Research Article

Stability of the $n$-Dimensional Mixed-Type Additive and Quadratic Functional Equation in Non-Archimedean Normed Spaces

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We will prove the stability of the functional equation $2f\left(\sum_{i=1}^{n} x_i\right) + \sum_{i<j\leq n} f(x_i - x_j) = (n + 1) \sum_{i=1}^{n} f(x_i) + (n - 1) \sum_{i=1}^{n} f(-x_i)$ in non-Archimedean normed spaces.

1. Introduction

A classical question in the theory of functional equations is “when is it true that a function, which approximately satisfies a functional equation, must be somehow close to an exact solution of the equation?” Such a problem, called a stability problem of the functional equation, was formulated by Ulam in 1940 (see [1]). In the following year, Hyers [2] gave a partial solution of Ulam’s problem for the case of approximate additive functions. Subsequently, his result was generalized by Aoki [3] for additive functions and by Rassias [4] for linear functions. Indeed, they considered the stability problem for unbounded Cauchy differences. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians (see [5–23]).

A non-Archimedean field is a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot| : \mathbb{K} \to [0, \infty)$ such that
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\((F_1)\) \(|r| = 0\) if and only if \(r = 0\);

\((F_2)\) \(|rs| = |r||s|\);

\((F_3)\) \(|r + s| \leq \max\{|r|, |s|\}\) for all \(r, s \in \mathbb{K}\).

Clearly, it holds that \(|1| = |−1| = 1\) and \(|n| \leq 1\) for all \(n \in \mathbb{N}\).

Let \(X\) be a vector space over a scalar field \(\mathbb{K}\) with a non-Archimedean and nontrivial valuation \(|\cdot|\). A function \(\|\cdot\| : X \to \mathbb{R}\) is a non-Archimedean norm (valuation) if it satisfies the following conditions:

\((N_1)\) \(\|x\| = 0\) if and only if \(x = 0\);

\((N_2)\) \(\|rx\| = |r|\|x\|\) for all \(r \in \mathbb{K}\) and \(x \in X\);

\((N_3)\) \(\|x + y\| \leq \max\{\|x\|, \|y\|\}\) for all \(x, y \in X\).

Then \((X, \|\cdot\|)\) is called a non-Archimedean space. Due to the fact that

\[\|x_n - x_m\| \leq \max_{m \leq i < n} \|x_i - x_{i-1}\| \quad (n > m), \quad (1.1)\]

a sequence \(\{x_n\}\) is Cauchy if and only if \(\{x_{n+1} - x_n\}\) converges to zero in a non-Archimedean space. A complete non-Archimedean space is a non-Archimedean space in which every Cauchy sequence is convergent.

Recently, Moslehian and Rassias [24] proved the Hyers-Ulam stability of the Cauchy functional equation

\[f(x + y) = f(x) + f(y), \quad (1.2)\]

and the quadratic functional equation

\[f(x + y) + f(x - y) = 2f(x) + 2f(y) \quad (1.3)\]

in non-Archimedean normed spaces.

We now consider the \(n\)-dimensional mixed-type quadratic and additive functional equation

\[2f\left(\sum_{i=1}^{n} x_i\right) + \sum_{1 \leq i < n, i \neq j} f(x_i - x_j) = (n + 1)\sum_{i=1}^{n} f(x_i) + (n - 1)\sum_{i=1}^{n} f(-x_i), \quad (1.4)\]

whose solution is called a quadratic-additive function.

In 2009, Towanlong and Nakmahachalasint [25] obtained a stability result for the functional equation (1.4), in which they constructed a quadratic-additive function \(F\) by composing an additive function \(A\) and a quadratic function \(Q\), where \(A\) and \(Q\) approximate the odd part and the even part of the given function \(f\), respectively.

In this paper, we investigate a general stability problem for the \(n\)-dimensional mixed-type quadratic and additive functional equation (1.4) in non-Archimedean normed spaces.
2. Solutions of (1.4)

In this section, we prove the generalized Hyers-Ulam stability of the $n$-dimensional mixed-type quadratic and additive functional equation (1.4). Assume that $H$ is an additive group and $X$ is a complete non-Archimedean space.

For a given function $f : H \to X$, we use the abbreviations

$$
\begin{align*}
    f_e(x) &:= \frac{f(x) + f(-x)}{2}, \\
    f_o(x) &:= \frac{f(x) - f(-x)}{2}, \\
    Af(x, y) &:= f(x + y) - f(x) - f(y), \\
    Qf(x, y) &:= f(x + y) + f(x - y) - 2f(x) - 2f(y), \\
    D_n f(x_1, x_2, \ldots, x_n) &:= 2f\left(\sum_{i=1}^{n} x_i\right) + \sum_{1 \leq i < j \leq n} f(x_i - x_j) \\
    &\quad - (n + 1) \sum_{i=1}^{n} f(x_i) - (n - 1) \sum_{i=1}^{n} f(-x_i)
\end{align*}
$$

for all $x, y, x_1, x_2, \ldots, x_n \in H$ and for an arbitrarily fixed $n \in \mathbb{N}$.

**Theorem 2.1.** Assume that $n \geq 2$ is an integer. Let $H$ and $X$ be an additive group and a complete non-Archimedean space, respectively. A function $f : H \to X$ is a solution of (1.4) if and only if $f_e$ is quadratic, $f_o$ is additive, and $f_e(0) = 0$.

**Proof.** If a function $f : H \to X$ is a solution of (1.4), then we have $f_e(0) = 0$,

$$
\begin{align*}
    Qf_e(x, y) &= f_e(x + y) + f_e(x - y) - 2f_e(x) - 2f_e(y) \\
    &= \frac{1}{2}D_n f_e(x, y, 0, \ldots, 0) + \frac{1}{2}(n - 2)(n + 3)f_e(0) \\
    &= 0, \\
    Af_o(x, y) &= f_o(x + y) - f_o(x) - f_o(y) = \frac{1}{2}D_n f_o(x, y, 0, \ldots, 0) = 0
\end{align*}
$$

for all $x, y \in H$, that is, $f_e$ is quadratic and $f_o$ is additive.

Conversely, assume that $f_e$ is quadratic, $f_o$ is additive, and $f_e(0) = 0$. We apply an induction on $j$ to prove $D_n f_e(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in H$. For $j = 2$, we have

$$
\begin{align*}
    D_n f_e(x_1, x_2, 0, \ldots, 0) &= 2f_e(x_1 + x_2) + 2f_e(x_1 - x_2) - 4f_e(x_1) - 4f_e(x_2) - (n - 2)(n + 3)f_e(0) \\
    &= 0.
\end{align*}
$$
If \( n > 2 \) and \( D_n f_e(x_1, x_2, \ldots, x_j, 0, \ldots, 0) = 0 \) for some integer \( j \) (\( 2 \leq j < n \)) and for all \( x_1, x_2, \ldots, x_j \in H \), then a routine calculation yields

\[
D_n f_e(x_1, x_2, \ldots, x_{j+1}, 0, \ldots, 0) \\
= Q f_e(x_1 + \cdots + x_j, x_{j+1} - x_j) + \frac{1}{2} D_n f_e(x_1, \ldots, x_{j-1}, 2x_j, 0, \ldots, 0) \\
+ \frac{1}{2} D_n f_e(x_1, \ldots, x_{j-1}, 2x_{j+1}, 0, \ldots, 0) - \sum_{k=1}^{j-1} (Q f_e(x_k, x_j) + Q f_e(x_k, x_{j+1})) \\
- \frac{j}{2} Q f_e(x_{j+1}, x_{j+1}) - \frac{j}{2} Q f_e(x_j, x_j) \\
= 0
\]

for all \( x_1, x_2, \ldots, x_{j+1} \in H \). Hence, we conclude that \( D_n f_e(x_1, x_2, \ldots, x_n) = 0 \) (2.5)

for all \( x_1, x_2, \ldots, x_n \in H \).

Since \( f_o \) is additive, a long calculation yields

\[
D_n f_o(x_1, x_2, \ldots, x_n) \\
= \sum_{1 \leq i < j \leq n, i \neq j} A f_o(x_i - x_j) + 2 \sum_{i=1}^{n-1} A f_o \left( \sum_{j=1}^{i} x_j \right) \\
= 0.
\]

Hence, it follows from (2.5) and (2.6) that

\[
D_n f(x_1, x_2, \ldots, x_n) = D_n f_e(x_1, x_2, \ldots, x_n) + D_n f_o(x_1, x_2, \ldots, x_n) = 0
\]

(2.7)

for all \( x_1, x_2, \ldots, x_n \in H \); that is, \( f \) is a solution of (1.4).

3. **Generalized Hyers-Ulam Stability of (1.4)**

In the following theorem, we will investigate the stability problem of the functional equation (1.4).

**Theorem 3.1.** Assume that \( n \geq 2 \) is an integer. Let \( H \) and \( X \) be an additive group and a complete non-Archimedean space, respectively. Assume that \( \varphi : H^n \to [0, \infty) \) is a function such that

\[
\lim_{m \to \infty} \varphi(n^m x_1, n^m x_2, \ldots, n^m x_n) = 0
\]

(3.1)
for all $x_1, x_2, \ldots, x_n \in H$. Moreover, assume that the limit

$$
\tilde{\phi}(x) := \lim_{m \to \infty} \max_{0 \leq i < m} \left\{ \frac{\varphi(n^i x, \ldots, n^i x)}{|4||n|^{2i+2}}, \frac{\varphi(-n^i x, \ldots, -n^i x)}{|4||n|^{2i+2}} \right\}
$$

exists for each $x \in H$. If a function $f : H \to X$ satisfies the inequality

$$
\|D_n f(x_1, x_2, \ldots, x_n)\| \leq \varphi(x_1, x_2, \ldots, x_n)
$$

for any $x_1, x_2, \ldots, x_n \in H$, then there exists a unique quadratic-additive function $T : H \to X$ such that

$$
\|f(x) - T(x)\| \leq \tilde{\phi}(x)
$$

for each $x \in H$. In particular, $T$ is given by

$$
T(x) = \lim_{m \to \infty} \left( \frac{f(n^m x) + f(-n^m x)}{2n^{2m}} + \frac{f(n^m x) - f(-n^m x)}{2n^{m}} \right)
$$

for all $x \in H$.

**Proof.** If we replace $x_i$ in (3.1) with 0 for each $i \in \{1, 2, \ldots, n\}$, then we have

$$
\lim_{m \to \infty} \frac{\varphi(0, 0, \ldots, 0)}{|n|^{2m}} = 0.
$$

Since $|n| \leq 1$, it holds that $\varphi(0, 0, \ldots, 0) = 0$ and

$$
\|(n - 1)(n + 2)f(0)\| = \|D_n f(0, 0, \ldots, 0)\| \leq \varphi(0, 0, \ldots, 0) = 0.
$$

Hence, we conclude that $f(0) = 0$.

Let $J_m f : H \to Y$ be a function defined by

$$
J_m f(x) = \frac{f(n^m x) + f(-n^m x)}{2n^{2m}} + \frac{f(n^m x) - f(-n^m x)}{2n^{m}}
$$
for all $x \in H$ and $m \in \{0, 1, 2, \ldots\}$. A tedious calculation, together with $(F_2)$, $(N_3)$, and (3.3), yields

$$
\| J_i f(x) - J_{i+1} f(x) \| = \left\| \frac{D_n f(n^i x, \ldots, n^i x)}{4 n^{2i+2}} - \frac{D_n f(-n^i x, \ldots, -n^i x)}{4 n^{2i+2}} + \frac{D_n f(n^i x, \ldots, n^i x)}{4 n^{i+1}} - \frac{D_n f(-n^i x, \ldots, -n^i x)}{4 n^{i+1}} \right\|
\leq \max \left\{ \frac{\| D_n f(n^i x, \ldots, n^i x) \|}{|4||n|^{2i+2}}, \frac{\| D_n f(-n^i x, \ldots, -n^i x) \|}{|4||n|^{2i+2}} \right\},
$$

(3.9)

for all $x \in H$ and $i \in \{0, 1, 2, \ldots\}$. It follows from (3.1) and (3.9) that the sequence $\{ J_m f(x) \}$ is Cauchy. Since $X$ is complete, we conclude that $\{ J_m f(x) \}$ is convergent.

Let us define

$$
T(x) := \lim_{m \to \infty} J_m f(x)
$$

(3.10)

for any $x \in H$. It follows from $(N_3)$ and (3.9) that

$$
\| f(x) - J_m f(x) \| = \left\| \sum_{i=0}^{m-1} (J_i f(x) - J_{i+1} f(x)) \right\|
\leq \max_{0 \leq i \leq m} \| J_i f(x) - J_{i+1} f(x) \|
\leq \max_{0 \leq i \leq m} \left\{ \frac{\| f(n^i x, \ldots, n^i x) \|}{|4||n|^{2i+2}}, \frac{\| f(-n^i x, \ldots, -n^i x) \|}{|4||n|^{2i+2}} \right\}
$$

(3.11)

for all $m \in \{0, 1, 2, \ldots\}$ and $x \in H$. In view of (3.2), if we let $m \to \infty$ in (3.11), then we obtain the inequality (3.4).

Replacing $x_i$ in (3.3) with $n^m x_i$ for $i \in \{1, 2, \ldots, n\}$ and considering $(F_2)$ and $(N_3)$, we get

$$
\| D_n J_m f(x_1, x_2, \ldots, x_n) \| = \left\| \frac{D_n f(n^m x_1, \ldots, n^m x_n) - D_n f(-n^m x_1, \ldots, -n^m x_n)}{2 n^m} + \frac{D_n f(n^m x_1, \ldots, n^m x_n) + D_n f(-n^m x_1, \ldots, -n^m x_n)}{2 n^{2m}} \right\|
$$
for all $m \in \{0, 1, 2, \ldots\}$ and $x_1, x_2, \ldots, x_n \in H$. If we let $m \to \infty$ in the last inequality, then it follows from the condition (3.1) that $D_n T(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in H$; that is, $T$ is a quadratic-additive function.

Assume that $T' : H \to X$ is another quadratic-additive function satisfying (3.4). By the definition of $D_n$, a routine calculation yields

$$
-D_n T'(n' x, \ldots, n' x) + D_n T'(-n' x, \ldots, -n' x) = \frac{1}{2n^{2(i+1)}} (T'(n^{i+1} x) - T'(-n^{i+1} x)) - \frac{1}{2n^i} (T'(n^i x) - T'(-n^i x))
$$

for each $i \in \{0, 1, 2, \ldots\}$ and $x \in H$. Hence, it follows from (3.8) that

$$
\sum_{i=0}^{k-1} \left( \frac{D_n T'(n' x, \ldots, n' x)}{4n^{2i+2}} - \frac{D_n T'(-n' x, \ldots, -n' x)}{4n^{2i+2}} - \frac{D_n T'(n' x, \ldots, n' x)}{4n^{i+1}} + \frac{D_n T'(-n' x, \ldots, -n' x)}{4n^{i+1}} \right) = T'(x) - J_k T'(x)
$$

for any $k \in \mathbb{N}$ and $x \in H$. Since $T'$ is a solution of (1.4), it follows from the last equality that

$$
T'(x) = J_k T'(x)
$$

for any $k \in \mathbb{N}$ and $x \in H$. Obviously, this equality also holds for $T$.

Consequently, by considering that $|n| \leq 1$, it follows from $(N_3)$, (3.1), (3.4), and (3.8) that

$$
\|T(x) - T'(x)\| = \lim_{k \to \infty} \|J_k T(x) - J_k T'(x)\| \leq \lim_{k \to \infty} \max\{\|J_k T(x) - J_k f(x)\|, \|J_k f(x) - J_k T'(x)\|\}
$$
\[
\leq \lim_{k \to \infty} |2|^{-1} n^{-2k} \max \left\{ \| T(n^k x) - f(n^k x) \|, \| T(-n^k x) - f(-n^k x) \|, \| f(n^k x) - T'(n^k x) \|, \| f(-n^k x) - T'(-n^k x) \| \right\}
\leq \lim_{k \to \infty} \lim_{m \to \infty} \max_{k \leq m+k} \left\{ \frac{\varphi(n^k x, \ldots, n^k x) - \varphi(-n^k x, \ldots, -n^k x)}{|8||n|^{2r+2}}, \frac{\varphi(n^k x, \ldots, n^k x) - \varphi(-n^k x, \ldots, -n^k x)}{|8||n|^{2r+2}} \right\}
= 0
\]
(3.16)

for all \( x \in H \). Therefore, \( T = T' \), which proves the uniqueness of \( T \).

\[\Box\]

**Corollary 3.2.** Let \( X \) and \( Y \) be non-Archimedean normed spaces over \( \mathbb{K} \) with \( |n| < 1 \). If \( Y \) is complete and \( f : X \to Y \) satisfies the inequality
\[
\| Df(x_1, x_2, \ldots, x_n) \| \leq \theta \sum_{i=1}^{n} \| x_i \|^r
\]
(3.17)

for all \( x_1, x_2, \ldots, x_n \in X \) and for some \( r > 2 \), then there exists a unique quadratic-additive function \( T : X \to Y \) such that
\[
\| f(x) - T(x) \| \leq \frac{n\theta}{|4||n|^2} \| x \|^r
\]
(3.18)

for all \( x \in X \).

**Proof.** Let \( \varphi(x_1, x_2, \ldots, x_n) = \theta \sum_{i=1}^{n} \| x_i \|^r \). Since \( |n| < 1 \) and \( r - 2 > 0 \), we get
\[
\lim_{m \to \infty} |n|^{-2m} \varphi(n^m x_1, n^m x_2, \ldots, n^m x_n) = \lim_{m \to \infty} |n|^{m(r-2)} \varphi(x_1, x_2, \ldots, x_n) = 0
\]
(3.19)

for all \( x_1, x_2, \ldots, x_n \in X \). Therefore, the conditions of Theorem 3.1 are satisfied. Indeed, it is easy to see that \( \varphi(x) = n\theta(|4||n|^2)\| x \|^r \). By Theorem 3.1, there exists a unique quadratic-additive function \( T : X \to Y \) such that (3.18) holds. \[\Box\]

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