The Expression of the Generalized Drazin Inverse of \( A - CB \)

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1. Introduction

Let \( \mathcal{X} \) and \( \mathcal{Y} \) be Banach spaces. We denote the set of all bounded linear operators from \( \mathcal{X} \) to \( \mathcal{Y} \) by \( B(\mathcal{X}, \mathcal{Y}) \). In particular, we write \( B(\mathcal{X}) \) instead of \( B(\mathcal{X}, \mathcal{X}) \).

For any \( A \in B(\mathcal{X}, \mathcal{Y}) \), \( \mathcal{R}(A) \) and \( \mathcal{N}(A) \) represent its range and null space, respectively.

If \( A \in B(\mathcal{X}) \), the symbols \( \sigma(A) \) and \( \text{acc}(\sigma(A)) \) stand for its spectrum and the set of all accumulation points of \( \sigma(A) \), respectively.

Recall the concept of the generalized Drazin inverse introduced by Koliha [1] that the element \( T_d \in B(\mathcal{X}) \) is called the generalized Drazin inverse of \( T \in B(\mathcal{X}) \) provided it satisfies

\[
TT_d = T_d T, \quad T_d TT_d = T_d, \quad T - T^2 T_d \text{ is quasinilpotent.} \tag{1.1}
\]

If it exists then it is unique. The Drazin index \( \text{Ind}(T) \) of \( T \) is the least positive integer \( k \) if \( (T - T^2 T_d)^k = 0 \), and otherwise \( \text{Ind}(T) = +\infty \).

From the definition of the generalized Drazin inverse, it is easy to see that if \( T \) is a quasinilpotent operator, then \( T_d \) exists and \( T_d = 0 \). It is well known that the generalized Drazin inverse of \( T \in B(\mathcal{X}) \) exists if and only if \( 0 \notin \text{acc}(\sigma(T)) \) (see [1, Theorem 4.2]).
If \( T \) is generalized Drazin invertible, then the spectral idempotent \( T^\pi \) of \( T \) corresponding to \( 0 \) is given by \( T^\pi = I - TT^d \).

The generalized Drazin inverse is widely investigated because of its applications in singular differential difference equations, Markov chains, (semi-) iterative method numerical analysis (see, for example, [1–5, 7], and references therein).

In this paper, we aim to discuss the generalized Drazin inverse of \( A - CB \) over Banach spaces. This question stems from the Drazin inverse of a modified matrix (see, e.g., [6]). In [3], Deng studied the generalized Drazin inverse of \( A - CB \). Here we research the problem under more general conditions than those in [3]. Our results extend the relative results in [3, 4].

In this section, we will list some lemmas. In next section, we will present the expressions of the generalized Drazin inverse of \( A - CB \). In final section, we illustrate a simple example.

**Lemma 1.1** (see [4, Theorem 2.3]). Let \( A, B \in \mathcal{B}(X) \) be the generalized Drazin invertible. If \( AB = 0 \), then \( A + B \) is generalized Drazin invertible and

\[
(A + B)_d = B^\pi \sum_{n=0}^{\infty} B^n A_n^d A^\pi_0 + \left( \sum_{n=0}^{\infty} B_n^d A_n^\pi \right) A^\pi.
\]  

**Lemma 1.2** (see [7, Theorem 5.1]). If \( A \in \mathcal{B}(X) \) and \( B \in \mathcal{B}(Y) \) are generalized Drazin invertible and \( C \in \mathcal{B}(Y, X) \), then

\[
M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}
\]  

is also generalized Drazin invertible and

\[
M^d = \begin{pmatrix} A^d & S \\ O & B^d \end{pmatrix},
\]  

where

\[
S = A_0^2 \left( \sum_{n=0}^{\infty} A_n^d CB^n \right) B^\pi + A^\pi \left( \sum_{n=0}^{\infty} A_n^d CB_n^d \right) B_0^2 - A_0^d CB_0^d.
\]

2. **Main Results**

We start with our main result.
Theorem 2.1. Let $A \in \mathcal{B}(\mathcal{X})$ be the generalized Drazin invertible, $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. Suppose that there exists a $P \in \mathcal{B}(\mathcal{X})$ such that $AP = PAP$ and $BP = 0$. If $R = (I - P)(A - CB)$ and $AP$ are generalized Drazin invertible, then $A - CB$ is generalized Drazin invertible and

$$
(A - CB)_d = \left[ \sum_{n=0}^{\infty} (AP)^{n+1} \left( R^n + VR^{n-1} + V^2 R^{n-2} \right) \right] R^\pi
$$

$$
- (AP)_d \left[ VR_d + V^2 R_d^2 + (AP)_d V^2 R_d \right]
$$

$$
+ (AP)^{2} \sum_{n=0}^{\infty} (AP)^{n} \left( R^{n+1} + VR^{n+2} + V^2 R^{n+3} \right),
$$

where $V = PA - PCB - AP$ and the symbols $V^i R^j = 0$, $i = 1, 2$, if $j < 0$.

Proof. Let $S := AP$ and $T := (A - CB)(I - P)$. Then

$$
TS = (A - CB)(I - P)AP = 0,
$$

$$
RP = (I - P)(A - CB)P = 0,
$$

$$
A - CB = AP + A(I - P) - CB(I - P) = S + T
$$

since $AP = PAP$ and $BP = 0$. So, by Lemma 1.1,

$$
(T + S)_d = S^{\pi} \sum_{m=0}^{\infty} S_d^n T_d^{n+1} + \sum_{m=0}^{\infty} S_d^{n+1} T_d^n T^{\pi}.
$$

Next, we will give the representations of $T_d$, $T^n$, and $T^n_d$. In order to obtain the expression of $T_d$, rewrite $T$ as

$$
T = R + PA - PCB - PAP = R + V.
$$

Since $VP = PAP - AP^2 = PAP(I - P)$,

$$
V^2 P = (PA - PCB - AP)PAP(I - P) = (PAPAP - APPAP)(I - P) = 0,
$$

and then $V^n = 0$ for $n > 2$ since $V = PA - CB - AP$. So $V_d$ exists and $V_d = 0$. By (2.3), $RV = RP(A - CB - AP) = 0$ and then $R_d V = R_d R_d RV = 0$. So, by Lemma 1.1,

$$
T_d = (R + V)_d = R_d + VR_d^2 + V^2 R_d^3,
$$

and then

$$
TT_d = RR_d + VR_d + V^2 R_d^2.
$$
Since $R(R + V)^k = R^{k+1}$ and $V^2(R + V)^k = V^2R^k$ for $k \geq 1$,

$$T^n = (R + V)^n = \left( R^2 + VR + V^2 \right) (R + V)^{n-2} = R^n + VR^{n-1} + V^2R^{n-2}, \quad n \geq 2. \quad (2.10)$$

From $R_dV = 0$, it is easy to verify that

$$T^n_d = \left( R_d + VR^2_d + V^2R^3_d \right)^n = R^n_d + VR^{n+1}_d + V^2R^{n+2}_d. \quad (2.11)$$

Hence,

$$\left( \sum_{n=0}^{\infty} S^{n+1}_d T^n \right) T^\pi = (AP)_d \left[ I + (AP)_d (R + V) + (AP)_d^2 \left( R^2 + VR + V^2 \right) \right]$$

$$\times \left( R^\pi - VR_d - V^2R^3_d \right) + \sum_{n=3}^{\infty} (AP)_d^{n+1} \left( R^n + VR^{n+1} + V^2R^{n-2} \right) R^\pi$$

$$= (AP)_d \left[ I + (AP)_d (R + V) + (AP)_d^2 \left( R^2 + VR + V^2 \right) \right] R^\pi$$

$$- (AP)_d \left( VR_d + V^2R^3_d + (AP)_d V^2R_d \right)$$

$$+ \sum_{n=3}^{\infty} (AP)_d^{n+1} \left( R^n + VR^{n+1} + V^2R^{n-2} \right) R^\pi,$$

$$S^\pi \sum_{n=0}^{\infty} S^n T^n_d = (AP)^\pi \sum_{n=0}^{\infty} (AP)^n \left( R^{n+1}_d + VR^{n+2}_d + V^2R^{n+3}_d \right). \quad (2.12)$$

Therefore, we reach (2.1).

When $\text{Ind}(AP)$, $\text{Ind}(R) < +\infty$, we have the following corollary.

**Corollary 2.2.** Let $A \in \mathcal{B}(\mathcal{X})$ be generalized Drazin invertible, $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. Suppose that there exists a $P \in \mathcal{B}(\mathcal{X})$ such that $AP = PAP$ and $BP = 0$. If $R = (I - P)(A - CB)$ and $AP$ are generalized Drazin invertible and $\text{Ind}(R) = k < +\infty$ and $\text{Ind}(AP) = h < +\infty$, then $A - CB$ is generalized Drazin invertible and

$$(A - CB)_d = \left[ \sum_{n=0}^{k-1} (AP)_d^{n+1} \left( R^n + VR^{n+1} + V^2R^{n-2} \right) \right] R^\pi$$

$$- (AP)_d \left[ VR_d + V^2R^3_d + (AP)_d V^2R_d \right]$$

$$+ (AP)^h \sum_{n=0}^{h-1} (AP)^n \left( R^{n+1}_d + VR^{n+2}_d + V^2R^{n+3}_d \right), \quad (2.13)$$

where $V = PA - PCB - AP$ and the symbols $V^i R^i = 0, i = 1, 2, \text{if } j < 0$.

If an operator $T$ is quasinilpotent, $T_d = 0$ and $T^\pi = I$. So, the following corollary follows from Theorem 2.1.
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**Corollary 2.3.** Let \( A \in \mathcal{B}(\mathcal{X}) \) be generalized Drazin invertible, \( C \in \mathcal{B}(\mathcal{X}, \mathcal{Y}) \), and \( B \in \mathcal{B}(\mathcal{Y}, \mathcal{X}) \). Suppose that there exists a \( P \in \mathcal{B}(\mathcal{X}) \) such that \( AP = PAP \) and \( BP = 0 \). If \( R = (I - P)(A - CB) \) is generalized Drazin invertible and \( AP \) is a quasinilpotent operator, then \( A - CB \) is generalized Drazin invertible and

\[
(A - CB)_d = \sum_{n=0}^{\infty} (AP)^n \left( R_d^{n+1} + VR_d^{n+2} + V^2 R_d^{n+3} \right),
\]

(2.14)

where \( V = PA - PCB - AP \).

**Theorem 2.4.** Let \( A \in \mathcal{B}(\mathcal{X}) \) be generalized Drazin invertible, \( C \in \mathcal{B}(\mathcal{X}, \mathcal{Y}) \), and \( B \in \mathcal{B}(\mathcal{Y}, \mathcal{X}) \). Suppose that there exists an idempotent \( P \in \mathcal{B}(\mathcal{X}) \) such that \( PA = PAP \) and \( BP = B \). If \( R = P(A - CB) \) is generalized Drazin invertible, then \( A - CB \) is generalized Drazin invertible and

\[
(A - CB)_d = R_d + A_d(I - P) + \sum_{n=0}^{\infty} A_d^{n+1} (I - P)(A - CB)P(A - CB)^n R^\tau
\]

\[
+ A^\tau \sum_{n=0}^{\infty} A^n (I - P)(A - CB)PR_d^{n+2} - A_d(I - P)(A - CB)R_d.
\]

(2.15)

**Proof.** Since \( P^2 = P \), we have \( \mathcal{X} = \mathcal{R}(P) \oplus \mathcal{N}(P) \) and can write \( P \) in the following matrix form:

\[
P = \begin{pmatrix}
I & 0 \\
0 & 0
\end{pmatrix}.
\]

(2.16)

The condition \( PA = PAP \), therefore, yields the matrix form of \( A \) as follows:

\[
A = \begin{pmatrix}
A_1 & 0 \\
A_3 & A_2
\end{pmatrix}.
\]

(2.17)

From \( \sigma(A) = \sigma(A_1) \cup \sigma(A_2) \) and the hypothesis that \( A_d \) exists, \( A_1 \in \mathcal{B}(\mathcal{R}(P)) \) and \( A_2 \in \mathcal{B}(\mathcal{N}(P)) \) are generalized Drazin invertible since \( 0 \notin \text{acc}(\sigma(A)) \) if and only if \( 0 \notin \text{acc}(\sigma(A_1)) \) and \( 0 \notin \text{acc}(\sigma(A_2)) \). And, by Lemma 1.2,

\[
A_d = \begin{pmatrix}
A_1^d & 0 \\
W & A_2^d
\end{pmatrix},
\]

(2.18)

where \( W \) is some operator. Since

\[
A(I - P) = \begin{pmatrix}
0 & 0 \\
0 & A_2
\end{pmatrix},
\]

(2.19)

\( (A(I - P))_d \) exists and

\[
(A(I - P))_d = \begin{pmatrix}
0 & 0 \\
0 & A_2^d
\end{pmatrix} = A_d(I - P).
\]

(2.20)
To use Theorem 2.1 to complete the proof, let $Q = (I - P)$. So $R = (I - Q)(A - CB)$ and $AQ$ are generalized Drazin invertible. And from the conditions $PA = PAP$ and $BP = B$, we can obtain $AQ = QAQ$ and $BQ = 0$. Thus, by Theorem 2.1, we have

\[
(A - CB)_d = (AQ)_d R^\pi + (AQ)^2_d (R + V) R^\pi + \left[ \sum_{n=2}^{\infty} (AQ)^{n+1}_d (R^n + VR^{n-1} + V^2 R^{n-2}) \right] R^\pi \\
- (AQ)^d \left[ VR_d + V^2 R^2_d + (AQ)_d VR_d \right] + (AQ)^\pi \left( R_d + VR_d^2 + V^2 R^3_d \right) \\
+ (AQ)^\pi \sum_{n=1}^{\infty} (AP)^n (R^{n+1}_d + VR^{n+2}_d + V^2 R^{n+3}_d),
\]

where $V = QA - QCB - AQ$.

Since $P^2 = P$ and $Q^2 = Q$ and then $VQ = 0$ and $V = QV$. So $V^2 = 0$. Note that $QR = 0$ and then $QR_d = 0$ and $(AQ)_d R = 0$. Thus it follows from (2.21) that

\[
(A - CB)_d = (AQ)_d + (AQ)_d^2 VR^\pi + \left[ \sum_{n=2}^{\infty} (AQ)_d^{n+1} VR^{n-1}_d \right] R^\pi - (AQ)_d VR_d \\
+ R_d + (AQ)^\pi VR_d^2 + (AQ)^\pi \sum_{n=1}^{\infty} (AQ)^n VR^{n+2}_d \\
= (AQ)_d + \left[ \sum_{n=0}^{\infty} (AQ)_d^{n+2} VR^n \right] R^\pi - (AQ)_d VR_d + R_d \\
+ (AQ)^\pi \sum_{n=0}^{\infty} (AQ)^n V(R_d)^{n+2}.
\]


Adding the condition $PC = C$ in Theorem 2.4 yields a result below.

**Corollary 2.5.** Let $A \in \mathcal{B}(X)$ be generalized Drazin invertible, $C \in \mathcal{B}(X, Y)$, and $B \in \mathcal{B}(Z, X)$. Suppose that there exists an idempotent $P \in \mathcal{B}(X)$ such that $PA = PAP$, $BP = B$, and $PC = C$. If $R = P(A - CB)$ is generalized Drazin invertible, then $A - CB$ is generalized Drazin invertible and

\[
(A - CB)_d = R_d + A_d (I - P) + \sum_{n=1}^{\infty} A^{n+2}_d (I - P) AP(A - CB)^n R^\pi \\
+ A^\pi \sum_{n=0}^{\infty} A^n (I - P) AR_d R^{n+2} - A_d (I - P) AR_d.
\]

Adding the condition $PC = 0$ in Theorem 2.4 yields $R = PA$. So similar to the proof of $(A(I - P))^d = A^d (I - P)$ in Theorem 2.4, we can gain $(PA)^d = PA^d$. 


Corollary 2.6. Let \( A \in \mathcal{B}(\mathcal{X}) \) be generalized Drazin invertible, \( C \in \mathcal{B}(\mathcal{X}, \mathcal{Y}) \), and \( B \in \mathcal{B}(\mathcal{Z}, \mathcal{X}) \). Suppose that there exists an idempotent \( P \in \mathcal{B}(\mathcal{X}) \) such that \( PA = PAP, BP = B, \) and \( PC = 0 \); then \( A - CB \) is generalized Drazin invertible and

\[
(A - CB)_d = A_d + \sum_{n=0}^{\infty} A_d^{n+2} (I - P)(A - CB)PA^n A^\pi + A^\pi \sum_{n=0}^{\infty} A^n (I - P)(A - CB)PA_d^{n+2} - A_d(I - P)(A - CB) PA_d.
\]

(2.24)

Analogously, we can deduce Theorem 2.7 and Corollary 2.9 below.

Theorem 2.7. Let \( A \in \mathcal{B}(\mathcal{X}) \) be generalized Drazin invertible, \( C \in \mathcal{B}(\mathcal{X}, \mathcal{Y}) \), and \( B \in \mathcal{B}(\mathcal{Y}, \mathcal{X}) \). Suppose that there exists an idempotent \( P \in \mathcal{B}(\mathcal{X}) \) such that \( AP = PAP \) and \( PC = C \). If \( R = (A - CB)P \) is generalized Drazin invertible, then \( A - CB \) is generalized Drazin invertible and

\[
(A - CB)_d = R_d + (I - P) A_d + \sum_{n=0}^{\infty} R_d^{n+2} P(A - CB)(I - P) A^n A^\pi + R^\pi \sum_{n=0}^{\infty} (A - CB)^n P(A - CB)(I - P) A_d^{n+2} - R_d(A - CB)(I - P) A_d.
\]

(2.25)

Remark 2.8 (see [4, Theorem 2.4]). It is a special case of Theorem 2.7.

Corollary 2.9. Let \( A \in \mathcal{B}(\mathcal{X}) \) be generalized Drazin invertible, \( C \in \mathcal{B}(\mathcal{X}, \mathcal{Y}) \), and \( B \in \mathcal{B}(\mathcal{Z}, \mathcal{X}) \). Suppose that there exists an idempotent \( P \in \mathcal{B}(\mathcal{X}) \) such that \( AP = PAP, PC = C, \) and \( BP = 0 \); then \( A - CB \) is generalized Drazin invertible and

\[
(A - CB)_d = A_d + \sum_{n=0}^{\infty} A_d^{n+2} P(A - CB)(I - P) A^n A^\pi + A^\pi \sum_{n=0}^{\infty} A^n P(A - CB)(I - P) A_d^{n+2} - A_d P(A - CB)(I - P) A_d.
\]

(2.26)

Similar to Theorem 2.1 and Corollary 2.2, we can show the following two results.

Theorem 2.10. Let \( A \in \mathcal{B}(\mathcal{X}) \) be generalized Drazin invertible, \( C \in \mathcal{B}(\mathcal{X}, \mathcal{Y}) \), and \( B \in \mathcal{B}(\mathcal{Y}, \mathcal{X}) \). Suppose that there exists a \( P \in \mathcal{B}(\mathcal{X}) \) such that \( PA = PAP \) and \( PC = 0 \). If \( R = (A - CB)(I - P) \) and \( PA \) are generalized Drazin invertible, then \( A - CB \) is generalized Drazin invertible and

\[
(A - CB)_d = R^\pi \sum_{n=0}^{\infty} \left( R^n + R^{n-1} V + R^{n-2} V^2 \right) (PA)_d^{n+1} - \left[ R_d V + R_d^2 V^2 + R_d V^2 (PA)_d \right] (PA)_d^n + \left[ \sum_{n=0}^{\infty} \left( R_d^{n+1} + R_d^{n+2} V + R_d^{n+3} V^2 \right) (PA)_d^n \right] (PA),
\]

(2.27)

where \( V = AP - CB - PA \) and the symbols \( RV^j = 0, j = 1, 2, \) if \( i < 0 \).
Suppose that there exists a $P \in \mathcal{B}(\mathcal{X})$ such that $PA = PAP$ and $PC = 0$. If $R = (A - CB)(I - P)$ and $PA$ are generalized Drazin invertible and $\text{Ind}(R) = k < +\infty$ and $\text{Ind}(PA) = h < +\infty$, then $A - CB$ is generalized Drazin invertible and

$$
(A - CB)_d = R^n \sum_{k=0}^{n-1} \left( R^k + R^{k+1}V + R^{k+2}V^2 \right) (PA)_d^{n+1} \\
- \left[ R_dV + R_d^2V^2 + R_dV^2(PA)_d \right] (PA)^d \\
+ \left[ \sum_{k=0}^{n-1} \left( R_d^{n+1} + R_d^{n+2}V + R_d^{n+3}V^2 \right) (PA)^n \right] (PA)^{n+1}.
$$

(2.28)

where $V = AP - CBP - PA$ and the symbols $R^jV^j = 0$, $j = 1, 2$, if $i < 0$.

When $PA = AP$ and $P^2 = P$ in Theorem 2.10, we can obtain the following result since $R^n = (A - CB)^n(I - P)$.

**Corollary 2.12** (see [3, Theorem 4.3]). Let $A \in \mathcal{B}(\mathcal{X})$ be the generalized Drazin invertible, $C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, and $B \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. Suppose that there exists an idempotent $P \in \mathcal{B}(\mathcal{X})$ commuting with $A$ such that $PC = 0$. If $R = (A - CB)(I - P)$ is generalized Drazin invertible, then $A - CB$ is the generalized Drazin invertible and

$$
(A - CB)_d = R_d + PA_d - R_dVd + R^n \sum_{n=0}^{\infty} (A - CB)^nV A_d^{n+2} + \sum_{n=0}^{\infty} R_d^{n+2} V A^n A^x,
$$

(2.29)

where $V = -CBP$.

**3. Example**

Before ending this paper, we give an example as follows.

**Example 3.1.** Let

$$
A = \begin{pmatrix}
1 & 2 & 4 & 1 \\
0 & -1 & 1 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
B = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
C = \begin{pmatrix}
1 \\
-1 \\
0 \\
0
\end{pmatrix}.
$$

(3.1)

Then

$$
CB = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
A - CB = \begin{pmatrix}
1 & 2 & 4 & 0 \\
0 & -1 & 1 & 1 \\
0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
$$

(3.2)
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We will compute the Drazin inverse of $A - CB$. To do this, we choose the matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$  \hspace{1cm} (3.3)

Apparently, $P$ is not idempotent and $PA \neq AP$. But $BP = 0$ and

$$AP = PAP = \begin{pmatrix} 1 & -2 & 8 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. $$ \hspace{1cm} (3.4)

Obviously, $\text{Ind}(AP) = 2$. Computing

$$R = (I - P)(A - CB) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \hspace{1cm} R_d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$ \hspace{1cm} (3.5)

$$V = PA - PCB - AP = \begin{pmatrix} 0 & 4 & -4 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$ \hspace{1cm} (3.6)

we have $\text{Ind}(R) = 2$. So, by Corollary 2.2, 

$$(A - CB)_d = \begin{pmatrix} 1 & -4 & 10 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. $$ \hspace{1cm} (3.7)

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