m, n \) with \( m > n \),

\[
g(\|x_m - x_n\|) \leq \phi(x_m, x_n) \leq \phi(x_m, x_1) - \phi(x_n, x_1),
\]

(3.13)

where \( g : [0, \infty) \rightarrow [0, \infty) \) is a continuous, strictly increasing, and convex function with \( g(0) = 0 \). Then the properties of the function \( g \) yield that \( \{x_n\} \) is a Cauchy sequence in \( C \), so there exists an \( x^* \in C \) such that

\[
x_n \rightarrow x^* \quad (n \rightarrow \infty).
\]

(3.14)

In view of \( x_{n+1} = \Pi_n x_1 \in C_n \) and the definition of \( C_n \), we also have

\[
\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n), \quad \forall n \geq 1.
\]

(3.15)

This implies that

\[
\lim_{n \rightarrow \infty} \phi(x_{n+1}, y_n) = \lim_{n \rightarrow \infty} \phi(x_{n+1}, x_n) = 0.
\]

(3.16)

It follows from Lemma 2.2 that

\[
\lim_{n \rightarrow \infty} \|x_{n+1} - y_n\| = \lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0.
\]

(3.17)

Since \( J \) is uniformly norm-to-norm continuous on bounded sets, we have

\[
\lim_{n \rightarrow \infty} \|Jx_{n+1} - Jy_n\| = \lim_{n \rightarrow \infty} \|Jx_{n+1} - Jx_n\| = 0.
\]

(3.18)

On the other hand, we have, for each \( n \geq 1 \),

\[
\|Jx_{n+1} - Jy_n\| = \|Jx_{n+1} - (\alpha_n Jx_n + (1 - \alpha_n) JT_i x_n)\| \\
= \|(1 - \alpha_n)(Jx_{n+1} - JT_i x_n) - \alpha_n (Jx_n - Jx_{n+1})\| \\
\geq (1 - \alpha_n) \|Jx_{n+1} - JT_i x_n\| - \alpha_n \|Jx_n - Jx_{n+1}\|,
\]

(3.19)
and hence
\[ \| Jx_{n+1} - JT_{i_n}x_n \| \leq \frac{1}{1 - \alpha_n} \| Jx_{n+1} - Jy_n \| + \frac{\alpha_n}{1 - \alpha_n} \| Jx_n - Jx_{n+1} \|. \] (3.20)

From (3.18) and \( \limsup_{n \to \infty} \alpha_n < 1 \), we obtain that
\[ \lim_{n \to \infty} \| Jx_{n+1} - JT_{i_n}x_n \| = 0. \] (3.21)

Since \( J^{-1} \) is uniformly norm-to-norm continuous on bounded sets, we have
\[ \lim_{n \to \infty} \| x_{n+1} - T_{i_n}x_n \| = \lim_{n \to \infty} \left\| J^{-1}(Jx_{n+1}) - J^{-1}(JT_{i_n}x_n) \right\| = 0. \] (3.22)

It follows from (3.17) that, as \( n \to \infty \),
\[ \| x_n - T_{i_n}x_n \| \leq \| x_n - x_{n+1} \| + \| x_{n+1} - T_{i_n}x_n \| \to 0. \] (3.23)

Now, set \( \mathcal{K}_i = \{ k \geq 1 : k = i + (m - 1)m/2, m \geq i, m \in \mathbb{Z}^+ \} \) for each \( i \geq 1 \). Note that \( T_{i_k} = T_i \) whenever \( k \in \mathcal{K}_i \). For example, by the definition of \( \mathcal{K}_1 \), we have \( \mathcal{K}_1 = \{ 1, 2, 4, 7, 11, 16, \ldots \} \) and \( i_1 = i_2 = i_4 = i_7 = i_{11} = i_{16} = \cdots = 1 \). Then it follows from (3.23) that
\[ \lim_{\mathcal{K}_i \ni k \to \infty} \| T_i x_k - x_k \| = 0, \quad \forall i \geq 1. \] (3.24)

Since \( \{ x_k \}_{k \in \mathcal{K}} \) is a subsequence of \( \{ x_n \} \), (3.14) implies that \( x_k \to x^* \) as \( \mathcal{K} \ni k \to \infty \). It immediately follows from (3.24) and the closedness of \( T_i \) that \( x^* \in F(T_i) \) for each \( i \geq 1 \), and hence \( x^* \in F \). Furthermore, by (3.10),
\[ \phi(x^*, x_1) = \lim_{n \to \infty} \phi(x_n, x_1) \leq \phi(p, x_1), \quad \forall p \in F. \] (3.25)

This implies that \( x^* = \Pi_F x_1 \). The proof is completed.

**Remark 3.2.** Note that the algorithm (3.1) is based on the projection onto an intersection of two closed and convex sets. An example [22] of how to compute such a projection is given as follows.
Dykstra’s Algorithm

Let $\Omega_1, \Omega_2, \ldots, \Omega_p$ be closed and convex subsets of $\mathbb{R}^n$. For any $i = 1, 2, \ldots, p$ and $x^0 \in \mathbb{R}^n$, the sequences $\{x_i^k\}$ are defined by the following recursive formulae:

$$
\begin{align*}
    x_0^k &= x_{p}^{k-1}, \\
    x_i^k &= P_{\Omega_i} \left( x_{i-1}^k - y_i^{k-1} \right), \quad i = 1, 2, \ldots, p, \\
    y_i^k &= x_i^k - \left( x_{i-1}^k - y_i^{k-1} \right), \quad i = 1, 2, \ldots, p,
\end{align*}
$$

(3.26)

for $k = 1, 2, \ldots$ with initial values $x_0^0 = x^0$ and $y_i^0 = 0$ for $i = 1, 2, \ldots, p$. If $\Omega := \bigcap_{i=1}^{p} \Omega_i \neq \emptyset$, then $\{x_i^k\}$ converges to $x^* = P_{\Omega}(x^0)$, where $P_{\Omega}(x) := \arg\inf_{y \in \Omega} \|y - x\|^2$, for all $x \in \mathbb{R}^n$.

4. Applications

The so-called convex feasibility problem for a family of mappings $\{T_i\}_{i=1}^{\infty}$ is to find a point in the nonempty intersection $\bigcap_{i=1}^{\infty} F(T_i)$, which exactly illustrates the importance of finding fixed points of infinite families. The following example also clarifies the same thing.

Example 4.1. Let $E$ be a smooth, strictly convex, and reflexive Banach space, $C$ a nonempty and closed convex subset of $E$, and $\{f_i\}_{i=1}^{\infty} : C \to C$ a countable family of bifunctions satisfying the conditions: for each $i \geq 1$,

(A1) $f_i(x, x) = 0$;
(A2) $f_i$ is monotone, that is, $f_i(x, y) + f_i(y, x) \leq 0$;
(A3) $\limsup_{t \to 0} f_i(x + t(z - x), y) \leq f_i(x, y)$;
(A4) the mapping $y \mapsto f_i(x, y)$ is convex and lower semicontinuous.

A system of equilibrium problems for $\{f_i\}_{i=1}^{\infty}$ is to find an $x^* \in C$ such that

$$
f_i(x^*, y) \geq 0, \quad \forall y \in C, \ i \geq 1,
$$

(4.1)

whose set of common solutions is denoted by $EP := \bigcap_{i=1}^{\infty} EP(f_i)$, where $EP(f_i)$ denotes the set of solutions to the equilibrium problem for $f_i$ $(i = 1, 2, \ldots)$. It will be shown in Theorem 4.3 that such a system of problems can be reduced to approximation of some fixed points of a countable family of nonexpansive mappings.

Example 4.2 (see [23]). Let $r > 0$. Define a countable family of mappings $\{T_{r,i}\}_{i=1}^{\infty} : E \to C$ as follows:

$$
T_{r,i}(x) = \left\{ z \in C : f_i(z, y) + \frac{1}{r} (y - z, Jz - Jx) \geq 0, \forall y \in C \right\}, \quad \forall i \geq 1.
$$

(4.2)
Then we have that

1. \( \{T_{r,i}\}_{i=1}^{\infty} \) is a sequence of single-valued mappings;

2. \( \{T_{r,i}\}_{i=1}^{\infty} \) is a sequence of closed relatively quasi-nonexpansive mappings;

3. \( F := \bigcap_{i=1}^{\infty} F(T_{r,i}) = EP. \)

Now, we have the following result.

**Theorem 4.3.** Let \( C, E, \) and \( \{a_n\} \) be the same as those in Theorem 3.1. Let \( \{f_i\}_{i=1}^{\infty} : C \to C \) be a countable family of bifunctions satisfying the conditions \( (A_1)-(A_4) \). Let \( \{T_{r,i}\}_{i=1}^{\infty} : E \to C \) be a countable family of mappings defined by (4.2). Let \( \{x_n\} \) be the sequence generated by

\[
x_1 \in C, \quad C_0 = Q_0 = C;
\]

\[
f_i(u_n, y) + \frac{1}{r} (y - u_n, J u_n - J x_n) \geq 0, \quad \forall y \in C,
\]

\[
y_n = J^{-1}(a_n J x_n + (1 - a_n) J u_n),
\]

\[
C_n = \{ z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n) \},
\]

\[
Q_n = \{ z \in C_{n-1} \cap Q_{n-1} : (x_n - z, J x_1 - J x_n) \geq 0 \},
\]

\[
x_{n+1} = \Pi_n x_1, \quad \forall n \geq 1,
\]

where \( i_n \) satisfies the positive integer equation: \( n = i + (m - 1)m/2 \) (\( m \geq i, n = 1, 2, \ldots \)). If \( F := \bigcap_{i=1}^{\infty} F(T_{r,i}) \neq \emptyset \), then \( \{x_n\} \) strongly converges to \( \Pi F x_1 \) which is a common solution of the system of equilibrium problems for \( \{f_i\}_{i=1}^{\infty} \).

**Proof.** Since each \( T_{r,i} \) is single-valued, \( u_n = T_{r,i_n} x_n \) for all \( n \geq 1 \). In addition, we have pointed out in Example 4.2 that \( F = EP \) and \( \{T_{r,i}\}_{i=1}^{\infty} \) is a sequence of closed relatively quasi-nonexpansive mappings. Hence, (4.3) can be rewritten as follows:

\[
x_1 \in C, \quad C_0 = Q_0 = C;
\]

\[
y_n = J^{-1}(a_n J x_n + (1 - a_n) J T_{r,i_n} x_n),
\]

\[
C_n = \{ z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n) \},
\]

\[
Q_n = \{ z \in C_{n-1} \cap Q_{n-1} : (x_n - z, J x_1 - J x_n) \geq 0 \},
\]

\[
x_{n+1} = \Pi_n x_1, \quad \forall n \geq 1.
\]

Therefore, this conclusion can be obtained immediately from Theorem 3.1. \( \square \)
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References


